Exact category structures and tensor products of topological vector spaces with linear topology

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The categories of topological abelian groups or topological vector spaces are rarely abelian. Constructing and using exact category structures (in Quillen's sense) in order to develop homological topological algebra or homological functional analysis is an old idea, going back to the 1960's–70's.

Part I

Topological Algebra

Topological Vectors Spaces with Linear Topology

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The completion topology on $V\widehat{\ }$ is the projective limit topology of discrete vector spaces V/U. Equivalently, the open subspaces in $V\widehat{\ }$ are the kernels of the projection maps $V\widehat{\ } \longrightarrow V/U$.

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So there is a natural bijection between the open subspaces in V and in $\mathfrak{V} = V^\frown$, assigning to an open subspace $U \subset V$ the kernel $\mathfrak{U}$ of the projection map $V^\frown \longrightarrow V/U$. Conversely, to an open subspace $\mathfrak{U} \subset \mathfrak{V}$ its full preimage $\lambda_V^{-1}(\mathfrak{U}) \subset V$ is assigned.

Complete and Separated Subspaces and Quotients

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Here $\mathfrak{U}$ ranges over all the open subspaces in $\mathfrak{A}$, while $\mathfrak{W}$ ranges over all the open subspaces in $\mathfrak{A}$ containing $\mathfrak{K}$.

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Let $\mathfrak{A}$ be a complete, separated topological vector space and $\mathfrak{K} \subset \mathfrak{A}$ be a closed subspace. What does it mean that the quotient space $C = \mathfrak{A}/\mathfrak{K}$ is complete (or not complete) in the quotient topology?

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The counterexamples show that, in general, this cannot be done.

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Lemma

For any family of topological vector spaces V_i, the product $\prod_{i\in I} V_i$ is a complete, separated topological vector space in the modified box topology.

Main Counterexample: Construction Cont'd

The modified coproduct topology

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The modified coproduct topology on the direct sum of a family of topological vector spaces $\bigoplus_{i \in I} V_i$

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For any family of separated topological vector spaces V_i, the subspace $\bigoplus_{i\in I} V_i \subset \prod_{i\in I} V_i$ is closed in the modified box topology on the product.

Corollary

For any family of separated topological vector spaces V_i, the direct sum $\bigoplus_{i\in I} V_i$ is separated and complete in the modified coproduct topology.

Main Counterexample: Construction Fin'd

Construction of the Main Counterexample.

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Let C be a separated topological vector space.

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Let C be a separated topological vector space. Choose an infinite set I, and consider the direct sum $\mathfrak{A}_I(C) = C^{(I)} = \bigoplus_{i \in I} C$, endowed with the modified coproduct topology. Let $\Sigma \colon C^{(I)} \longrightarrow C$ be the summation map.

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These tensor product operations in functional analysis were studied by Grothendieck in 1950's.

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The definition of the $*$-topology looks complicated, but it has a very natural universal property. Let U, V, and W be topological vector spaces, and let $\phi\colon V \times W \longrightarrow U$ be a bilinear map. Let $\phi^{\otimes}\colon V \otimes_k W \longrightarrow U$ be the related map from the tensor product.

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These are observations from Beilinson's 2008 paper.

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Part II

Exact Category Structures

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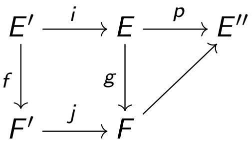

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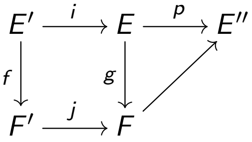

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exists, and the morphism $j\colon F' \longrightarrow F$ is an admissible monomorphism. Simply put, admissible monomorphisms are preserved by pushouts.

If $E' \longrightarrow E \longrightarrow E''$ is a short exact sequence, then it follows from Ex0–Ex1 and Ex2(a) that $F' \longrightarrow F \longrightarrow E''$ is a short exact sequence, too. Conversely, assuming Ex0–Ex1, the axiom Ex2(a) can be equivalently reformulated

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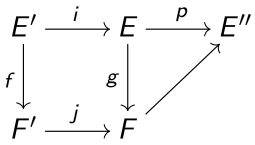

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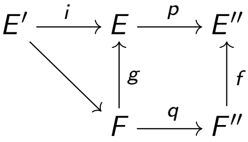

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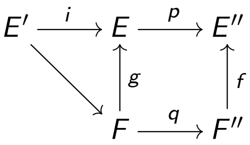

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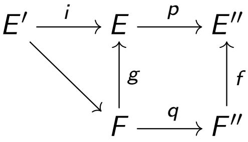

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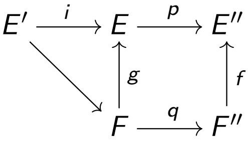

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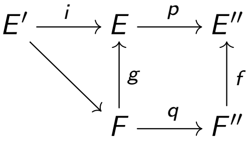

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The axiom Ex2 is Ex2(a) + Ex2(b).

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For example, the category of even-dimensional finite-dimensional vector spaces is weakly idempotent-complete, but not idempotent-complete. The category of vector spaces of dimension different from 1 (or from 1, 2, and 3) is additive, but not weakly idempotent-complete.

The assumption of (at least) weak idempotent-completeness simplifies the exact category theory considerably.

Axioms of Exact Category Cont'd

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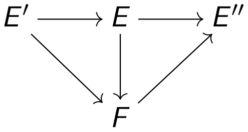

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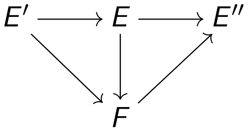

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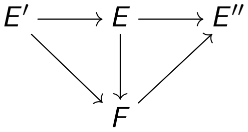

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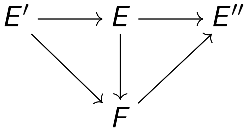

both $E' \longrightarrow E \longrightarrow E''$ and $E' \longrightarrow F \longrightarrow E''$ is are short exact sequences, then $E \longrightarrow F$ is an isomorphism.

Any one of the axioms Ex2(a) or Ex2(b) implies Ex2'(c).

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This is the end of the list of exact category axioms Ex0–Ex3.

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Quasi-abelian categories form the class of additive categories closest to the abelian ones.

Let A be an additive category in which all morphisms have kernels and cokernels (such categories are sometimes called preabelian).

We will say that a morphism in A is a kernel if it is the kernel of some morphism in A. Similarly, a morphism in A is a cokernel if it is a the cokernel of some morphism in A. Any kernel is the kernel of its cokernel, and any cokernel is the cokernel of its kernel.

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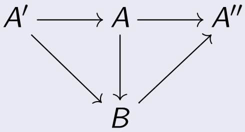

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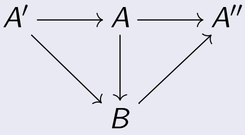

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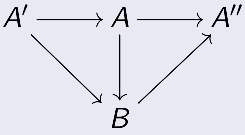

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shows that condition (2) is not satisfied for A.

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Exactness Properties of Categories of VSLTs

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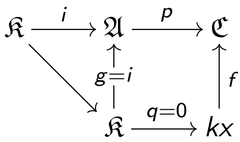

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Moreover, there does not exist an exact category structure on $\mathrm{Top}_k^{\mathrm{sc}}$ in which the closed embedding $i\colon \mathfrak{K} \longrightarrow \mathfrak{A}$ would be an admissible monomorphism.

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Maximal Exact Category Structure

According to Rump's paper [“On the maximal exact category structure of an additive category”, 2011], every additive category has a maximal exact category structure. This means a class of short exact sequences satisfying Ex0–Ex3 which contains any other class of short exact sequences satisfying Ex0–Ex3.

For a weakly idempotent-complete additive category A, the maximal exact category structure has a rather simple description, which is convenient to work with [Sieg–Wegner, “Maximal exact structures on additive categories”, 2011; Crivei, “Maximal exact structures on additive categories revisited”, 2012].

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For any weakly idempotent-complete additive category A, the class of all stable kernel-cokernel pairs $K \longrightarrow A \longrightarrow C$ is an exact category structure on A.

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Now let A be a right quasi-abelian category. Then all kernels in A are semi-stable, hence all semi-stable cokernels in A are stable.

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A morphism *i* in A is said to be a stable kernel if *i* is a semi-stable kernel and the cokernel of *i* is a semi-stable cokernel. Dually, *p* is a stable cokernel if *p* is a semi-stable cokernel and the kernel of *p* is a semi-stable kernel.

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Now let A be a right quasi-abelian category. Then all kernels in A are semi-stable, hence all semi-stable cokernels in A are stable. The short exact sequences in the maximal exact category structure are the kernel-cokernels pairs $K \xrightarrow{i} A \xrightarrow{p} C$ in which p is a semi-stable cokernel.

Maximal Exact Structure on Complete, Sep'd VSLTs

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It appears that Beilinson's intended meaning was "the completed tensor products are exact functors in the quasi-abelian exact structure on $\mathrm{Top}_k^{\mathrm{sc}}$" (which does not exist, as we have seen). Next we will see that in the maximal exact structure on $\mathrm{Top}_k^{\mathrm{sc}}$, at least two of the three tensor products are not exact, either. However, these functors can be thought of as being "exact" in some other sense.

Exactness Properties of Uncompleted Tensor Products

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Then it follows from the above good properties of the tensor product topologies that the functor F_V preserves kernels and cokernels in Top_k.

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Then it follows from the above good properties of the tensor product topologies that the functor F_V preserves kernels and cokernels in Top_k. Hence $F_V : \mathrm{Top}_k \longrightarrow \mathrm{Top}_k$ is an exact functor between exact categories in the quasi-abelian exact structure,

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Then the functor F^{s}_V preserves kernels and cokernels in $\mathrm{Top}^{\mathrm{s}}_k$. Hence $F^{\mathrm{s}}_V : \mathrm{Top}^{\mathrm{s}}_k \longrightarrow \mathrm{Top}^{\mathrm{s}}_k$ is also an exact functor between exact categories in the quasi-abelian exact structure on $\mathrm{Top}^{\mathrm{s}}_k$.

Exactness Properties of Completed Tensor Products

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Thus the functors $-\mathbin{\widehat{\otimes}^!}\mathfrak{W}$ and $-\mathbin{\widehat{\otimes}^{\leftarrow}}\mathfrak{W}$ are not exact in the maximal exact category structure on $\mathrm{Top}_k^{\mathrm{sc}}$.

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The End

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