Exact category structures and tensor products of topological vector spaces with linear topology

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Introduction

Topological vector spaces, such as Banach spaces or locally convex spaces over the topological/normed fields of real or complex numbers, are a central concept in functional analysis.

Topological tensor products of topological vector spaces in functional analysis were studied by Grothendieck in 1950's.

The topological vector spaces with linear topology form the most "algebraic" class of topological vector spaces.

Topological vector spaces with linear topology can be considered over an arbitrary field k. The field k is viewed as endowed with the discrete topology.

The categories of topological abelian groups or topological vector spaces are rarely abelian. Constructing and using exact category structures (in Quillen's sense) in order to develop homological topological algebra or homological functional analysis is an old idea, going back to the 1960's–70's.

Part I

Topological Algebra

Topological Vectors Spaces with Linear Topology

We fix once and for all a ground field k, and endow it with the discrete topology. A topological vector space V is a k-vector space endowed with a topological space structure such that the summation map $+\colon V \times V \longrightarrow V$ is continuous (as a function of two variables) and the homothety map $a* : V \longrightarrow V$ is continuous for every $a \in k$.

A topological vector space V is said to have a linear topology if open vector subspaces form a base of neighborhoods of zero in V. Any filter of vector subspaces in a vector space V defines a linear topology on V in which the open subspaces are the ones belonging to the filter.

In the rest of this talk, all "topological vector spaces" will be presumed to have linear topology. The abbreviation "VSLT" means a "(topological) vector space with linear topology".

Subspaces and Quotients of VSLTs

Let V be a topological vector space. Given a vector subspace $K \subset V$, the induced topology on K is defined by the rule that the open subspaces in K are the intersections $K \cap U$, where U ranges over the open subspaces in V.

An injective linear map of topological vector spaces $i\colon K \longrightarrow V$ is a closed continuous map if and only if $i(K)$ is a closed subspace in V and the topology of K is induced from the topology of V via i.

Given a surjective map of vector spaces $p\colon V \longrightarrow C$, the quotient topology on C is defined by the rule that the open subspaces in C are the images $p(U)$ of the open subspaces $U \subset V$.

A surjective linear map of topological vector spaces $p\colon V \longrightarrow C$ is an open continuous map if and only if the topology of C is the quotient topology of the topology of V.

Products and Coproducts of VSLTs

Let $\{V_i\}_{i\in I}$ be a family of topological vector spaces. Then the product topology on the product $\prod_{i\in I} V_i$ is defined by the following rule. A base of neighborhoods of zero in $\prod_{i\in I} V_i$ is formed by the vector subspaces $\prod_{j\in J} U_j \times \prod_{s\in I\setminus J} V_s$, where $J\subset I$ is an arbitrary finite subset of indices and $U_j\subset V_j$ are open subspaces.

More generally, given a projective system of topological vector spaces $(V_\gamma)_{\gamma\in\Gamma}$ indexed by a poset Γ, the projective limit topology on $\varprojlim_{\gamma\in\Gamma} V_\gamma$ is the induced topology on $\varprojlim_{\gamma\in\Gamma} V_\gamma \subset \prod_{\gamma\in\Gamma} V_\gamma$, where $\prod_{\gamma\in\Gamma} V_\gamma$ is endowed with the product topology.

The coproduct topology on the direct sum $\bigoplus_{i\in I} V_i$ is defined by the rule that a subspace $U\subset \bigoplus_{i\in I} V_i$ is open if and only if the intersection $V_i\cap U$ is an open subspace in V_i for every $i\in I$. Equivalently, a base of neighborhoods of zero in $\bigoplus_{i\in I} V_i$ is formed by the subspaces $\bigoplus_{i\in I} U_i$, where $U_i\subset V_i$ are arbitrary open subspaces.

Separated and Complete VSLTs

The completion of a topological vector space V is defined as the projective limit of its discrete quotient spaces $V^\frown = \varprojlim_{U \subset V} V/U$, where U ranges over the open subspaces in V. There is a natural linear map $\lambda_V \colon V \longrightarrow V^\frown$, called the completion map.

A topological vector space V is called separated (or Hausdorff) if the map λ_V is injective, and V is called complete if λ_V is surjective.

The completion topology on $V^\frown$ is the projective limit topology of discrete vector spaces V/U. Equivalently, the open subspaces in $V^\frown$ are the kernels of the projection maps $V^\frown \longrightarrow V/U$.

So there is a natural bijection between the open subspaces in V and in $\mathfrak{V} = V^\frown$, assigning to an open subspace $U \subset V$ the kernel $\mathfrak{U}$ of the projection map $V^\frown \longrightarrow V/U$. Conversely, to an open subspace $\mathfrak{U} \subset \mathfrak{V}$ its full preimage $\lambda_V^{-1}(\mathfrak{U}) \subset V$ is assigned.

Complete and Separated Subspaces and Quotients

Let V be a topological vector space and $K \subset V$ be a subspace. Then the quotient space V/K is separated in the quotient topology if and only if K is a closed subspace in V.

Lemma

Let $\mathfrak{V}$ be a complete, separated topological vector space and $K \subset \mathfrak{V}$ be a vector subspace, endowed with the induced topology. Then the completion of K with its completion topology is naturally isomorphic to the closure of K in $\mathfrak{V}$ with its induced topology, $K^{\frown} \simeq \overline{K}_{\mathfrak{V}}$. In particular, K is complete if and only if it is closed in $\mathfrak{V}$.

Complete and Separated Products and Coproducts

Let $(V_i)_{i\in I}$ be a family of topological vector spaces.

Lemma

(a) *If the topological vector spaces V_i are separated, then the product $\prod_{i\in I} V_i$ is separated in the product topology.*
(b) *If the topological vector spaces V_i are complete, then the product $\prod_{i\in I} V_i$ is complete in the product topology.*

More generally, the projective limit of a diagram of separated (resp., complete) topological vector spaces is separated (resp., complete) in the projective limit topology.

Lemma

(a) *If the topological vector spaces V_i are separated, then the direct sum $\bigoplus_{i\in I} V_i$ is separated in the coproduct topology.*
(b) *If the topological vector spaces V_i are complete, then the direct sum $\bigoplus_{i\in I} V_i$ is complete in the coproduct topology.*

Countably Based VSLTs

Proposition

Let V be a topological vector space and $U \subset V$ be a vector subspace, endowed with the induced topology. Let $\mathfrak{W}$ be a complete, separated topological vector space with a countable base of neighborhoods of zero. Then any continuous linear map $U \longrightarrow \mathfrak{W}$ can be extended to a continuous linear map $V \longrightarrow \mathfrak{W}$.

Corollary

Let V be a topological vector space and $\mathfrak{K} \subset V$ be a vector subspace. Suppose that the topological vector space $\mathfrak{K}$ is complete and separated, and has a countable base of neighborhoods of zero. Then the inclusion map $\mathfrak{K} \longrightarrow V$ makes $\mathfrak{K}$ a direct summand of V.

Lemma

Let $(V_i)_{i=1}^{\infty}$ be a countable family of nondiscrete, separated topological vector spaces. Then the direct sum $\bigoplus_{i=1}^{\infty} V_i$ with its coproduct topology does not have a countable base of neighborhoods of zero.

Main Counterexample: Formulation

Let $\mathfrak{V}$ be a complete, separated topological vector space and $\mathfrak{K} \subset \mathfrak{V}$ be a closed subspace. Then the quotient space $\mathfrak{V}/\mathfrak{K}$ is separated in the quotient topology, but it need not be complete.

Relevant counterexamples are known in functional analysis, at least, since the 1950's. A counterexample which works for topological vector spaces with linear topology can be found in the book [Kelly, Namioka "Linear topological spaces", 1963–76, Problem 20D]. A very general construction of counterexamples can be found in the book [Arnautov, Glavatsky, Mikhalev "Introduction to the theory of topological rings and modules", 1996, Theorem 4.1.48]. The following assertion is its particular case.

Main Counterexample (Arnautov *et al.*)

For any separated topological vector space C there exists a complete, separated topological vector space $\mathfrak{A}(C)$ with a closed subspace $\mathfrak{K}(C) \subset \mathfrak{A}(C)$ such that the quotient space $\mathfrak{A}(C)/\mathfrak{K}(C)$ is isomorphic to C as a topological vector space.

Main Counterexample: Discussion

Let $\mathfrak{A}$ be a complete, separated topological vector space and $\mathfrak{K} \subset \mathfrak{A}$ be a closed subspace. What does it mean that the quotient space $C = \mathfrak{A}/\mathfrak{K}$ is complete (or not complete) in the quotient topology?

The topological vector space C is separated, since $\mathfrak{K}$ is a closed subspace in $\mathfrak{A}$. Consider the completion $\mathfrak{C} = C\widehat{\ }$. Then the question is about surjectivity of the natural map $\mathfrak{A} \longrightarrow \mathfrak{C}$. This map can be interpreted as the natural map of projective limits

$$\mathfrak{A} = \varprojlim\nolimits_{\mathfrak{U} \subset \mathfrak{A}} \mathfrak{A}/\mathfrak{U} \longrightarrow \varprojlim\nolimits_{\mathfrak{K} \subset \mathfrak{W} \subset \mathfrak{A}} \mathfrak{A}/\mathfrak{W} = \mathfrak{C}.$$

Here $\mathfrak{U}$ ranges over all the open subspaces in $\mathfrak{A}$, while $\mathfrak{W}$ ranges over all the open subspaces in $\mathfrak{A}$ containing $\mathfrak{K}$.

The problem, therefore, consists in the following. Let $(\bar{a}_{\mathfrak{W}} \in \mathfrak{A}/\mathfrak{W})_{\mathfrak{K} \subset \mathfrak{W} \subset \mathfrak{A}}$ be a compatible system of vectors in the quotient spaces $\mathfrak{A}/\mathfrak{W}$. Can one extend it to a compatible system of vectors $(\bar{a}_{\mathfrak{U}} \in \mathfrak{A}/\mathfrak{U})_{\mathfrak{U} \subset \mathfrak{A}}$ in the quotient spaces $\mathfrak{A}/\mathfrak{U}$?

The counterexamples show that, in general, this cannot be done.

Main Counterexample: Construction

Let $(V_i)_{i \in I}$ be a family of topological vector spaces. Besides the product topology on $\prod_{i \in I} V_i$, the vector space $\prod_{i \in I} V_i$ can be also endowed with the box topology. A base of neighborhoods of zero in the box topology is formed by the subspaces $\prod_{i \in I} U_i \subset \prod_{i \in I} V_i$, where $U_i \subset V_i$ are arbitrary open subspaces.

The coproduct topology on $\bigoplus_{i \in I} V_i \subset \prod_{i \in I} V_i$ is induced from the box topology on the product.

Lemma

(a) *If the topological vector spaces V_i are separated, then the product $\prod_{i \in I} V_i$ is separated in the box topology.*
(b) *If the topological vector spaces V_i are complete, then the product $\prod_{i \in I} V_i$ is complete in the box topology.*

Main Counterexample: Construction Cont'd

The construction of the counterexamples uses a modified version of the box (or coproduct) topology. The box topology is finer than the product topology, and the modified box topology is finer still.

The modified box topology on the product of topological vector spaces $\prod_{i \in I} V_i$ is defined as follows. A base of neighborhoods of zero in the modified box topology is formed by the vector subspaces $\prod_{j \in J} \{0\} \times \prod_{s \in I \setminus J} U_s$, where J ranges over the finite subsets of I and $U_s \subset V_s$, $s \in I \setminus J$, are arbitrary open subspaces.

In particular, if the set I is finite, then the modified box topology on $\prod_{i \in I} V_i$ is discrete.

Lemma

For any family of topological vector spaces V_i, the product $\prod_{i \in I} V_i$ is a complete, separated topological vector space in the modified box topology.

Main Counterexample: Construction Cont'd

The modified coproduct topology on the direct sum of a family of topological vector spaces $\bigoplus_{i\in I} V_i$ is defined as the topology on the subspace $\bigoplus_{i\in I} V_i \subset \prod_{i\in I} V_i$ induced by the modified box topology on the product. In other words, a base of neighborhoods of zero in the modified coproduct topology is formed by the subspaces $\bigoplus_{j\in J}\{0\} \oplus \bigoplus_{s\in I\setminus J} U_s$, where J ranges over the finite subsets of I and $U_s \subset V_s$, $s \in I \setminus J$, are open subspaces.

Lemma

For any family of separated topological vector spaces V_i, the subspace $\bigoplus_{i\in I} V_i \subset \prod_{i\in I} V_i$ is closed in the modified box topology on the product.

Corollary

For any family of separated topological vector spaces V_i, the direct sum $\bigoplus_{i\in I} V_i$ is separated and complete in the modified coproduct topology.

Main Counterexample: Construction Fin'd

Construction of the Main Counterexample.

Let C be a separated topological vector space. Choose an infinite set I, and consider the direct sum $\mathfrak{A}_I(C) = C^{(I)} = \bigoplus_{i \in I} C$, endowed with the modified coproduct topology. Let $\Sigma\colon C^{(I)} \longrightarrow C$ be the summation map.

One can easily see that Σ is continuous. In fact, the map Σ is even continuous in the coproduct topology on $C^{(I)}$, which is coarser than the modified coproduct topology.

When the set I is infinite, the map Σ is also open. Indeed, for any choice of a finite subset $J \subset I$ and open subspaces $U_s \subset C$, $s \in I \setminus J$, the subspace $\Sigma\big(\bigoplus_{j \in J}\{0\} \oplus \bigoplus_{s \in I \setminus J} U_s\big) = \sum_{s \in I \setminus J} U_s$ is open in C. Thus the topology of C is the quotient topology of the modified coproduct topology on $C^{(I)}$. $\square$

Zero-Convergent Families of Vectors

Let $\mathfrak{V}$ be a complete, separated topological vector space. A family of vectors $(v_x \in \mathfrak{V})_{x \in X}$ indexed by some set X is said to converge to zero in the topology of $\mathfrak{V}$ if, for every open subspace $\mathfrak{U} \subset \mathfrak{V}$, the set $\{x \in X \mid v_x \notin \mathfrak{U}\}$ is finite.

Fix a set of indices X. Then the X-indexed zero-convergent families of elements in $\mathfrak{V}$ form a vector subspace in $\mathfrak{V}^X = \prod_{x \in X} \mathfrak{V}$, which we denote by $\mathfrak{V}[[X]] \subset \mathfrak{V}^X$. The vector space $\mathfrak{V}[[X]]$ can be computed as the projective limit

$$\mathfrak{V}[[X]] = \varprojlim_{\mathfrak{U} \subset \mathfrak{V}} (\mathfrak{V}/\mathfrak{U})[X],$$

where $\mathfrak{U}$ ranges over all the open subspaces in $\mathfrak{V}$, while $A[X] = A^{(X)} = \bigoplus_{x \in X} A$ is a notation for the direct sum of X copies of a (discrete) vector space A.

We endow the vector space $\mathfrak{V}[[X]]$ with the topology of projective limit of discrete vector spaces $(\mathfrak{V}/\mathfrak{U})[X]$. So $\mathfrak{V}[[X]]$ is a complete, separated topological vector space.

Strongly Surjective Maps

A continuous linear map of complete, separated topological vector spaces $f\colon \mathfrak{V} \longrightarrow \mathfrak{C}$ is said to be strongly surjective if any zero-convergent family of vectors in $\mathfrak{C}$ can be lifted to a zero-convergent family of vectors in $\mathfrak{V}$. In other words, this means that the map $f[[X]]\colon \mathfrak{V}[[X]] \longrightarrow \mathfrak{C}[[X]]$ is surjective for all sets X.

A surjective continuous open map need not be strongly surjective.

Counterexample

For any separated topological vector space C and any infinite set I, consider the topological vector space $\mathfrak{A}_I(C) = C^{(I)}$ with the modified coproduct topology. It is claimed that there exist no infinite families of nonzero vectors in $\mathfrak{A}_I(C)$ converging to zero in the modified coproduct topology.

Let $\mathfrak{C}$ be a complete, separated topological vector space where an infinite zero-convergent family of nonzero vectors does exist. Then it follows that the summation map $\Sigma\colon \mathfrak{A}_I(\mathfrak{C}) \longrightarrow \mathfrak{C}$ is continuous, surjective, and open, but not strongly surjective.

Tensor Product Topologies

Let V and W be topological vector spaces (with linear topologies). How can one define a topology on the tensor product $V \otimes_k W$?

Beilinson discovered that there are three natural tensor product topologies. Two topological tensor products, denoted by $\otimes^*$ and $\otimes^!$, are associative and commutative; they are the polar cases. The third construction is intermediate between these two. It is associative, but not commutative; so it is denoted by a notation with an arrow ($\overleftarrow{\otimes}$ or $\otimes^{\leftarrow}$).

Two of three Beilinson's topologies have "naïve" versions, which were introduced and briefly discussed by Beilinson and Drinfeld in the book ["Chiral Algebras", AMS, 2004, Section 2.7.7]. In the subsequent paper ["Remarks on topological algebras", Moscow Math. Journ. 2008], Beilinson defines the non-naïve (correct) topologies and suggests to discard the naïve ones.

Tensor Product Topologies: Naïve Definitions

Let V and W be topological vector spaces. The two naïve constructions of topologies on $V \otimes_k W$ are

1. Let a base of neighborhoods of zero in $V \otimes_k W$ consist of all the subspaces $P \otimes Q \subset V \otimes_k W$, where $P \subset V$ and $Q \subset W$ are open subspaces.
2. Let a base of neighborhoods of zero in $V \otimes_k W$ consist of all the subspaces $P \otimes W + V \otimes Q \subset V \otimes_k W$, where $P \subset V$ and $Q \subset W$ are open subspaces.

Why are these constructions naïve? The second one is actually OK, and it defines what we will call the topological vector space $V \otimes^! W$. What is the problem with the first one?

Consider the particular case when the topological vector space V is discrete. Then the first construction defines a discrete topology on $V \otimes_k W$ (as one can take $P = 0$). Now observe that the map

$$V \times W \longrightarrow V \otimes_k W, \qquad (v, w) \longmapsto v \otimes w$$

is not continuous (as a function of two variables) with respect to the first topology on $V \otimes_k W$. This is not good.

Tensor Product Topologies: Naïve Definitions

The naïve definition of the noncommutative topological tensor product is

3. Let a base of neighborhoods of zero in $V \otimes_k W$ consist of all the subspaces of the form $P \otimes W \subset V \otimes_k W$, where $P \subset V$ is an open subspace.

In addition to the above-mentioned discontinuity problem, this definition is also strange in that it only uses the topology of V and ignores the topology of W.

Nevertheless, it is helpful to keep this naive version of the $\leftarrow$-topology in mind. It gives a good first approximation to the correct definition and illustrates some of its properties (viz., strong dependence of the first argument's topology and weak dependence on the second one's).

Tensor Product Topologies: Proper Definitions

Denote by $V \otimes^{\leftarrow} W$ the vector space $V \otimes_k W$ with the following topology. A vector subspace $E \subset V \otimes^{\leftarrow} W$ is open if and only if two conditions hold:

- there exists an open subspace $P \subset V$ such that $P \otimes W \subset E$;
- for every vector $v \in V$, there exists an open subspace $Q_v \subset W$ such that $v \otimes Q_v \subset E$.

Denote by $V \otimes^{*} W$ the vector space $V \otimes_k W$ with the following topology. A vector subspace $E \subset V \otimes^{*} W$ is open if and only if three conditions hold:

- there exist open subspaces $P \subset V$ and $Q \subset W$ such that $P \otimes Q \subset E$;
- for every vector $v \in V$, there exists an open subspace $Q_v \subset W$ such that $v \otimes Q_v \subset E$;
- for every vector $w \in W$, there exists an open subspace $P_w \subset V$ such that $P_w \otimes w \subset E$.

Tensor Product Topologies: Functional Analysis Analogy

From the three tensor product topologies, the $*$-topology is the finest one. The !-topology is the coarsest one, and the $\leftarrow$-topology is in between.

The $*$-topology on the tensor product of topological vector spaces with linear topologies is an analogue of what is called the projective tensor product of Banach spaces or locally convex spaces in functional analysis.

The !-topology on the tensor product of VSLTs is an analogue of the injective tensor product of Banach spaces or locally convex spaces.

These tensor product operations in functional analysis were studied by Grothendieck in 1950's.

Tensor Product Topologies: Universal Properties

The definition of the $*$-topology looks complicated, but it has a very natural universal property. Let U, V, and W be topological vector spaces, and let $\phi\colon V \times W \longrightarrow U$ be a bilinear map. Let $\phi^{\otimes}\colon V \otimes_k W \longrightarrow U$ be the related map from the tensor product.

Then the map ϕ is continuous (as a function of two variables) if and only if the map $\phi^{\otimes}\colon V \otimes^* W \longrightarrow U$ is continuous.

Let A be an associative algebra over k whose underlying vector space is endowed with a linear topology. Then A is a topological algebra (i.e., its multiplication map $m\colon A \times A \longrightarrow A$ is continuous) if and only if the map $m^{\otimes}\colon A \otimes^* A \longrightarrow A$ is continuous.

Next, A is a topological algebra with a base of neighborhoods of zero consisting of open right ideals if and only if the map $m^{\otimes}\colon A \otimes^{\leftarrow} A \longrightarrow A$ is continuous. A is a topological algebra with a base of neighborhoods of zero consisting of open two-sided ideals if and only if the map $m^{\otimes}\colon A \otimes^! A \longrightarrow A$ is continuous.

These are observations from Beilinson's 2008 paper.

Good Properties of the Tensor Product Topologies

Lemma

Let $f\colon V \longrightarrow C$ and $g\colon W \longrightarrow D$ be continuous linear maps of topological vector spaces. Then, for any symbol $? = *, \leftarrow$, or $!$,
(a) *$f \otimes g\colon V \otimes^? W \longrightarrow C \otimes^? D$ is a continuous map of topological vector spaces;*
(b) *If f and g are surjective open maps, then $f \otimes g$ is also a (surjective) open map.*

Lemma

Let V and W be topological vector spaces, and let $K \subset V$ and $L \subset W$ be vector subspaces. Then, for any $? = *, \leftarrow$, or $!$,
(a) *the induced topology on $K \otimes_k L \subset V \otimes^? W$ coincides with the topology of $K \otimes^? L$;*
(b) *if K is dense in V and L is dense in W, then $K \otimes_k L$ is dense in $V \otimes^? W$.*
(c) *if K is closed in V and L is closed in W, then $K \otimes_k L$ is closed in $V \otimes^? W$.*

Completed Tensor Products

If V and W are separated VSLTs, then their tensor products $V \otimes^? W$ are separated as well. But the tensor products of complete VSLTs need not be complete; so one has to take their completions.

For any complete, separated topological vector spaces $\mathfrak{V}$ and $\mathfrak{W}$ and any symbol $? = *$, $\leftarrow$, or $!$, we denote by $\mathfrak{V} \mathbin{\widehat{\otimes}}^? \mathfrak{W}$ the completion of the topological vector space $\mathfrak{V} \otimes^? \mathfrak{W}$.

Example

Let $\mathfrak{V}$ and $\mathfrak{W}$ be profinite-dimensional (linearly compact) topological vector spaces, i.e., projective limits of finite-dimensional discrete vector spaces endowed with the projective limit topologies. Then all the three topologies on the tensor product $\mathfrak{V} \otimes_k \mathfrak{W}$ coincide. Assuming that both $\mathfrak{V}$ and $\mathfrak{W}$ are infinite-dimensional, the topological vector space $\mathfrak{V} \otimes^* \mathfrak{W} = \mathfrak{V} \otimes^{\leftarrow} \mathfrak{W} = \mathfrak{V} \otimes^! \mathfrak{W}$ is incomplete. Its completion $\mathfrak{V} \mathbin{\widehat{\otimes}}^* \mathfrak{W} = \mathfrak{V} \mathbin{\widehat{\otimes}}^{\leftarrow} \mathfrak{W} = \mathfrak{V} \mathbin{\widehat{\otimes}}^! \mathfrak{W}$ is the usual completed tensor product of linearly compact topological vector spaces.

Examples of Topological Tensor Products

1. Let $\mathfrak{V}$ be a discrete vector space, and let $\mathfrak{W}$ be a complete, separated topological vector space. Let I be a set indexing a basis in $\mathfrak{V}$. Then the $*$-topology and the $\leftarrow$-topology on $\mathfrak{V} \otimes_k \mathfrak{W}$ coincide, and the topology of $\mathfrak{V} \otimes^* \mathfrak{W} = \mathfrak{V} \otimes^{\leftarrow} \mathfrak{W}$ is the coproduct topology of $\mathfrak{W}^{(I)} = \bigoplus_{i \in I} \mathfrak{W}$.

It follows that the topological vector space $\mathfrak{V} \otimes^* \mathfrak{W} = \mathfrak{V} \otimes^{\leftarrow} \mathfrak{W}$ is complete, so $\mathfrak{V} \otimes^* \mathfrak{W} = \mathfrak{V} \otimes^{\leftarrow} \mathfrak{W} = \mathfrak{W}^{(I)} = \mathfrak{V} \mathbin{\widehat{\otimes}^*} \mathfrak{W} = \mathfrak{V} \mathbin{\widehat{\otimes}^{\leftarrow}} \mathfrak{W}$.

2. Let $\mathfrak{V}$ be a complete, separated topological vector space, and let $\mathfrak{W}$ be a discrete vector space. Let X be a set indexing a basis in $\mathfrak{W}$. Then the $\leftarrow$-topology and the $!$-topology on $\mathfrak{V} \otimes_k \mathfrak{W}$ coincide. A topology base in $\mathfrak{V} \otimes^{\leftarrow} \mathfrak{W} = \mathfrak{V}^{(X)} = \mathfrak{V}[X] = \mathfrak{V} \otimes^! \mathfrak{W}$ is formed by the vector subspaces $\mathfrak{U}[X] \subset \mathfrak{V}[X]$, where $\mathfrak{U}$ ranges over the open subspaces of $\mathfrak{V}$.

It follows that the related completed tensor products are $\mathfrak{V} \mathbin{\widehat{\otimes}^{\leftarrow}} \mathfrak{W} = \varprojlim_{\mathfrak{U} \subset \mathfrak{V}} (\mathfrak{V}/\mathfrak{U})[X] = \mathfrak{V}[[X]] = \mathfrak{V} \mathbin{\widehat{\otimes}^!} \mathfrak{W}$.

Bad Properties of Topological Tensor Products

Corollary

The class of (complete, separated) topological vector spaces with a countable base of neighborhoods of zero is not preserved by the tensor products $\widehat{\otimes}^*$ *or* $\widehat{\otimes}^{\leftarrow}$.

In fact, even the tensor products $\mathfrak{V}\,\widehat{\otimes}^*\,-$ and $\mathfrak{V}\,\widehat{\otimes}^{\leftarrow}\,-$ with a countably-dimensional discrete vector space $\mathfrak{V}$ lead outside of the class of VSLTs with a countable base of neighborhoods of zero.

Corollary

For any complete, separated topological vector space $\mathfrak{C}$ *where some infinite family of nonzero vectors converges to zero, there exists a surjective open (continuous linear) map of complete, separated topological vector spaces* $\mathfrak{A} \longrightarrow \mathfrak{C}$ *which the completed tensor product functors* $-\,\widehat{\otimes}^{\leftarrow}\,\mathfrak{W}$ *and* $-\,\widehat{\otimes}^{!}\,\mathfrak{W}$ *(with a countably-dimensional discrete vector space* $\mathfrak{W}$*) take to a nonsurjective map.*

Part II

Exact Category Structures

Axioms of Exact Category

An exact category E is an additive category endowed with a class of short exact sequences (or conflations) $E' \longrightarrow E \longrightarrow E''$ satisfying the following axioms Ex0–Ex3.

A morphism $E' \longrightarrow E$ appearing in some short exact sequence $E' \longrightarrow E \longrightarrow E''$ is called an admissible monomorphism (or inflation), and a morphism $E \longrightarrow E''$ apearing in some short exact sequence is called an admissible epimorphism (or deflation).

Ex0: The short sequence $0 \longrightarrow 0 \longrightarrow 0$ is exact. Any short sequence isomorphic to a short exact sequence is exact.

Ex1: Any short exact sequence $E' \xrightarrow{i} E \xrightarrow{p} E''$ is a kernel-cokernel pair, i.e., the morphism i is a kernel of the morphism p and the morphism p is a cokernel of the morphism i.

Axioms of Exact Category Cont'd

Ex2(a): For any admissible monomorphism $i\colon E' \longrightarrow E$ and any morphism $f\colon E' \longrightarrow F'$, a pushout square on the diagram

$$\begin{tikzcd} E' \arrow[r, "i"] \arrow[d, "f"'] & E \arrow[r, "p"] \arrow[d, "g"] & E'' \\ F' \arrow[r, "j"] & F \arrow[ur] & \end{tikzcd}$$

exists, and the morphism $j\colon F' \longrightarrow F$ is an admissible monomorphism. Simply put, admissible monomorphisms are preserved by pushouts.

If $E' \longrightarrow E \longrightarrow E''$ is a short exact sequence, then it follows from Ex0–Ex1 and Ex2(a) that $F' \longrightarrow F \longrightarrow E''$ is a short exact sequence, too. Conversely, assuming Ex0–Ex1, the axiom Ex2(a) can be equivalently reformulated by saying that any short exact sequence $E' \longrightarrow E \longrightarrow E''$ and a morphism $E' \longrightarrow F'$ can be included into a commutative diagram as above with a short exact sequence $F' \longrightarrow F \longrightarrow E''$. The square is then a pushout square.

Axioms of Exact Category Cont'd

Ex2(b): Dually, for any admissible epimorphism $p\colon E \longrightarrow E''$ and any morphism $f\colon F'' \longrightarrow E''$, a pullback square on the diagram

$$\begin{tikzcd} E' \arrow[r, "i"] \arrow[dr] & E \arrow[r, "p"] & E'' \\ & F \arrow[u, "g"] \arrow[r, "q"] & F'' \arrow[u, "f"] \end{tikzcd}$$

exists, and the morphism $q\colon F \longrightarrow F''$ is an admissible epimorphism. Simply put, admissible epimorphisms are preserved by pullbacks.

If $E' \longrightarrow E \longrightarrow E''$ is a short exact sequence, then it follows from Ex0–Ex1 and Ex2(b) that $E' \longrightarrow F \longrightarrow F''$ is a short exact sequence, too.

The axiom Ex2 is $\mathrm{Ex2(a)} + \mathrm{Ex2(b)}$.

Weak Idempotent-Completeness

An additive category E is called idempotent-complete if any idempotent endomorphism $e\colon A \longrightarrow A$ of an object $A \in \mathrm{E}$ comes from a direct sum decomposition $A = B \oplus C$ in E.

An additive category E is called weakly idempotent-complete if any pair of morphisms $i\colon B \longrightarrow A$ and $p\colon A \longrightarrow B$ with $p \circ i = \mathrm{id}_B$ comes from a direct sum decomposition $A = B \oplus C$.

For example, the category of even-dimensional finite-dimensional vector spaces is weakly idemponent-complete, but not idempotent-complete. The category of vector spaces of dimension different from 1 (or from 1, 2, and 3) is additive, but not weakly idempotent-complete.

The assumption of (at least) weak idempotent-completeness simplifies the exact category theory considerably.

Axioms of Exact Category Cont'd

For a class of short exact sequences satisfying Ex0–Ex1 in a weakly idempotent-complete additive category E, axiom Ex2 is equivalent to the conjuction of the following three conditions:

Ex2'(a): If a composition fg is an admissible monomorphism, then g is an admissible monomorphism.

Ex2'(b): If a composition fg is an admissible epimorphism, then f is an admissible epimorphism.

Ex2'(c): If in the commutative diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[dr] & E \arrow[r] \arrow[d] & E'' \\ & F \arrow[ur] & \end{tikzcd}$$

both $E' \longrightarrow E \longrightarrow E''$ and $E' \longrightarrow F \longrightarrow E''$ is are short exact sequences, then $E \longrightarrow F$ is an isomorphism.

Any one of the axioms Ex2(a) or Ex2(b) implies Ex2'(c).

Axioms of Exact Category Fin'd

For any class of short exact sequences satisfying Ex0–Ex2 in an additive category E, the following two conditions are equivalent to each other:

Ex3(a): Every composition of two admissible monomorphisms is an admissible monomorphism.

Ex3(b): Every composition of two admissible epimorphisms is an admissible epimorphism.

The axiom Ex3 is Ex3(a) or Ex3(b).

This is the end of the list of exact category axioms Ex0–Ex3.

Quasi-Abelian Categories

Quasi-abelian categories form the class of additive categories closest to the abelian ones.

Let A be an additive category in which all morphisms have kernels and cokernels (such categories are sometimes called preabelian).

We will say that a morphism in A is a kernel if it is the kernel of some morphism in A. Similarly, a morphism in A is a cokernel if it is a the cokernel of some morphism in A. Any kernel is the kernel of its cokernel, and any cokernel is the cokernel of its kernel.

The category A is called quasi-abelian if the class of all short sequences satisfying Ex1 (i.e., all the kernel-cokernel pairs) is an exact category structure on A. This holds if and only if the class of all kernel-cokernel pairs satisfies Ex2. In other words, A is quasi-abelian if and only if the class of all kernels in A is stable under pushouts and the class of all cokernels in A is stable under pullbacks.

Semi-Abelian Categories

Let A be an additive category with kernels and cokernels, and let $f: A \longrightarrow B$ be a morphism in A. Let K be the kernel of f and C be the cokernel of f. Denote by $\operatorname{coim}(f)$ the cokernel of the morphism $K \longrightarrow A$, and by $\operatorname{im}(f)$ the kernel of the morphism $B \longrightarrow C$. Then there is a natural morphism $\operatorname{coim}(f) \longrightarrow \operatorname{im}(f)$ appearing in the textbook definition of an abelian category.

The category A is called semi-abelian if, for any morphism f in A, the morphism $\operatorname{coim}(f) \longrightarrow \operatorname{im}(f)$ is both a monomorphism and an epimorphism.

Quasi-abelianity is a stronger property than semi-abelianity: any quasi-abelian category is semi-abelian. The question whether the converse holds came to be known as Raikov's conjecture (with the reference to 1969 and 1976 papers of Raikov). Both analytic and algebraic counterexamples to Raikov's conjecture have been found between 2005–2012.

Semi-Abelian Categories

Proposition

For any additive category A *with kernels and cokernels, the following four conditions are equivalent:*

1. *for any morphism* f *in* A, *the natural morphism* $\mathrm{coim}(f) \longrightarrow \mathrm{im}(f)$ *is an epimorphism;*
2. *the composition of any two kernels is a kernel;*
3. *if the composition* fg *is a kernel, then* g *is a kernel;*
4. *any pushout of a kernel is a monomorphism.*

A category satisfying the equivalent conditions of the proposition is called right semi-abelian. An additive category with kernels and cokernels in which any pushout of a kernel is a kernel is called right quasi-abelian. So any right quasi-abelian category is right semi-abelian. The left semi-abelianity and left quasi-abelianity are the dual properties. Quasi-abelianity = right + left quasi-abelianity (and similarly for semi-abelianity).

Semi-Abelian and Quasi-Abelian Categories

Proposition

An additive category A with kernels and cokernels is quasi-abelian if and only if two conditions hold:

1. A *is semi-abelian;*
2. *for any commutative diagram*

$$\begin{tikzcd} A' \arrow[r] \arrow[dr] & A \arrow[r] \arrow[d] & A'' \\ & B \arrow[ur] & \end{tikzcd}$$

in which both the sequences $A' \longrightarrow A \longrightarrow A''$ *and* $A' \longrightarrow B \longrightarrow A''$ *are kernel-cokernel pairs, the morphism* $A \longrightarrow B$ *is an isomorphism.*

If A is either left or right quasi-abelian, then condition (2) is satisfied. Thus any right quasi-abelian, left semi-abelian category is quasi-abelian, and similarly any left quasi-abelian, right semi-abelian category.

Semi-Abelian and Quasi-Abelian Categories

Here is perhaps the simplest algebraic example of a semi-abelian, but not quasi-abelian category (i.e., a simple counterexample to Raikov's conjecture).

A is the category of morphisms of vector spaces $a\colon V_1 \longrightarrow V_2$ with the following additional datum. In the vector space $\mathrm{im}(a)$, a vector subspace $V \subset \mathrm{im}(a)$ is chosen. Let us denote the objects of A by $(V_1 \xrightarrow[V]{\mathrm{a}} V_2)$. Morphisms in A are morphisms of morphisms (i.e., commutative squares) such that the induced morphism of the images takes the chosen subspace into the chosen subspace. The diagram

$$\begin{tikzcd}
(0 \xrightarrow[0]{0} k) \arrow[r] \arrow[dr] & (k \xrightarrow[0]{\mathrm{id}} k) \arrow[r] \arrow[d] & (k \xrightarrow[0]{0} 0) \\
& (k \xrightarrow[k]{\mathrm{id}} k) \arrow[ur] &
\end{tikzcd}$$

shows that condition (2) is not satisfied for A.

Mistake in Beilinson's Paper

On the first page of Beilinson's 2008 paper [Moscow Math. J. **8**, #1], it is claimed that (in our notation introduced below):

"The category of [complete, separated] topological vector spaces $\mathrm{Top}_k^{\mathrm{sc}}$ is quasi-abelian [...]. In particular, it is naturally an exact category: the admissible monomorphisms are closed embeddings, the admissible epimorphisms are open surjections."

These assertions are not true, as we will now see. The category $\mathrm{Top}_k^{\mathrm{sc}}$ is not quasi-abelian, and it does not have an exact category structure in which all closed embeddings are admissible monomorphisms. (Though an exact category structure in which the admissible epimorphisms are the open surjections does actually exist on $\mathrm{Top}_k^{\mathrm{sc}}$.)

The important observation that the categories of incomplete topological vector spaces are quasi-abelian, while the categories complete ones aren't, goes back (at least) to F. Prosmans ["Derived categories for functional analysis", 2000], who worked in the context of locally convex topologies.

Additive Categories of Topological Vector Spaces

Denote by Top_k the category of topological vector spaces (with linear topology) over a field k. The morphisms in Top_k are the continuous linear maps. Let $\mathrm{Top}_k^{\mathrm{s}} \subset \mathrm{Top}_k$ denote the full subcategory of separated VSLTs, and let $\mathrm{Top}_k^{\mathrm{sc}} \subset \mathrm{Top}_k^{\mathrm{s}}$ be the full subcategory of complete, separated VSLTs.

The functor of forgetting the topology $\mathrm{Top}_k \longrightarrow \mathrm{Vect}_k$ preserves all kernels and cokernels, as well as all products and coproducts (hence all limits and colimits).

The kernel K of a morphism $f\colon V \longrightarrow W$ in Top_k is the kernel of f in the category of vector spaces Vect_k, endowed with the induced topology as a subspace in V. The cokernel of f is the cokernel of f in Vect_k, endowed with the quotient topology as a quotient space of W. The products and coproducts are as per the discussion of the product and coproduct topologies in the beginning of this talk.

Additive Categories of Topological Vector Spaces

The full subcategories $\mathrm{Top}_k^{\mathrm{s}}$ and $\mathrm{Top}_k^{\mathrm{sc}}$ are closed under kernels and products (as well as coproducts) in Top_k. So the kernels, products (hence all limits), and coproducts in the three categories agree.

Moreover, the full subcategories $\mathrm{Top}_k^{\mathrm{s}}$ and $\mathrm{Top}_k^{\mathrm{sc}}$ are reflective in Top_k, i.e., the inclusion functors $\mathrm{Top}_k^{\mathrm{s}} \longrightarrow \mathrm{Top}_k$ and $\mathrm{Top}_k^{\mathrm{sc}} \longrightarrow \mathrm{Top}_k$ have left adjoint functors (the reflectors). The reflector $\mathrm{Top}_k \longrightarrow \mathrm{Top}_k^{\mathrm{s}}$ takes a topological vector space U to its quotient space $U/\overline{\{0\}}_U$ by the closure of the zero subspace in U, endowed with the quotient topology. The reflector $\mathrm{Top}_k \longrightarrow \mathrm{Top}_k^{\mathrm{sc}}$ takes a topological vector space U to its completion $U\hat{}$.

The cokernel of a morphism $f\colon V \longrightarrow W$ in $\mathrm{Top}_k^{\mathrm{s}}$ is the quotient space $W/\overline{f(V)}_W$ of W by the closure of the subspace $f(V) \subset W$, endowed with the quotient topology. The cokernel of a morphism $f\colon \mathfrak{V} \longrightarrow \mathfrak{W}$ in $\mathrm{Top}_k^{\mathrm{sc}}$ is the completion $\mathfrak{C} = C\hat{}$ of the quotient space $C = \mathfrak{W}/f(\mathfrak{V})$. Here $C = \mathfrak{W}/f(\mathfrak{V})$ is endowed with the quotient topology and $\mathfrak{C} = C\hat{}$ with the completion topology.

Exactness Properties of Categories of VSLTs

The category of topological vector spaces Top_k and the category of separated (Hausdorff) topological vector spaces $\mathrm{Top}_k^{\mathrm{s}}$ are quasi-abelian. But the category of complete, separated topological vector spaces $\mathrm{Top}_k^{\mathrm{sc}}$ is not quasi-abelian. In fact, it is right but not left quasi-abelian.

Here is the counterexample. Let C be any separated, but incomplete topological vector space. As we have seen, there exists a complete, separated topological vector space $\mathfrak{A}$ with a closed subspace $\mathfrak{K} \subset \mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{K} \simeq C$. Let $\mathfrak{C} = C^{\frown}$ be the completion of C. Then $0 \longrightarrow \mathfrak{K} \xrightarrow{i} \mathfrak{A} \xrightarrow{p} \mathfrak{C} \longrightarrow 0$ is a kernel-cokernel pair in $\mathrm{Top}_k^{\mathrm{sc}}$ (where $p\colon \mathfrak{A} \longrightarrow \mathfrak{C}$ is the composition $\mathfrak{A} \longrightarrow C \longrightarrow \mathfrak{C}$). The morphism $p\colon \mathfrak{A} \longrightarrow \mathfrak{C}$ is a cokernel, hence an epimorphism in $\mathrm{Top}_k^{\mathrm{sc}}$, but it is not a surjective map.

Exactness Properties of Complete, Separated VSLTs

Let $x \in \mathfrak{C} \setminus C$ be a vector in the complement, and let kx be the one-dimensional vector space spanned by x, endowed with the discrete topology. Let $f\colon kx \longrightarrow \mathfrak{C}$ be the inclusion map. Taking the pullback of p by f, we obtain the diagram

$$\begin{tikzcd}
\mathfrak{K} \arrow[r, "i"] \arrow[dr] & \mathfrak{A} \arrow[r, "p"] & \mathfrak{C} \\
& \mathfrak{K} \arrow[u, "g=i"] \arrow[r, "q=0"] & kx \arrow[u, "f"']
\end{tikzcd}$$

Since the images of p and f in $\mathfrak{C}$ only intersect at zero, the pullback $q\colon \mathfrak{K} \longrightarrow kx$ of a cokernel $p\colon \mathfrak{A} \longrightarrow \mathfrak{C}$ is the zero map. The zero map $q = 0$ is not even an epimorphism. Thus the category of complete, separated topological vector spaces $\mathrm{Top}_k^{\mathrm{sc}}$ is not left quasi-abelian (and not even left semi-abelian).

Moreover, there does not exist an exact category structure on $\mathrm{Top}_k^{\mathrm{sc}}$ in which the closed embedding $i\colon \mathfrak{K} \longrightarrow \mathfrak{A}$ would be an admissible monomorphism.

Maximal Exact Category Structure

According to Rump's paper ["On the maximal exact category structure of an additive category", 2011], every additive category has a maximal exact category structure. This means a class of short exact sequences satisfying Ex0–Ex3 which contains any other class of short exact sequences satisfying Ex0–Ex3.

For a weakly idempotent-complete additive category A, the maximal exact category structure has a rather simple description, which is convenient to work with [Sieg–Wegner, "Maximal exact structures on additive categories", 2011; Crivei, "Maximal exact structures on additive categories revisited", 2012].

Maximal Exact Category Structure

Let A be a weakly idempotent-complete additive category. A morphism $i\colon K \longrightarrow A$ in A is called a semi-stable kernel if all pushouts of i exist, and they are kernels in A. Dually, a morphism $p\colon A \longrightarrow C$ is called a semi-stable cokernel if all pullbacks of p exist, and they are cokernels in A. Clearly, any semi-stable kernel has a cokernel, and any semi-stable cokernel has a kernel.

A kernel-cokernel pair $K \xrightarrow{i} A \xrightarrow{p} C$ in A is said to be stable if i is a semi-stable kernel and p is a semi-stable cokernel.

Proposition (Sieg–Wegner, Crivei)

For any weakly idempotent-complete additive category A, the class of all stable kernel-cokernel pairs $K \longrightarrow A \longrightarrow C$ is an exact category structure on A. It is the maximal exact category structure on A.

Maximal Exact Category Structure

A morphism i in A is said to be a stable kernel if i is a semi-stable kernel and the cokernel of i is a semi-stable cokernel. Dually, p is a stable cokernel if p is a semi-stable cokernel and the kernel of p is a semi-stable kernel.

It follows from the proposition that, in the maximal exact structure on A, the admissible monomorphisms are the stable kernels and the admissible epimorphisms are the stable cokernels.

Now let A be a right quasi-abelian category. Then all kernels in A are semi-stable, hence all semi-stable cokernels in A are stable. The short exact sequences in the maximal exact category structure are the kernel-cokernels pairs $K \xrightarrow{i} A \xrightarrow{p} C$ in which p is a semi-stable cokernel.

Maximal Exact Structure on Complete, Sep'd VSLTs

We have seen that nonsurjective cokernels in $\mathrm{Vect}_k^{\mathrm{sc}}$ are not semi-stable. One can check that all surjective cokernels are semi-stable.

So in the maximal exact structure on $\mathrm{Vect}_k^{\mathrm{sc}}$:

- the admissible epimorphisms are the open surjective (continuous linear) maps;
- the admissible monomorphisms (= the stable kernels) are the closed embeddings $i\colon \mathfrak{K} \longrightarrow \mathfrak{A}$ for which the quotient space $\mathfrak{A}/i(\mathfrak{K})$ is complete in the quotient topology;
- the short exact sequences (stable kernel-cokernel pairs) $0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{C} \longrightarrow 0$ are the kernel-cokernel pairs in $\mathrm{Top}_k^{\mathrm{sc}}$ which remain kernel-cokernel pairs in $\mathrm{Top}_k^{\mathrm{s}}$ (or equivalently in Top_k or in Vect_k).

Mistake in Beilinson's Paper II

On the second page of Beilinson's 2008 paper, after the definitions of three (completed) tensor product operations, there is an

"**Exercise.** The tensor products are exact."

Let us give a relevant definition: an additive functor between exact categories $F\colon \mathrm{E}' \longrightarrow \mathrm{E}''$ is said to be exact if it takes short exact sequences (conflations) in E' to short exact sequences (conflations) in E''.

It appears that Beilinson's intended meaning was "the completed tensor products are exact functors in the quasi-abelian exact structure on $\mathrm{Top}_k^{\mathrm{sc}}$" (which does not exist, as we have seen). Next we will see that in the maximal exact structure on $\mathrm{Top}_k^{\mathrm{sc}}$, at least two of the three tensor products are not exact, either. However, these functors can be thought of as being "exact" in some other sense.

Exactness Properties of Uncompleted Tensor Products

Let V be a topological vector space. Let F_V denote one of the topological tensor product functors

$$V \otimes^* - : \mathrm{Top}_k \longrightarrow \mathrm{Top}_k$$
$$V \otimes^{\leftarrow} - : \mathrm{Top}_k \longrightarrow \mathrm{Top}_k$$
$$- \otimes^{\leftarrow} V : \mathrm{Top}_k \longrightarrow \mathrm{Top}_k$$
$$V \otimes^! - : \mathrm{Top}_k \longrightarrow \mathrm{Top}_k$$

Then it follows from the above good properties of the tensor product topologies that the functor F_V preserves kernels and cokernels in Top_k. Hence $F_V \colon \mathrm{Top}_k \longrightarrow \mathrm{Top}_k$ is an exact functor between exact categories in the quasi-abelian exact structure, i.e., it takes kernel-cokernel pairs to kernel-cokernel pairs.

Exactness Properties of Uncompleted Tensor Products

Let V be a separated topological vector space. Let F^{s}_V denote one of the topological tensor product functors (the restriction of F_V to $\mathrm{Top}^{\mathrm{s}}_k \subset \mathrm{Top}_k$)

$$\begin{aligned} V \otimes^* - : \mathrm{Top}^{\mathrm{s}}_k &\longrightarrow \mathrm{Top}^{\mathrm{s}}_k \\ V \otimes^{\leftarrow} - : \mathrm{Top}^{\mathrm{s}}_k &\longrightarrow \mathrm{Top}^{\mathrm{s}}_k \\ - \otimes^{\leftarrow} V : \mathrm{Top}^{\mathrm{s}}_k &\longrightarrow \mathrm{Top}^{\mathrm{s}}_k \\ V \otimes^{!} - : \mathrm{Top}^{\mathrm{s}}_k &\longrightarrow \mathrm{Top}^{\mathrm{s}}_k \end{aligned}$$

Then the functor F^{s}_V preserves kernels and cokernels in $\mathrm{Top}^{\mathrm{s}}_k$. Hence $F^{\mathrm{s}}_V : \mathrm{Top}^{\mathrm{s}}_k \longrightarrow \mathrm{Top}^{\mathrm{s}}_k$ is also an exact functor between exact categories in the quasi-abelian exact structure on $\mathrm{Top}^{\mathrm{s}}_k$.

Exactness Properties of Completed Tensor Products

Let $\mathfrak{W}$ be a complete, separated topological vector space. Let $F^{\mathrm{sc}}_{\mathfrak{W}}$ denote one of the completed topological tensor product functors

$$\begin{aligned} \mathfrak{W}\,\widehat{\otimes}^{*}\,-\,&\colon \mathrm{Top}_k^{\mathrm{sc}} \longrightarrow \mathrm{Top}_k^{\mathrm{sc}} \\ \mathfrak{W}\,\widehat{\otimes}^{\leftarrow}\,-\,&\colon \mathrm{Top}_k^{\mathrm{sc}} \longrightarrow \mathrm{Top}_k^{\mathrm{sc}} \\ -\,\widehat{\otimes}^{\leftarrow}\,\mathfrak{W}&\colon \mathrm{Top}_k^{\mathrm{sc}} \longrightarrow \mathrm{Top}_k^{\mathrm{sc}} \\ \mathfrak{W}\,\widehat{\otimes}^{!}\,-\,&\colon \mathrm{Top}_k^{\mathrm{sc}} \longrightarrow \mathrm{Top}_k^{\mathrm{sc}} \end{aligned}$$

Then the functor $F^{\mathrm{sc}}_{\mathfrak{W}}$ preserves cokernels in $\mathrm{Top}_k^{\mathrm{sc}}$. We do not know whether it preserves kernels, generally speaking.

We do know that the functor $F^{\mathrm{sc}}_{\mathfrak{W}}\colon \mathrm{Top}_k^{\mathrm{sc}} \longrightarrow \mathrm{Top}_k^{\mathrm{sc}}$ takes kernel-cokernel pairs to kernel-cokernel pairs. However, the class of all kernel-cokernel pairs is not well-behaved in $\mathrm{Top}_k^{\mathrm{sc}}$ (not preserved by the pullbacks), as we have seen.

Exactness Properties of Completed Tensor Products

The class of all stable kernel-cokernel pairs in $\mathrm{Top}_k^{\mathrm{sc}}$ is better behaved (forming the maximal exact category structure), but it is not preserved by the functors $F_{\mathfrak{W}}^{\mathrm{sc}}$, generally speaking.

The problem is with the functors $-\mathbin{\widehat{\otimes}^!}\mathfrak{W}$ and $-\mathbin{\widehat{\otimes}^{\leftarrow}}\mathfrak{W}$, where $\mathfrak{W}$ is a discrete vector space. According to the discussion above, we have $\mathfrak{V}\mathbin{\widehat{\otimes}^!}\mathfrak{W}\simeq\mathfrak{V}\mathbin{\widehat{\otimes}^{\leftarrow}}\mathfrak{W}\simeq\mathfrak{V}[[X]]$ for any $\mathfrak{V}\in\mathrm{Top}_k^{\mathrm{sc}}$, where X is a set indexing a basis in $\mathfrak{W}$.

For any topological vector space $\mathfrak{C}\in\mathrm{Top}_k^{\mathrm{sc}}$, there exists a stable kernel-cokernel pair $0\longrightarrow\mathfrak{K}\longrightarrow\mathfrak{A}\longrightarrow\mathfrak{C}\longrightarrow 0$ in $\mathrm{Top}_k^{\mathrm{sc}}$ such that no infinite family of nonzero elements converges to zero in $\mathfrak{A}$. If there exists an infinite family of nonzero elements converging to zero in $\mathfrak{C}$, then the induced map $\mathfrak{A}[[X]]\longrightarrow\mathfrak{C}[[X]]$ is not surjective, so the kernel-cokernel pair
$0\longrightarrow\mathfrak{K}[[X]]\longrightarrow\mathfrak{A}[[X]]\longrightarrow\mathfrak{C}[[X]]\longrightarrow 0$ in $\mathrm{Top}_k^{\mathrm{sc}}$ is not stable.

Thus the functors $-\mathbin{\widehat{\otimes}^!}\mathfrak{W}$ and $-\mathbin{\widehat{\otimes}^{\leftarrow}}\mathfrak{W}$ are not exact in the maximal exact category structure on $\mathrm{Top}_k^{\mathrm{sc}}$.

Conclusions

One can modify or restrict the maximal exact category structure on $\mathrm{Top}_k^{\mathrm{sc}}$ by imposing the condition of preservation of exactness by the tensor products. In particular, in the strong exact structure on $\mathrm{Top}_k^{\mathrm{sc}}$ the admissible epimorphisms are the strongly surjective open maps.

Restricting the class of short exact sequences (possibly) even further, one arrives to the tensor-refined exact structure, which is maximal among the exact structures on $\mathrm{Top}_k^{\mathrm{sc}}$ in which the tensor product functors are exact. We know very little about this exact structure beyond its existence and uniqueness.

Conclusions

The general feeling is that the categories of incomplete VSLTs have good exactness properties, while the restriction to the complete VSLTs destroys such good properties. However, it is the complete, separated topological vector spaces that are the main object of interest. Perhaps the conclusion should be that the language of exact categories does not provide the most suitable category-theoretic point of view on topological algebra.

One possible alternative might be to work with the quasi-abelian category Top_k or $\mathrm{Top}_k^{\mathrm{s}}$ of (separated or nonseparated) incomplete VSLTs, with the exact functors of uncompleted tensor products defined on it. Endow this category with the class of weak equivalences consisting of all the morphisms which become isomorphisms after the completion. Inverting such weak equivalences produces the category $\mathrm{Top}_k^{\mathrm{sc}}$.

The End

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