Abelian and exact DG-categories

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Summary of two blackboard talks

Two branches of homological algebra

(here homological algebra $=$ study of derived categories)

0. $<$ I. $\wedge$ II. — common roots of two branches: modules, sheaves, comodules, … (or rather, complexes of these)

I. — first branch: abelian categories, (Quillen's) exact categories (or rather, complexes in these)

II. — second branch: DG-modules, sheaves of DG-modules, DG-comodules (DG-contramodules, …)

Curved generalizations:

I. $<$ I$'$: factorization categories (factorizations of an endomorphism of the identity functor or of a morphism from the identity functor to an auto-equivalence of an abelian/exact category)

II. $<$ II$'$: curved DG-modules, sheaves of CDG-modules, CDG-comodules (CDG-contramodules, …)

Two branches of homological algebra

$I'. \vee II'. < III.$ — Common generalization of the two branches: abelian DG-categories, exact DG-categories

Caveat: Derived categories of the 2^{nd} kind (coderived, contraderived, absolute derived categories) are naturally defined for abelian/exact DG-categories.

Conventional derived categories (of the 1^{st} kind) — not so clear how to define.

Extra remark on derived categories

One clear problem/feature is that the abelian DG-category of DG-modules over any DG-ring $R^\bullet$ is equivalent to the abelian DG-category of DG-modules over a suitable acyclic DG-ring $\widehat{\widehat{R}}^\bullet$ (with a vanishing cohomology ring $H^*(\widehat{\widehat{R}}^\bullet) = 0$). So the derived category $\mathrm{D}(\widehat{\widehat{R}}^\bullet\text{-}\mathbf{Mod}) = 0$ but $\mathrm{D}(R^\bullet\text{-}\mathbf{Mod}) \neq 0$ (if $H^*(R^\bullet) \neq 0$).

Still the coderived category $\mathrm{D}^{\mathrm{co}}(\widehat{\widehat{R}}^\bullet\text{-}\mathbf{Mod}) = \mathrm{D}^{\mathrm{co}}(R^\bullet\text{-}\mathbf{Mod})$ and the contraderived category $\mathrm{D}^{\mathrm{ctr}}(\widehat{\widehat{R}}^\bullet\text{-}\mathbf{Mod}) = \mathrm{D}^{\mathrm{ctr}}(R^\bullet\text{-}\mathbf{Mod})$. From our ("second kind") point of view, $\widehat{\widehat{R}}^\bullet$ is an acyclic DG-ring Morita equivalent to $R^\bullet$.

For example, if $R^\bullet = R$ is just a ring (viewed as a DG-ring with $R^i = 0$ for $i \neq 0$, $R^0 = R$, and zero differential), then $\widehat{\widehat{R}}^\bullet$ is the DG-ring of endomorphisms of the acyclic complex of right R-modules $\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\mathrm{id}} R \longrightarrow 0 \longrightarrow \cdots$.

What is an exact DG-category?

Main point: An exact DG-category has two underlying exact categories.

Example

Let $R^\bullet = (R, d)$ be a DG-ring. We denote by $R^\bullet\text{-}\mathbf{Mod}$ the abelian DG-category of DG-modules over $R^\bullet$. The abelian DG-category $R^\bullet\text{-}\mathbf{Mod}$ has two underlying abelian categories:

- The abelian category $R\text{-}\mathrm{Mod}$ of graded modules over the graded ring R. Objects: graded R-modules (no differential). Morphisms: homogeneous R-module morphisms of degree 0.
- The abelian category $\mathrm{Z}^0(R^\bullet\text{-}\mathbf{Mod})$ of DG-modules over $R^\bullet$. Objects: DG-modules over $R^\bullet$. Morphisms: closed morphisms of degree 0 (preserving the grading, and commuting with the actions of R and the differentials on the DG-modules).

The notation $\mathrm{Z}^0(\mathbf{A})$ for a DG-category $\mathbf{A}$ will be explained below.

A DG-category is a category enriched in complexes of abelian groups. In other words, a DG-category $\mathbf{A}$ consists of a class of objects $\mathcal{O}b\,\mathbf{A}$ (we will write simply $X \in \mathbf{A}$ instead of $X \in \mathcal{O}b\,\mathbf{A}$), complexes of abelian groups $\mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y)$ defined for all X, $Y \in \mathbf{A}$, and associative, unital multiplication maps

$$\mathrm{Hom}^\bullet_{\mathbf{A}}(Y, Z) \otimes_{\mathbb{Z}} \mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y) \longrightarrow \mathrm{Hom}^\bullet_{\mathbf{A}}(X, Z).$$

defined for all X, Y, $Z \in \mathbf{A}$. The identity morphisms $\mathrm{id}_X \in \mathrm{Hom}^0_{\mathbf{A}}(X, X)$ must be cocycles, i.e., $d(\mathrm{id}_X) = 0$.

To a DG-category $\mathbf{A}$ one can simply assign three preadditive categories $\mathbf{A}^0$, $\mathrm{Z}^0(\mathbf{A})$, and $\mathrm{H}^0(\mathbf{A})$ with $\mathcal{O}b\,\mathbf{A}^0 = \mathcal{O}b\,\mathrm{Z}^0(\mathbf{A}) = \mathcal{O}b\,\mathrm{H}^0(\mathbf{A}) = \mathcal{O}b\,\mathbf{A}$ and

- $\mathrm{Hom}_{\mathbf{A}^0}(X, Y) = \mathrm{Hom}^0_{\mathbf{A}}(X, Y)$
- $\mathrm{Hom}_{\mathrm{Z}^0(\mathbf{A})}(X, Y) = Z^0\,\mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y)$
- $\mathrm{Hom}_{\mathrm{H}^0(\mathbf{A})}(X, Y) = H^0\,\mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y)$

being the groups of degree 0 cochains, cocycles, and cohomology (respectively) of the complex $\mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y)$.

Strict theory warning

Our theory of DG-categories is a strict one. We consider complexes up to isomorphism and not up to quasi-isomorphism. We work with DG-categories up to an equivalence and not up to quasi-equivalence. A "fully faithful DG-functor" for us is a DG-functor inducing isomorphisms on the complexes of morphisms rather than quasi-isomorphisms.

For example, assigning to a DG-category $\mathbf{A}$ the preadditive categories $\mathbf{A}^0$ and $\mathrm{Z}^0(\mathbf{A})$ would not be possible if $\mathbf{A}$ were considered up to quasi-equivalence. To a DG-category $\mathbf{A}$ viewed up to quasi-equivalence, only the preadditive category $\mathrm{H}^0(\mathbf{A})$ and its variations can be assigned; this is not what we are doing.

Recovering R-Mod from $R^\bullet$-$\mathbf{Mod}$

Let $R^\bullet = (R, d)$ be a DG-ring with the underlying graded ring R. Suppose given the DG-category of DG-modules $\mathbf{A} = R^\bullet\text{-}\mathbf{Mod}$.

There are two abelian categories associated with $\mathbf{A}$, as mentioned above. One of them is simply $\mathrm{Z}^0(\mathbf{A}) = \mathrm{Z}^0(R^\bullet\text{-}\mathbf{Mod})$ (we have already explained this notation). The other one is the abelian category R-Mod of graded R-modules.

Question: Given the DG-category $\mathbf{A} = R^\bullet\text{-}\mathbf{Mod}$, how can one recover the abelian category R-Mod ?

Naïve attempt to recover R-Mod from $R^\bullet$-$\mathbf{Mod}$: consider the additive category $\mathbf{A}^0 = (R^\bullet\text{-}\mathbf{Mod})^0$. In the category $\mathbf{A}^0$:

- the objects are the DG-modules over $R^\bullet$;
- the morphisms are the homogeneous R-module morphisms of degree 0, not necessarily commuting with the differentials.

How does the category $(R^\bullet\text{-}\mathbf{Mod})^0$ compare to R-Mod ? It is a badly behaved full subcategory.

Recovering R-Mod from $R^\bullet$-Mod

There is a natural additive functor

$$(R^\bullet\text{-}\mathbf{Mod})^0 \longrightarrow R\text{-}\mathrm{Mod}$$

assigning to a DG-module $M^\bullet = (M, d_M)$ over $R^\bullet = (R, d)$ its underlying graded R-module M (with the differential forgotten),

$$M^\bullet = (M, d_M) \longmapsto M.$$

This functor is fully faithful, but not essentially surjective.

Not every graded R-module admits a DG-module structure over $R^\bullet$!

The full subcategory $(R^\bullet\text{-}\mathbf{Mod})^0 \subset R\text{-}\mathrm{Mod}$ consists of all the graded R-modules admitting some (at least one) DG-module structure over $R^\bullet$. This full subcategory in the abelian category R-Mod is not well-behaved.

Recovering R-Mod from $R^\bullet$-Mod

The full additive subcategory $(R^\bullet\text{-}\mathbf{Mod})^0 \subset R\text{-Mod}$ is, generally speaking,

- not closed under direct summands,
- not closed under kernels (not even under the kernels of epimorphisms),
- not closed under cokernels (not even under the cokernels of monomorphisms),
- not closed under extensions.

Even the direct summand closure of $(R^\bullet\text{-}\mathbf{Mod})^0$ in R-Mod is, in general,

- not closed under kernels (not even under the kernels of epimorphisms),
- not closed under cokernels (not even under the cokernels of monomorphisms),
- not closed under extensions

in the ambient abelian category R-Mod.

Recovering R-Mod from $R^\bullet$-$\mathbf{Mod}$

Given a graded R-module M, how to assign to it a DG-module over $R^\bullet$? Consider the DG-module $G^+(M)$ freely generated by M. This is a contractible DG-module over $R^\bullet$!

The degree n component $G^+(M)^n$ of the DG-module $G^+(M)$ consists of all formal expressions $m' + d(m'')$, where $m' \in M^n$ and $m'' \in M^{n-1}$. No relations are imposed on such formal expressions. The differential and the action of R on $G^+(M)$ are defined by the obvious formulas $d(m' + d(m'')) = 0 + d(m')$ and

$$r(m' + d(m'')) = (rm' - (-1)^{|r|} d(r)m'') + (-1)^{|r|} d(rm'')$$

for all $r \in R$. The map $m' + d(m'') \longmapsto m'' + d(0)$ is a natural R-linear contracting homotopy on $G^+(M)$.

The abelian category R-Mod can be recovered from the DG-category $R^\bullet$-$\mathbf{Mod}$ as a certain category of contractible objects in $R^\bullet$-$\mathbf{Mod}$ endowed with a fixed contracting homotopy with zero square. Abstracting from this construction, one arrives to the following procedure assigning a new DG-category to a given one.

The $\natural$ construction

To any DG-category $\mathbf{A}$ we assign the following DG-category $\mathbf{A}^\natural$. The objects of $\mathbf{A}^\natural$ are pairs $X^\natural = (X, \sigma_X)$, where

- X is an object of $\mathbf{A}$,
- $\sigma_X \in \mathrm{Hom}_{\mathbf{A}}^{-1}(X, X)$ is a cochain of cohomological degree -1 in the complex of endomorphisms,
- $d(\sigma_X) = \mathrm{id}_X$ in $\mathrm{Hom}_{\mathbf{A}}^0(X, X)$,
- $\sigma_X^2 = 0$ in $\mathrm{Hom}_{\mathbf{A}}^{-2}(X, X)$.

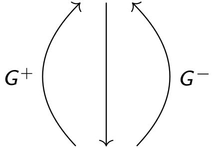
The complex of morphisms $\mathrm{Hom}_{\mathbf{A}^\natural}^\bullet((X, \sigma_X), (Y, \sigma_Y))$ is constructed by the rules

- $\mathrm{Hom}_{\mathbf{A}^\natural}^n((X, \sigma_X), (Y, \sigma_Y)) = Z^{-n}\,\mathrm{Hom}_{\mathbf{A}}^\bullet(X, Y)$ is the group of degree $-n$ cocycles in the complex $\mathrm{Hom}_{\mathbf{A}}^\bullet(X, Y)$,
- $d^\natural(f) =$ "$[\sigma, f]$" $= \sigma_Y \circ f - (1)^n f \circ \sigma_X$ for all $f \in \mathrm{Hom}_{\mathbf{A}^\natural}^n((X, \sigma_X), (Y, \sigma_Y))$,
- the composition of morphisms in $\mathbf{A}^\natural$ is induced by the composition of moprhisms in $\mathbf{A}$.

The $\mathbf{A} = R^\bullet\text{-}\mathbf{Mod}$ example

Let $R^\bullet = (R, d)$ be a DG-ring, and let $\mathbf{A} = R^\bullet\text{-}\mathbf{Mod}$ be the DG-category of DG-modules. Then

$$\mathrm{Z}^0(\mathbf{A}) = \text{the abelian category of DG-modules over } R^\bullet$$



$$\mathrm{Z}^0(\mathbf{A}^\natural) \simeq \text{the abelian category } R\text{-}\mathrm{Mod} \text{ of graded } R\text{-modules}$$

Here $\simeq$ denotes a natural equivalence of additive/abelian categories. The straight downward arrow denotes the forgetful functor, and the two curvilinear upward arrows denote its left and right adjoint functors. The two adjoint functors G^+ and G^- only differ by a shift: there is a natural (closed) isomorphism of DG-modules $G^-(M) \simeq G^+(M)[1]$ for every graded R-module M.

Basic definitions and constructions II

Let $\mathbf{A}$ be a DG-category. The following operations on the objects of $\mathbf{A}$ are relevant for us.

Finite direct sums: let X, $Y \in \mathbf{A}$ be two objects. By definition, $U = X \oplus Y$ in $\mathbf{A}$ if $U = X \oplus Y$ in $\mathrm{Z}^0(\mathbf{A})$.

In other words, this means that there exist morphisms $\iota_X \in \mathrm{Hom}^0_{\mathbf{A}}(X, U)$, $\iota_Y \in \mathrm{Hom}^0_{\mathbf{A}}(Y, U)$, $\pi_X \in \mathrm{Hom}^0_{\mathbf{A}}(U, X)$, $\pi_Y \in \mathrm{Hom}^0_{\mathbf{A}}(U, Y)$ satisfying the equations

$$d(\iota_X) = 0 = d(\iota_Y), \quad d(\pi_X) = 0 = d(\pi_Y),$$
$$\pi_X \iota_X = \mathrm{id}_X, \quad \pi_Y \iota_Y = \mathrm{id}_Y,$$
$$\pi_X \iota_Y = 0 = \pi_Y \iota_X, \quad \iota_X \pi_X + \iota_Y \pi_Y = \mathrm{id}_U\,.$$

Shifts: let X, $W \in \mathbf{A}$ be two objects. We write $W = X[n]$ (for some $n \in \mathbb{Z}$) if there are morphisms $s \in \mathrm{Hom}^{-n}(X, W)$ and $t \in \mathrm{Hom}^n(W, X)$ satisfying the equations

$$d(s) = 0 = d(t), \quad ts = \mathrm{id}_X, \quad st = \mathrm{id}_W\,.$$

Basic definitions and constructions II

Cones: let $f\colon X \longrightarrow Y$ be a closed morphism of degree 0 in $\mathbf{A}$, i.e., $f \in \mathrm{Hom}^0_{\mathbf{A}}(X, Y)$ and $d(f) = 0$. We say that $C = \mathrm{cone}(f)$ if there are morphisms

$$Y \underset{\iota}{\overset{\pi'}{\rightleftarrows}} C \underset{\pi}{\overset{\iota'}{\rightleftarrows}} X[1]$$

$\iota \in \mathrm{Hom}^0_{\mathbf{A}}(Y, C)$, $\pi \in \mathrm{Hom}^1_{\mathbf{A}}(C, X)$, $\iota' \in \mathrm{Hom}^{-1}_{\mathbf{A}}(X, C)$, $\pi' \in \mathrm{Hom}^0_{\mathbf{A}}(C, Y)$ satisfying the equations

$$\pi'\iota = \mathrm{id}_Y, \quad \pi\iota' = \mathrm{id}_X,$$
$$\pi\iota = 0 = \pi'\iota', \quad \iota\pi' + \iota'\pi = \mathrm{id}_C,$$
$$d(\iota) = 0 = d(\pi), \quad d(\iota') = \iota f, \quad d(\pi') = -f\pi.$$

Maurer–Cartan twists: let $X \in \mathbf{A}$ be an object. A morphism $a \in \mathrm{Hom}^1_{\mathbf{A}}(X, X)$ is called a Maurer–Cartan cochain if the Maurer–Cartan equation $d(a) + a^2 = 0$ is satisfied.

Let $W \in \mathbf{A}$ be another object. We write $W = X(a)$ and say that W is the Maurer–Cartan twist of X by a if there are morphisms $s \in \mathrm{Hom}^0_{\mathbf{A}}(X, W)$ and $t \in \mathrm{Hom}^0_{\mathbf{A}}(W, X)$ satisfying the equations

$$ts = \mathrm{id}_X, \quad st = \mathrm{id}_W, \quad d(s) = sa, \quad d(t) = -at.$$

Let X, $W \in \mathbf{A}$ be two objects. Then there exists a Maurer–Cartan cochain $a \in \mathrm{Hom}^1_{\mathbf{A}}(X, X)$ such that $W = X(a)$ if and only if the objects X and W are isomorphic in the category $\mathbf{A}^0$.

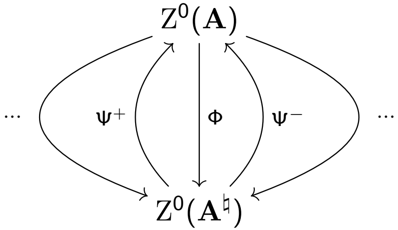
The cone of a closed morphism $f\colon X \longrightarrow Y$ of degree 0 can be expressed in terms of a shift, a finite direct sum, and a twist:

$$\mathrm{cone}(f) = (Y \oplus X[1])(a_f),$$

where $a_f \in \mathrm{Hom}^1_{\mathbf{A}}(Y \oplus X[1],\ Y \oplus X[1])$ is constructed in terms of f. (In fact, one has $d(a_f) = 0 = a_f^2$ in this case.)

Ladder of adjoint additive functors

Let $\mathbf{A}$ be a DG-category with shifts and cones (as defined above, i.e., in the strict sense). Then there is an infinite ladder of adjoint functors between additive categories



Here $\Psi^+(X, \sigma_X) := X$ is "constructed as the forgetful functor". The functor Ψ^- only differs by a shift: $\Psi^-(X^\natural) = \Psi^+(X^\natural)[1]$.

In the above example of $\mathbf{A} = R^\bullet\text{-}\mathbf{Mod}$, under the identification $\mathrm{Z}^0(\mathbf{A}^\natural) \simeq R\text{-}\mathrm{Mod}$ the functor Φ becomes the forgetful functor, Ψ^+ is identified with G^+, and Ψ^- is identified with G^-.

Thus the functor Φ is "interpreted as the forgetful functor" (in this example, and also generally).

Fully faithful additive functors $\widetilde{\Phi}$ and $\widetilde{\Psi}^{\pm}$

Let us emphasize that Φ and $\Psi^{\pm}$ are not DG-functors. They only act between the additive categories $\mathsf{Z}^0(\mathbf{A})$ and $\mathsf{Z}^0(\mathbf{A}^{\natural})$.

Similarly as in the example of $\mathbf{A} = R^{\bullet}\text{-}\mathbf{Mod}$, the additive category $\mathbf{A}^0$ is in general a full subcategory in $\mathsf{Z}^0(\mathbf{A}^{\natural})$, and the additive category $(\mathbf{A}^{\natural})^0$ is a full subcategory in $\mathsf{Z}^0(\mathbf{A})$. In other words, there are commutative diagrams of additive functors

$$\begin{array}{ccc} \mathsf{Z}^0(\mathbf{A}) & \hookrightarrow & \mathbf{A}^0 \\ {\scriptstyle\Phi}\downarrow & \swarrow {\scriptstyle\widetilde{\Phi}} & \\ \mathsf{Z}^0(\mathbf{A}^{\natural}) & & \end{array} \qquad \begin{array}{ccc} \mathsf{Z}^0(\mathbf{A}^{\natural}) & \hookrightarrow & (\mathbf{A}^{\natural})^0 \\ {\scriptstyle\Psi^{+}}\downarrow & \swarrow {\scriptstyle\widetilde{\Psi}^{+}} & \\ \mathsf{Z}^0(\mathbf{A}) & & \end{array}$$

Here $\mathsf{Z}^0(\mathbf{A}) \hookrightarrow \mathbf{A}^0$ and $\mathsf{Z}^0(\mathbf{A}^{\natural}) \hookrightarrow (\mathbf{A}^{\natural})^0$ are the (faithful, but not full!) identity inclusions (bijective on objects), while the functors $\widetilde{\Phi}\colon \mathbf{A}^0 \rightarrowtail \mathsf{Z}^0(\mathbf{A}^{\natural})$ and $\widetilde{\Psi}^{+}\colon (\mathbf{A}^{\natural})^0 \rightarrowtail \mathsf{Z}^0(\mathbf{A})$ are fully faithful.

Extra remark on kernels and cokernels

The following little observation can be viewed as a simple application of the techniques described above.

Lemma

Let $\boldsymbol{A}$ be a DG-category with shifts and cones. Let $K \xrightarrow{k} X \xrightarrow{f} Y \xrightarrow{c} C$ be three morphisms in the category $\mathrm{Z}^0(\boldsymbol{A})$. Then

- (a) *if k is the kernel of f in the category $\mathrm{Z}^0(\boldsymbol{A})$, then k is also the kernel of f in the category $\boldsymbol{A}^0$;*
- (b) *if c is the cokernel of f in the category $\mathrm{Z}^0(\boldsymbol{A})$, then c is also the cokernel of f in the category $\boldsymbol{A}^0$.*

Proof.

The functor $\Phi\colon \mathrm{Z}^0(\mathbf{A}) \longrightarrow \mathrm{Z}^0(\mathbf{A}^\natural)$ preserves kernels and cokernels, since it has adjoints on both sides. Since the functor $\widetilde{\Phi}\colon \mathbf{A}^0 \rightarrowtail \mathrm{Z}^0(\mathbf{A}^\natural)$ is fully faithful and the triangular diagram above is commutative, it follows that the identity inclusion functor $\mathrm{Z}^0(\mathbf{A}) \hookrightarrow \mathbf{A}^0$ also preserves kernels and cokernels. □

The DG-functor $\natural\natural$

The discussion above illustrates that the roles of the DG-categories $\mathbf{A}$ and $\mathbf{A}^\natural$ are almost interchangeable. In fact, the assignment

$$\natural\colon \mathrm{DG\text{-}Cats} \longrightarrow \mathrm{DG\text{-}Cats}$$

is an "almost involution" in the following precise sense.

For any DG-category $\mathbf{A}$ with shifts and cones, there is a natural fully faithful DG-functor

$$\natural\natural\colon \mathbf{A} \longrightarrow \mathbf{A}^{\natural\natural}.$$

For any idempotent-complete DG-category $\mathbf{A}$ with shifts, finite direct sums, and Maurer–Cartan twists, the DG-functor $\natural\natural$ is an equivalence of DG-categories.

Here the idempotent completeness condition (as well as all the other conditions) is understood in the strict sense: a DG-category $\mathbf{A}$ with finite direct sums is said to be idempotent-complete if the additive category $\mathrm{Z}^0(\mathbf{A})$ is idempotent-complete.

Extra remarks on the $\natural$ construction

Not every DG-category $\mathbf{B}$ has the form $\mathbf{A}^\natural$ for some DG-category $\mathbf{A}$. In particular, all the Maurer–Cartan twists exist in the DG-category $\mathbf{A}^\natural$, for any DG-category $\mathbf{A}$.

Let $\mathbf{A}$ be a DG-category with shifts and cones. Then, generally speaking, the passage from $\mathbf{A}$ to $\mathbf{A}^{\natural\natural}$ adjoins to $\mathbf{A}$ all the Maurer–Cartan twists and some of their direct summands (i.e., some images of idempotent closed endomorphisms of the Maurer–Cartan twists of the objects of $\mathbf{A}$).

The passage from $\mathbf{A}$ to $\mathbf{A}^\natural$ preserves equivalences, but not quasi-equivalences of DG-categories. In particular, if $\mathbf{A}$ is the DG-category of complexes in an idempotent-complete additive category A, then all objects of the DG-category $\mathbf{A}^\natural$ are contractible, while the DG-category $\mathbf{A}^{\natural\natural}$ is equivalent to $\mathbf{A}$.

Abelian DG-categories

A DG-category $\mathbf{A}$ with finite direct sums, shifts, and cones is called abelian if both $\mathrm{Z}^0(\mathbf{A})$ and $\mathrm{Z}^0(\mathbf{A}^\natural)$ are abelian categories.

Theorem

Let $\boldsymbol{A}$ be a DG-category with finite direct sums, shifts, and cones. Then the following conditions are equivalent:

1. *$\boldsymbol{A}$ is an abelian DG-category;*
2. *$\mathrm{Z}^0(\boldsymbol{A})$ is an abelian category;*
3. *$\mathrm{Z}^0(\boldsymbol{A}^\natural)$ is an abelian category, and the additive category $\mathrm{Z}^0(\boldsymbol{A})$ has kernels and cokernels;*
4. *$\mathrm{Z}^0(\boldsymbol{A}^\natural)$ is an abelian category, and the DG-category $\boldsymbol{A}$ is idempotent-complete and has Maurer–Cartan twists.*

The last item (4) emphasizes the point that all the Maurer–Cartan twists exist in any abelian DG-category. Consequently, the DG-functor $\natural\natural\colon \mathbf{A} \longrightarrow \mathbf{A}^{\natural\natural}$ is an equivalence of DG-categories for any abelian DG-category $\mathbf{A}$.

Exact DG-categories

A DG-category $\mathbf{E}$ with finite direct sums, shifts, and cones is called exact if both the additive categories $\mathrm{Z}^0(\mathbf{E})$ and $\mathrm{Z}^0(\mathbf{E}^\natural)$ are endowed with exact category structures, preserved by the shift functors $[1]$, $[-1]$ and such that both the additive functors Φ and Ψ^+ preserve and reflect admissible short exact sequences.

Theorem

Let $\boldsymbol{E}$ be a DG-category with finite direct sums, shifts, and cones. Then specifying an exact DG-category structure on $\boldsymbol{E}$ is equivalent to specifying an exact category structure on the additive category $\mathrm{Z}^0(\boldsymbol{E})$ such that the following conditions are satisfied:

- *the exact category structure on $\mathrm{Z}^0(\boldsymbol{E})$ is preserved by the shift functors $[1]$, $[-1]$;*
- *the additive functor*

  $$\Xi = \Psi^+ \circ \Phi = (X \longmapsto \mathrm{cone}(\mathrm{id}_{X[-1]}))\colon \mathrm{Z}^0(\boldsymbol{E}) \longrightarrow \mathrm{Z}^0(\boldsymbol{E})$$

  preserves and reflects admissible short exact sequences in $\mathrm{Z}^0(\boldsymbol{E})$.

L. Positselski. Exact DG-categories and fully faithful triangulated inclusion functors. Electronic preprint arXiv:2110.08237 [math.CT], 2021–22, 148 pp.