Exotic derived categories

Leonid Positselski – Czech Academy of Sciences, Prague

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Introduction

This is a talk about the foundations of homological algebra.

The main object of study in homological algebra is the derived category. It is defined as the category of complexes (say, of modules over a ring R) with quasi-isomorphisms inverted. One considers the abelian category of complexes of R-modules, or the category of complexes of R-modules up to chain homotopy, and adjoins formal inverse morphisms for the morphisms inducing isomorphisms on the cohomology modules.

Many people nowadays are studying one particular aspect of the foundations: to what class of categories does the derived category belong, and what structure does it carry? This involves the difference between triangulated categories, derivators, and $(\infty, 1)$-categories, various notions of enhancements, triangulated categories of algebraic or topological origin, etc. These questions are not the subject of this talk.

Introduction

Instead, we ask a more basic question: What is a complex?

Specifically:

1. What is a differential? What is the square of a differential?
2. What is a quasi-isomorphism? Can one define quasi-isomorphisms when the cohomology is undefined?
3. What are the terms of a complex? What can they be, beyond modules and sheaves?

I will try to convince you that there is a whole half of algebra, or at least half of homological algebra, which was either missed by the classical authors, or forgotten and left undeveloped by their followers, up until recently.

What is a differential? What is its square?

The first step on the way to the derived category is the homotopy category, meaning the category of complexes up to chain homotopy. More generally, people often consider the homotopy and derived categories of DG-modules.

Let (A, d_A) be a DG-ring, i.e., a graded ring $A = \bigoplus_{n\in\mathbb{Z}} A^n$ with an odd derivation $d_A\colon A^n \longrightarrow A^{n+1}$ satisfying $d^2 = 0$.

Then, for any two (say, left) DG-modules (L, d_L) and (M, d_M) over A, one constructs a complex of abelian groups $\mathrm{Hom}_A(L, M)$. The degree-n component $\mathrm{Hom}_A^n(L, M)$ is the group of all homogeneous A-linear maps $f\colon L \longrightarrow M$ of degree n. (A suitable sign rule is presumed in saying that f is A-linear.) So f has to commute with the action of A, but not with the differentials on L and M.

The rule $d(f)(x) = d_M(f(x)) - (-1)^{|f|} f(d_L(x))$ defines the differential on $\mathrm{Hom}_A(L, M)$. With this complex of morphisms between any two DG-modules, one obtains the DG-category of DG-modules over (A, d_A), denoted by $\mathrm{DG}(A\text{-}\mathrm{Mod})$.

What is a differential? What is its square?

The homotopy category of DG-modules $\mathrm{K}(A\text{-}\mathrm{Mod})$ is defined as $\mathrm{K}(A\text{-}\mathrm{Mod}) = H^0\mathrm{DG}(A\text{-}\mathrm{Mod})$. So the objects of $\mathrm{K}(A\text{-}\mathrm{Mod})$ are the left DG-modules over (A, d_A), and the morphisms are homomorphisms of DG-modules up to chain homotopy.

Here the homomorphisms of DG-modules $L \longrightarrow M$ are the degree-0 cocycles in $\mathrm{Hom}_A(L, M)$, which means maps commuting with both the action of A and the differentials. The morphisms homotopic to zero are the coboundaries in $\mathrm{Hom}_A(L, M)$.

The important observation I want to share is that one does not need A to be a DG-ring for this construction to work. In fact, one does not need d_A^2 to vanish, nor d_L^2 and d_M^2 to vanish. The DG-category $\mathrm{DG}(A\text{-}\mathrm{Mod})$ can be constructed for a wider class of algebraic structures than DG-rings (A, d_A). This wider class of algebraic structures is called curved DG-rings.

Curved DG-Rings

A curved DG-ring (B, d_B, h_B) is a graded ring $B = \bigoplus_{n \in \mathbb{Z}} B^n$ endowed with an odd derivation $d_B \colon B^n \longrightarrow B^{n+1}$ (i.e., $d_B(bc) = d_B(b)c + (-1)^{|b|} b d_B(c)$ for all $b \in B^{|b|}$ and $c \in B^{|c|}$) and an element $h_B \in B^2$ such that

- $d_B^2(b) = hb - bh$ for all $b \in B$, and
- $d_B(h_B) = 0$.

The map d_B is called the differential (or the "predifferential"), and the element h_B is called the curvature element of B.

Morphisms of DG-rings include both the obvious ("strict") morphisms one could immediately think of, and the change-of-connection morphisms. All the change-of-connection morphisms are isomorphisms. For any CDG-ring (B, d, h) and an element $a \in B^1$, there is a change-of-connection isomorphism connecting (B, d, h) with (B, d', h'), where

- $d'(b) = d(b) + [a, b]$ (where $[x, y] = xy - (-1)^{|x||y|} xy$), and
- $h' = h + d(a) + a^2$.

More precisely, a morphism of CDG-rings $(B', d', h') \longrightarrow (B, d, h)$ is a pair (f, a), where $f\colon B' \longrightarrow B$ is a homomorphism of graded rings and $a \in B^1$ is an element such that

- $f(d'(b)) = d(f(b)) + [a, f(b)]$ for all $b \in B$, and
- $f(h') = h + d(a) + a^2$.

The element a is called the change-of-connection element.

A left CDG-module M over a CDG-ring (B, d_B, h_B) is a graded left B-module $M = \bigoplus_{n\in\mathbb{Z}} M^n$ endowed with an odd derivation $d_M\colon M^n \longrightarrow M^{n+1}$ compatible with the derivation d_B on B (i.e., $d_M(bx) = d_B(b)x + (-1)^{|b|} b d_M(x)$ for all $b \in B^{|b|}$ and $x \in M^{|x|}$) such that

- $d_M^2(x) = h_B x$ for all $x \in M$.

Right CDG-modules N are defined similarly; the related formula for the square of the differential is

- $d_N^2(y) = -y h_B$ for all $y \in N$.

Curved DG-Rings

The key observation is that B-linear morphisms between two CDG-modules form a complex. Let (L, d_L) and (M, d_M) be two CDG-modules over (B, d_B, h_B). Denote by $\mathrm{Hom}_B(L, M)$ the graded abelian group of homogeneous B-linear maps $L \longrightarrow M$, and define the differential $d\colon \mathrm{Hom}_B^n(L, M) \longrightarrow \mathrm{Hom}_B^{n+1}(L, M)$ by the usual formula $d(f)(x) = d_M(f(x)) - (-1)^n f(d_L(x))$. Then

$$d^2(f)(x) = d_M^2(f(x)) - f(d_L^2(x)) = h_B f(x) - f(h_B x) = 0.$$

So the curvature-related terms cancel each other, and we obtain the DG-category of CDG-modules $\mathrm{DG}(B\text{-}\mathrm{Mod})$. Hence the homotopy category $\mathrm{K}(B\text{-}\mathrm{Mod}) = H^0\mathrm{DG}(B\text{-}\mathrm{Mod})$, which is a triangulated category with infinite direct sums and products.

What can one do with this homotopy category?

Example: de Rham differential for a nonflat connection

Let M be a smooth real manifold and E be a vector bundle over M. Let $\mathrm{End}(E) = E \otimes E^*$ denote the vector bundle of endomorphisms of E. Consider the graded algebra $\Omega(M)$ of differential forms on M and the graded algebra $\Omega(M, \mathrm{End}(E))$ of differential forms with the coefficients in $\mathrm{End}(E)$. The graded space $\Omega(M, E)$ of differential forms with the coefficients in E is a graded module over $\Omega(M, \mathrm{End}(E))$.

Choose a connection ∇_E in E, and denote by $\nabla_{\mathrm{End}(E)}$ the induced connection in $\mathrm{End}(E)$. Then there is the connection-determined de Rham differential $d_{\nabla_E}\colon \Omega^n(M, E) \longrightarrow \Omega^{n+1}(M, E)$ and similarly $d_{\nabla_{\mathrm{End}(E)}}\colon \Omega^n(M, \mathrm{End}(E)) \longrightarrow \Omega^{n+1}(M, \mathrm{End}(E))$.

Let $h_{\nabla_E} \in \Omega^2(M, \mathrm{End}(E))$ be the curvature form of the connection ∇_E in E. Then

- $(B, d, h) = (\Omega(M, \mathrm{End}(E)), d_{\nabla_{\mathrm{End}(E)}}, h_{\nabla_E})$ is a CDG-ring;
- $(M, d_M) = (\Omega(M, E), d_{\nabla_E})$ is a CDG-module over (B, d, h);
- changing the connection in E leads to a change-of-connection isomorphism of CDG-rings.

Curved DG-Rings

Any morphism of CDG-rings $(f, a)\colon (B', d', h') \longrightarrow (B, d, h)$ induces a DG-functor of "restriction of scalars" $\mathrm{DG}(B\text{-}\mathrm{Mod}) \longrightarrow \mathrm{DG}(B'\text{-}\mathrm{Mod})$. In particular, a change-of-connection isomorphism $(\mathrm{id}, a)\colon (B, d', h') \longrightarrow (B, d, h)$ induces an equivalence of DG-categories $\mathrm{DG}((B, d, h)\text{-}\mathrm{Mod}) \simeq \mathrm{DG}((B, d', h')\text{-}\mathrm{Mod})$ assigning to a CDG-module (M, d_M) over (B, d, h) the CDG-module (M, d'_M) over (B, d', h') with the twisted differential

- $d'_M(x) = d_M(x) + ax$.

The inclusion of the category of DG-rings into the category of CDG-rings is faithful, but not fully faithful. There are both more objects and more morphisms in (the category of) CDG-rings than in DG-rings. The DG- and homotopy categories of DG-modules over CDG-isomorphic DG-rings are naturally equivalent. Change-of-connection isomorphisms between DG-rings correspond to Maurer–Cartan elements, and induce the related twists of the differentials on the DG-modules.

What is a quasi-isomorphism?

We have defined the homotopy category of CDG-modules over a CDG-ring (B, d_B, h_B), but what is the derived category? The derived category of DG-modules over a DG-ring is constructed by inverting the quasi-isomorphisms, and quasi-isomorphisms are defined as DG-module morphisms inducing isomorphisms on the cohomology. But CDG-modules have no cohomology modules . . .

Besides the definition in terms of quasi-isomorphisms, in order to work with the derived categories one needs resolutions. In the classical homological algebra of 1950's–60's, people usually considered either bounded above or bounded below complexes.

For suitably bounded complexes of modules over a ring R, one has triangulated equivalences

$$\mathrm{K}^+(R\text{-}\mathrm{Mod}_{\mathrm{inj}}) \simeq \mathrm{D}^+(R\text{-}\mathrm{Mod}),$$
$$\mathrm{K}^-(R\text{-}\mathrm{Mod}_{\mathrm{proj}}) \simeq \mathrm{D}^-(R\text{-}\mathrm{Mod}).$$

What is a quasi-isomorphism?

Similar equivalences hold for similarly bounded DG-modules over a DG-ring (A, d) which is nonpositively cohomologically graded, that is $A^n = 0$ for $n > 0$:

$$\mathrm{K}^+(A\text{-}\mathrm{Mod}_{\mathrm{inj}}) \simeq \mathrm{D}^+(A\text{-}\mathrm{Mod}),$$
$$\mathrm{K}^-(A\text{-}\mathrm{Mod}_{\mathrm{proj}}) \simeq \mathrm{D}^-(A\text{-}\mathrm{Mod}).$$

Notice that nonpositively cohomologically graded DG-rings have no curved versions: there is no room for the curvature if $A^2 = 0$.

There is also a version for connected, simply connected positively cohomologically graded DG-rings: if $A^n = 0$ for $n < 0$, A^0 is a field, and $A^1 = 0$, then

$$\mathrm{K}^-(A\text{-}\mathrm{Mod}_{\mathrm{inj}}) \simeq \mathrm{D}^-(A\text{-}\mathrm{Mod}),$$
$$\mathrm{K}^+(A\text{-}\mathrm{Mod}_{\mathrm{proj}}) \simeq \mathrm{D}^+(A\text{-}\mathrm{Mod}).$$

For unbounded complexes of modules, or for bounded DG-modules over other kinds of DG-rings (such as the de Rham DG-algebra), these equivalences fail.

Counterexamples

Here are some counterexamples. Let k be a field, and let $\Lambda = k[\epsilon]/(\epsilon^2)$ be the ring of dual numbers (the exterior algebra with one generator) over k. Then the unbounded complex

$$\cdots \longrightarrow \Lambda \xrightarrow{\epsilon *} \Lambda \xrightarrow{\epsilon *} \Lambda \longrightarrow \cdots$$

is a complex of projective-injective Λ-modules, it is acyclic, but it is not contractible. So it is a nonzero object in $\mathrm{K}(\Lambda\text{-}\mathrm{Mod}_{\mathrm{inj}})$ and in $\mathrm{K}(\Lambda\text{-}\mathrm{Mod}_{\mathrm{proj}})$, but a zero object in $\mathrm{D}(\Lambda\text{-}\mathrm{Mod})$.

Let (A, d_A) be the DG-algebra over k with $A = k[x]/(x^2)$, where cohom. $\deg x = 1$. Put $d_A = 0$. Consider the DG-module (M, d_M) over (A, d_A), where $M = Am$ is the free A-module with one generator m in degree 0, and $d_M(m) = \lambda xm$, where $\lambda \in k$.

Then M is a projective-injective graded A-module, the DG-module (M, d_M) is not contractible, but it is acyclic whenever $\lambda \neq 0$. Once again, (M, d_M) is a nonzero object in the homotopy category of DG-modules over (A, d) with projective/injective underlying graded modules, but a zero object in the derived category.

What is a quasi-isomorphism?

Let (B, d, h) be a CDG-ring. The coderived category in the sense of Becker of CDG-modules over (B, d, h) is defined as the homotopy category of CDG-modules whose underlying graded B-modules are injective. Dually, the contraderived category in the sense of Becker of CDG-modules over (B, d, h) is defined as the homotopy category of CDG-modules whose underlying graded B-modules are projective.

The homotopy category of complexes of projective modules over a ring was first considered by Jørgensen, and the homotopy category of complexes of injective modules (or quasi-coherent sheaves) was considered by Krause. Becker studied the CDG-module case.

The coderived category in the sense of Becker may be also called the "coderived category in the injective sense", and Becker's contraderived category may be called the "contraderived category in the projective sense".

What is a quasi-isomorphism?

For any kind of exotic derived category, it is of key importance to have it represented as a quotient category (and not only as a subcategory) of the homotopy category. It is best to have both types of descriptions. The language of abelian model category structures on the categories of complexes or CDG-modules is convenient for such results.

A CDG-module (L, d_L) over (B, d, h) is said to be coacyclic in the sense of Becker if, for any CDG-module (J, d_J) such that J is an injective graded B-module, the complex of abelian groups $\mathrm{Hom}_B(L, J)$ is acyclic. Dually, a CDG-module (M, d_M) over (B, d, h) is said to be contraacyclic in the sense of Becker if, for any CDG-module (P, d_P) such that P is a projective graded B-module, the complex $\mathrm{Hom}_B(P, M)$ is acyclic.

One can say that a morphism is a "co-quasi-isomorphism" if its cone is coacyclic and a "contra-quasi-isomorphism" if its cone is contraacyclic.

Becker's coderived and contraderived categories

Denote the full subcategories of coacyclic and contraacyclic CDG-modules in the sense of Becker by $\mathrm{Ac}^{\mathrm{bco}}(B\text{-}\mathrm{Mod})$ and $\mathrm{Ac}^{\mathrm{bctr}}(B\text{-}\mathrm{Mod}) \subset \mathrm{K}(B\text{-}\mathrm{Mod})$.

Theorem (Becker, 2012–14)

For any CDG-ring (B, d, h):
(a) $\mathrm{K}(B\text{-}\mathrm{Mod}_{\mathrm{inj}}) \simeq \mathrm{K}(B\text{-}\mathrm{Mod})/\mathrm{Ac}^{\mathrm{bco}}(B\text{-}\mathrm{Mod})$;
(b) $\mathrm{K}(B\text{-}\mathrm{Mod}_{\mathrm{proj}}) \simeq \mathrm{K}(B\text{-}\mathrm{Mod})/\mathrm{Ac}^{\mathrm{bctr}}(B\text{-}\mathrm{Mod})$.

The proof uses abelian model category structures and the small object argument.

The upside of Becker's approach to coderived and contraderived categories is that powerful set-theoretic methods, such as the small object argument, can be used for their study. The downside of Becker's approach is that it requires existence of enough injective or projective objects.

What is a quasi-isomorphism?

There is another and more elementary approach, which requires less assumptions. Let $0 \longrightarrow (K, d_K) \longrightarrow (L, d_L) \longrightarrow (M, d_M) \longrightarrow 0$ be a short exact sequence of CDG-modules over (B, d, h). Then a construction similar to the totalization of a bicomplex produces the "total CDG-module" $\mathrm{Tot}(K \to L \to M)$ with the components $\mathrm{Tot}^n = K^{n+1} \oplus L^n \oplus M^{n-1}$.

In any category worthy of the name "exotic derived category", the CDG-module $\mathrm{Tot}(K \to L \to M)$ should represent a zero object. More generally, a CDG-module is called absolutely acyclic if it belongs to the thick subcategory of $\mathrm{K}(B\text{-}\mathrm{Mod})$ generated by the CDG-modules of the form $\mathrm{Tot}(K \to L \to M)$.

A CDG-module is said to be coacyclic in the sense of Positselski if it belongs to the minimal triangulated subcategory of $\mathrm{K}(B\text{-}\mathrm{Mod})$ containing the CDG-modules $\mathrm{Tot}(K \to L \to M)$ and closed under infinite direct sums. A CDG-module is contraacyclic in the sense of Positselski if it belongs to the similar minimal subcategory closed under infinite products.

Totalization coderived and contraderived categories

The co/contraderived categories in the sense of Positselski were introduced by me and independently by Keller–Lowen–Nicolas. They can be also called the "co/contraderived categories in the sense of totalization". The full subcategories of coacyclic and contraacyclic CDG-modules in the sense of Positselski are denoted by $\mathrm{Ac}^{\mathrm{pco}}(B\text{-}\mathrm{Mod})$ and $\mathrm{Ac}^{\mathrm{pctr}}(B\text{-}\mathrm{Mod}) \subset \mathrm{K}(B\text{-}\mathrm{Mod})$.

Theorem (L.P., 2009–11)

Let $B = (B, d, h)$ be a CDG-ring.
(a) *If countable direct sums of injective graded left B-modules have finite injective dimensions, then $\mathrm{Ac}^{\mathrm{bco}}(B\text{-}\mathrm{Mod}) = \mathrm{Ac}^{\mathrm{pco}}(B\text{-}\mathrm{Mod})$.*
(b) *If countable products of projective graded left B-modules have finite projective dimensions, then*
$\mathrm{Ac}^{\mathrm{bctr}}(B\text{-}\mathrm{Mod}) = \mathrm{Ac}^{\mathrm{pctr}}(B\text{-}\mathrm{Mod})$.

Part (a) includes left Noetherian graded rings B. Part (b) includes right coherent graded rings B over which all flat modules have finite projective dimensions.

Coderived and contraderived categories

It is an open problem whether for an arbitrary CDG-ring $B = (B, d, h)$ one has $\mathrm{Ac}^{\mathrm{bco}}(B\text{-}\mathrm{Mod}) = \mathrm{Ac}^{\mathrm{pco}}(B\text{-}\mathrm{Mod})$ and $\mathrm{Ac}^{\mathrm{bctr}}(B\text{-}\mathrm{Mod}) = \mathrm{Ac}^{\mathrm{pctr}}(B\text{-}\mathrm{Mod})$. The "$\supset$" inclusion is easy.

The construction of the totalization coderived category also makes sense for any exact category E (in the sense of Quillen) with exact infinite coproduct functors. Similarly, the construction of the totalization contraderived category makes sense for any exact category E with exact infinite product functors.

All coacyclic and all contraacyclic complexes (and DG-modules), either in Becker's or Positselski's sense, are acyclic.

Theorem

(a) *Over an exact category of finite homological dimension, the classes of acyclic and absolutely acyclic complexes coincide.*
(b) *Over a CDG-ring (B, d, h) whose underlying graded ring B has finite left global dimension, the classes of coacyclic, contraacyclic, and absolutely acyclic CDG-modules coincide.*

What is a quasi-isomorphism?

Can one define "the derived categories of CDG-modules over CDG-rings" in such a way that for DG-rings one would obtain the usual derived categories of DG-modules?

The answer to this question is: Not if one wants the construction to be invariant under isomorphisms of CDG-rings.

Let (A, d) be a DG-ring and $c \in A^1$ be a Maurer–Cartain element, i.e., $d(c) + c^2 = 0$. Then the differential $d'(x) = d(x) + [c, x]$ defines another DG-ring structure on A. The DG-rings (A, d) and (A, d') are isomorphic as CDG-rings (by the change-of-connection isomorphism associated with c). But the derived categories $\mathrm{D}((A, d)\text{-}\mathrm{Mod})$ and $\mathrm{D}((A, d')\text{-}\mathrm{Mod})$ can be totally different.

The coderived, contraderived, and absolute derived categories are called collectively the derived categories of the second kind. All derived categories of the second kind are invariant under isomorphisms of CDG-rings.

What is a quasi-isomorphism?

The conventional derived category (called the derived category of the first kind) is well-known to be invariant under quasi-isomorphisms of DG-rings.

These are two incompatible kinds of invariance. Nothing is invariant simultaneously under quasi-isomorphisms of DG-rings and under the Maurer–Cartan twists.

In fact, let A' and A'' be two DG-algebras over a field k. Then there exists a DG-algebra (B, d') and an element $c \in B^1$ such that, setting $d'' = d' + [c, -]\colon B \longrightarrow B$, the DG-algebra (B, d') is quasi-isomorphic to A' and the DG-algebra (B, d'') is quasi-isomorphic to A''.

So any two DG-algebras over k can be connected by a chain of transformations some of which are quasi-isomorphisms and others are CDG-algebra isomorphisms.

What are the terms of a complex?

Beyond modules, people consider complexes in abelian or exact categories. The discussion above suggests the importance of abelian/exact categories with enough injective or projective objects for the definitions of derived categories of the second kind. Failing that, one needs at least a category with exact functors of infinite coproduct or product.

Notice that any exact category with infinite coproducts and enough injectives has exact coproduct functors, and any exact category with infinite products and enough projectives has exact product functors. In this sense, having exact coproducts is a weaker condition than having enough injectives (which makes the totalization coderived category a more generally applicable construction than the injective coderived category).

What are the terms of a complex?

A popular belief among the students of homological algebra nowadays is that "naturally occuring" abelian categories have enough injective objects, but rarely enough projectives. For example, Grothendieck categories, which are the most popular class of abelian categories, have enough injectives, but usually not projectives.

I would argue that this misconception persists because people do not know contramodules. There is a natural class of abelian categories dual-analogous or "covariantly dual" to the Grothendieck categories; namely, the locally presentable abelian categories with enough projective objects. Various contramodule categories provide examples of abelian categories in this class.

What are the terms of a complex?

For about every class of abelian categories of comodules, or discrete, smooth, or torsion modules (which are Grothendieck categories), there is a much less familiar, but no less natural dual-analogous class of abelian categories of contramodules (which are locally presentable with enough projectives).

The categories of sheaves are the hardest ones to “covariantly dualize”. In particular, the dual-analogous category to quasi-coherent sheaves on a scheme is the category of contraherent cosheaves. It is an exact, but not abelian, category with exact infinite products. Under mild assumptions on the scheme, it has enough projective objects.

One is supposed to consider derived or coderived categories of Grothendieck categories (like modules or comodules), and derived or contraderived categories of locally presentable abelian categories with enough projectives (like modules or contramodules).

Example: Contramodules over coalgebras over fields

A coalgebra $\mathcal{C}$ over a field k is a k-vector space endowed with k-linear maps of comultiplication $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and counit $\epsilon\colon \mathcal{C} \longrightarrow k$ satisfying the coassociativity and counitality equations. A left $\mathcal{C}$-comodule $\mathcal{M}$ is a k-vector space endowed with a k-linear map of left $\mathcal{C}$-coaction $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ satisfying the coassociativity and counitality equations.

A left $\mathcal{C}$-contramodule $\mathfrak{P}$ is a k-vector space endowed with a k-linear map $\pi\colon \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ satisfying the following contraassociativity and contraunitality equations.

There are two linear maps

$$\operatorname{Hom}_k(\mathcal{C}, \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P})) \simeq \operatorname{Hom}_k(\mathcal{C} \otimes_k \mathcal{C},\ \mathfrak{P}) \rightrightarrows \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}),$$

one of them induced by μ and the other one by π. These two maps must have equal compositions with the contraaction map π.

There is a map $\mathfrak{P} \longrightarrow \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P})$ induced by the counit map $\epsilon\colon \mathcal{C} \longrightarrow k$. The composition $\mathfrak{P} \longrightarrow \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) \xrightarrow{\pi} \mathfrak{P}$ must be equal to the identity map $\mathrm{id}_{\mathfrak{P}}$.

Example: Contramodules over coalgebras over fields

For any coalgebra $\mathcal{C}$ over k, the category of $\mathcal{C}$-comodules $\mathcal{C}\text{-}\mathrm{Comod}$ is a locally Noetherian Grothendieck abelian category. The injective $\mathcal{C}$-comodules are the direct summands of the cofree $\mathcal{C}$-comodules $\mathcal{C} \otimes_k V$, where V ranges over the k-vector spaces. In particular, any coproduct of injective $\mathcal{C}$-comodules is injective.

The category of $\mathcal{C}$-contramodules $\mathcal{C}\text{-}\mathrm{Contra}$ is a locally presentable abelian category with enough projective objects. The projective $\mathcal{C}$-contramodules are the direct summands of the free $\mathcal{C}$-contramodules $\mathrm{Hom}_k(\mathcal{C}, V)$. In particular, any product of projective $\mathcal{C}$-contramodules is projective.

The additive categories of injective $\mathcal{C}$-comodules and projective $\mathcal{C}$-contramodules are naturally equivalent. The covariant equivalence $\mathcal{C}\text{-}\mathrm{Comod}_{\mathrm{inj}} \simeq \mathcal{C}\text{-}\mathrm{Contra}_{\mathrm{proj}}$ takes the cofree comodule $\mathcal{C} \otimes_k V$ to the free contramodule $\mathrm{Hom}_k(\mathcal{C}, V)$. This equivalence is the simplest (underived) example of the comodule-contramodule correspondence phenomenon.

Example: Contramodules over coalgebras over fields

For coalgebras over fields, the co/contraderived categories in the sense of Becker and in the sense of Positselski coincide. So one has

$$\mathrm{K}(\mathcal{C}\text{-Comod}_{\mathrm{inj}}) = \mathrm{D}^{\mathrm{bco}}(\mathcal{C}\text{-Comod}) \simeq \mathrm{D}^{\mathrm{pco}}(\mathcal{C}\text{-Comod});$$
$$\mathrm{K}(\mathcal{C}\text{-Contra}_{\mathrm{proj}}) = \mathrm{D}^{\mathrm{bctr}}(\mathcal{C}\text{-Contra}) \simeq \mathrm{D}^{\mathrm{pctr}}(\mathcal{C}\text{-Contra}).$$

On the other hand, there is the equivalence of homotopy categories

$$\mathrm{K}(\mathcal{C}\text{-Comod}_{\mathrm{inj}}) \simeq \mathrm{K}(\mathcal{C}\text{-Contra}_{\mathrm{proj}})$$

induced by $\mathcal{C}\text{-Comod}_{\mathrm{inj}} \simeq \mathcal{C}\text{-Contra}_{\mathrm{proj}}$. The resulting equivalence of exotic derived categories

$$\mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-Comod}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-Contra})$$

is the simplest form of derived co-contra correspondence.

All these triangulated equivalences remain valid for curved DG-coalgebras over fields.

Derived nonhomogeneous Koszul dulality

The definition of a curved DG-coalgebra over k is obtained by dualizing the definition of a curved DG-algebra. A curved DG-coalgebra $(\mathcal{C}, d, h)$ is a graded coalgebra $\mathcal{C} = \bigoplus_{n\in\mathbb{Z}} \mathcal{C}^n$ with an odd coderivation $d\colon \mathcal{C}^n \longrightarrow \mathcal{C}^{n+1}$ and a curvature linear function $h\colon \mathcal{C}^{-2} \longrightarrow k$.

A coalgebra is said to be coaugmented if it is endowed with a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra is conilpotent if for every $\bar{c} \in \mathcal{C}/\gamma(k)$ there exists $m \geq 1$ such that the image of $\bar{c}$ under the iterated comultiplication map $\mathcal{C}/\gamma(k) \longrightarrow \mathcal{C}/\gamma(k) \otimes_k \cdots \otimes_k \mathcal{C}/\gamma(k)$ (m factors) vanishes. A conilpotent coalgebra has a unique coaugmentation. A curved DG-coalgebra $(\mathcal{C}, d, h)$ is conilpotent if the graded coalgebra $\mathcal{C}$ is coaugmented and conilpotent, and $d \circ \gamma = 0$.

Derived nonhomogeneous Koszul dulality

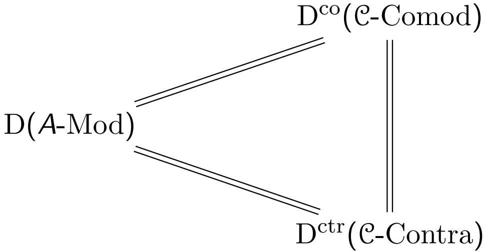
Curved DG-coalgebras are considered up to an equivalence relation similar to the one for comodules or curved DG-comodules in the coderived category. One cannot speak of quasi-isomorphisms of curved DG-coalgebras, as $d^2 \neq 0$ on $\mathcal{C}$, but there is a notion of filtered quasi-isomorphism of conilpotent curved DG-coalgebras. For usual (uncurved) DG-coalgebras, the filtered quasi-isomorphism is a more delicate equivalence relation than the conventional quasi-isomorphism.

The category of (associative, noncommutative) DG-algebras over k with quasi-isomorphisms inverted is equivalent to the category of conilpotent curved DG-coalgebras over k with filtered quasi-isomorphisms inverted,

$$\mathsf{DG\text{-}Alg}_k[\mathrm{Quis}^{-1}] \simeq \mathsf{CDG\text{-}Coalg}_k^{\mathrm{conilp}}[\mathrm{FQuis}^{-1}].$$

Suitable versions of the bar- and cobar-constructions are the functors providing this equivalence. A curvature in the coalgebra appears when the related algebra is not augmented.

Let $A = (A, d_A)$ be a DG-algebra and $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h_{\mathcal{C}})$ be a curved DG-coalgebra corresponding to each other under the above equivalence. Then there is a natural equivalence between three triangulated categories ("Koszul triality")



connecting the derived category of DG-modules over A,
the coderived category of curved DG-comodules over $\mathcal{C}$, and
the contraderived category of curved DG-contamodules over $\mathcal{C}$.
The vertical equivalence is the derived co-contra correspondence.

Example: Contramodules over the p-adic integers

Let p be a prime number. An abelian group A is said to be p-torsion (or "p-primary torsion") if for every $a \in A$ there exists $n \geq 1$ such that $p^n a = 0$. The full subcategory of p-torsion abelian groups in $\mathbb{Z}\text{-}\mathrm{Mod}$, denoted by $\mathbb{Z}\text{-}\mathrm{Mod}_{p\text{-}\mathrm{tors}}$, is a locally Noetherian Grothendieck abelian category with no nonzero projective objects.

The related contramodule category is known as the category of "Ext-p-complete" or "weakly p-complete" or "derived p-complete" abelian groups. I call them p-contramodule abelian groups. It is the full subcategory in $\mathbb{Z}\text{-}\mathrm{Mod}$ consisting of all the abelian groups C such that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[p^{-1}], C) = 0 = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[p^{-1}], C)$. The category of such groups, denoted by $\mathbb{Z}\text{-}\mathrm{Mod}_{p\text{-}\mathrm{ctra}}$, is a locally $\aleph_1$-presentable abelian category with enough projective objects, but no nonzero injectives.

The category of injective objects in $\mathbb{Z}\text{-}\mathrm{Mod}_{p\text{-}\mathrm{tors}}$ is naturally equivalent to the category of projective objects in $\mathbb{Z}\text{-}\mathrm{Mod}_{p\text{-}\mathrm{ctra}}$. This is a classical result essentially due to Harrison (1959).

Example: Contramodules over the p-adic integers

The following illuminating description of p-contramodule abelian groups is not so widely known yet. An abelian group C is a p-contramodule if and only if it can be endowed with a p-power infinite summation operation assigning to every sequence of elements $c_0, c_1, c_2, \ldots \in C$ an element denoted by $\sum_{n=0}^{\infty} p^n c_n$ such that the following axioms hold:

$$\sum_{n=0}^{\infty} p^n(a_n+b_n) = \sum_{n=0}^{\infty} p^n a_n + \sum_{n=0}^{\infty} p^n b_n \quad \text{for all } a_n,\, b_n \in C \text{ (additivity)},$$

$$\sum_{n=0}^{\infty} p^n c_n = c_0 + pc_1 \quad \text{when } c_n = 0 \text{ for } n \geq 2 \text{ (unitality+compatibility)},$$

$$\sum_{i=0}^{\infty} p^i \sum_{j=0}^{\infty} p^j c_{ij} = \sum_{n=0}^{\infty} p^n \sum_{i+j=n}^{i,j\geq 0} c_{ij} \quad \text{for all } c_{ij} \in C \text{ (contraassociativity)}.$$

Example: Contramodules over the p-adic integers

A p-power infinite summation operation on a given abelian group C is always unique if it exists. In fact, the element $\sum_{n=0}^\infty p^n c_n = b_0$ can be recovered as the 0-th component of a unique solution of the infinite system of nonhomogeneous linear equations in C:

$$b_n = pb_{n+1} + c_n, \qquad n \geq 0.$$

On the other hand, the p-power infinite summation operation cannot be understood as any kind of limit of finite partial sums. In fact, one can construct a p-contramodule abelian group C with a sequence of elements $c_n \in C$ such that $p^n c_n = 0$ for all $n \geq 0$ but $\sum_{n=0}^\infty p^n c_n \neq 0$ in C.

A related fact is that p-contramodule abelian groups C are always p-adically complete, but they need not be p-adically separated: the natural map $C \longrightarrow \varprojlim_{n\geq 0} C/p^n C$ is surjective, but it need not be injective.

Locally presentable abelian categories with proj. generator

The idea of describing contramodule categories in terms of infinite summation operations has suprisingly general applicability. The whole concept of a locally presentable abelian category with enough projective objects can be interpreted as additive infinitary universal algebra.

A locally presentable abelian category has enough projectives if and only if it has a projective generator. Let B be a cocomplete abelian category with a projective generator $P \in \mathrm{B}$. For any set X, put $\mathbb{M}_P(X) = \mathrm{Hom}_{\mathrm{B}}(P, P^{(X)})$, where $P^{(X)}$ is the coproduct of X copies of P. Then the functor $\mathbb{M} = \mathbb{M}_P \colon \mathrm{Sets} \longrightarrow \mathrm{Sets}$ has a natural structure of a monad on the category of sets.

This means that there are morphisms of functors $\phi \colon \mathbb{M} \circ \mathbb{M} \longrightarrow \mathbb{M}$ and $\epsilon \colon \mathrm{Id}_{\mathrm{Sets}} \longrightarrow \mathbb{M}$ satisfying the associativity and unitality axioms. Given an monad $\mathbb{M}$ on Sets, an $\mathbb{M}$-module (usually called an "$\mathbb{M}$-algebra") is a set C endowed with a map of sets $\mu \colon \mathbb{M}(C) \longrightarrow C$ satisfying the associativity and unitality axioms (with respect to ϕ and ϵ).

Locally presentable abelian categories with proj. generator

Any cocomplete abelian category B with a projective generator P is naturally equivalent to the category of modules over the monad $\mathbb{M}_P$. The functor assigning to an object $B \in \mathrm{B}$ the set $C = \mathrm{Hom}_{\mathrm{B}}(P, B)$ with its natural $\mathbb{M}_P$-module structure provides the equivalence.

The theory of modules over monads on Sets is a categorical language for universal algebra. For any set X, elements of the set $\mathbb{M}(X)$ are interpreted as X-ary operations acting in all the $\mathbb{M}$-modules, while the monad multiplication morphism ϕ describes the law of composition for such operations.

The category of modules over a monad $\mathbb{M} \colon \mathrm{Sets} \longrightarrow \mathrm{Sets}$ is additive if and only if it is abelian. Then the monad $\mathbb{M}$ is called additive. This holds if and only if there are operations "$x + y$"$\in \mathbb{M}(\{x, y\})$, "$-x$"$\in \mathbb{M}(\{x\})$, and $0 \in \mathbb{M}(\varnothing)$ in the monad $\mathbb{M}$ (where x and y are formal symbols) satisfying suitable equations and commuting with all the other operations in $\mathbb{M}$.

Locally presentable abelian categories with proj. generator

The category of modules over a monad $\mathbb{M}\colon \mathrm{Sets} \longrightarrow \mathrm{Sets}$ is locally presentable if and only if the functor $\mathbb{M}$ is accessible, i.e., there exists a cardinal λ such that $\mathbb{M}$ preserves λ-filtered colimits in Sets.

The locally presentable abelian categories with a fixed projective generator are precisely the categories of modules over accessible additive monads on Sets.

Many (though not all) accessible additive monads on Sets arise from topological rings $\mathfrak{R}$ via the following construction, which involves infinite summation operations with the families of coeffients from $\mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$.

In fact, there are two abelian categories one can assign to a topological ring: the category of discrete modules and the category of contramodules. The former is a Grothendieck category, and the latter is locally presentable with a projective generator.

Example: Contramodules over a topological ring

Let R be a topological associative ring in which open right ideals form a base of neighborhoods of zero. (Such topological rings are called right linear.) A right R-module N is said to be discrete if for every element $x \in N$ the annihilator of x is an open right ideal in R. Equivalently, this means that the action map $N \times R \longrightarrow N$ is continuous in the given topology on R and the discrete topology on N. The full subcategory of discrete right R-modules $\mathrm{Discr}\text{-}R \subset \mathrm{Mod}\text{-}R$ is closed under submodules, quotients, and infinite direct sums, so $\mathrm{Discr}\text{-}R$ is a Grothendieck abelian category.

The completion of R is the topological ring $R^{\frown} = \varprojlim_{I \subset R} R/I$, where I ranges over the open right ideals in R. The topological ring R is said to be separated if the completion map $\lambda_R \colon R \longrightarrow R^{\frown}$ is injective, and complete if the map λ_R is surjective. The topological ring $R^{\frown}$ is always separated and complete, and λ_R is a continuous ring homomorphism. The categories of discrete right modules over R and $R^{\frown}$ are naturally equivalent.

Example: Contramodules over a topological ring

Let $\mathfrak{R}$ be a complete, separated topological ring in which open right ideals form a base of neighborhoods of zero. The monad $\mathbb{M}_{\mathfrak{R}}\colon \mathrm{Sets} \longrightarrow \mathrm{Sets}$ is constructed in the following way. Let X be a set. Then $\mathbb{M}_{\mathfrak{R}}(X) = \mathfrak{R}[[X]]$ is the set of all infinite formal linear combinations $\sum_{x\in X} r_x x$ of elements $x \in X$ with the coefficients $r_x \in \mathfrak{R}$ such that the family $(r_x)_{x\in X}$ converges to zero in the topology of $\mathfrak{R}$. The latter condition means that, for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$, the set $\{x \in X \mid r_x \notin \mathfrak{U}\}$ is finite.

The monad unit map $\epsilon_X\colon X \longrightarrow \mathfrak{R}[[X]]$ assigns to any element $x_0 \in X$ the formal linear combination $\sum_{x\in X} r_x x$ with $r_{x_0} = 1$ and $r_x = 0$ for $x \neq x_0$. The monad multiplication map $\phi_X\colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$ "opens the parentheses", assigning a formal linear combination to a formal linear combination of formal linear combinations. This uses the multiplication in $\mathfrak{R}$ and infinite sums of elements of $\mathfrak{R}$, understood as the topological limits of finite partial sums. The condition that open right ideals form a base of neighborhoods of zero guarantees the convergence.

Example: Contramodules over a topological ring

A left $\mathfrak{R}$-contramodule $\mathfrak{P}$ is a module over the monad $\mathbb{M}_{\mathfrak{R}} = \mathfrak{R}[[-]]$ on Sets. Explicitly, $\mathfrak{P}$ is a set endowed with a map of sets $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$, called the contraaction map, satisfying the following (contra)associativity and (contra)unitality equations. The two compositions

$$\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{P}]] \xrightarrow{\pi} \mathfrak{P},$$

where one of the two maps $\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{P}]]$ is $\phi_{\mathfrak{P}}$ and the other one induced by π, must be equal to each other; and the composition

$$\mathfrak{P} \xrightarrow{\epsilon_{\mathfrak{P}}} \mathfrak{R}[[\mathfrak{P}]] \xrightarrow{\pi} \mathfrak{P}$$

must be equal to $\mathrm{id}_{\mathfrak{P}}$.

The category of left $\mathfrak{R}$-contramodules $\mathfrak{R}\text{-}\mathrm{Contra}$ is abelian and locally presentable with enough projective objects. For a discrete ring R (when the only zero-convergent families of coefficients in R are the finite ones), one has $R\text{-}\mathrm{Contra} = R\text{-}\mathrm{Mod}$.

Example: Contramodules over a coring

Corings are a generalization of coalgebras. Let A be an associative ring. A coring $\mathcal{C}$ over A is an A-A-bimodule endowed with comultiplication map $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$ and a counit map $\epsilon\colon \mathcal{C} \longrightarrow A$. Both μ and ϵ must be A-A-bimodule morphisms, and they must satisfy the coassociativity and counitality equations.

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a left A-module endowed with a coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M}$, which must be a left A-module morphism satisfying the coassociativity and counitality equations.

A left $\mathcal{C}$-contramodule $\mathfrak{P}$ is a left A-module endowed with a contraaction map $\pi\colon \operatorname{Hom}_A(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$, which must be a left A-module morphism satisfying the contraassociativity and contraunitality equations:

$$\operatorname{Hom}_A(\mathcal{C}, \operatorname{Hom}_A(\mathcal{C}, \mathfrak{P})) \simeq \operatorname{Hom}_A(\mathcal{C} \otimes_A \mathcal{C}, \mathfrak{P}) \rightrightarrows \operatorname{Hom}_A(\mathcal{C}, \mathfrak{P}) \xrightarrow{\pi} \mathfrak{P},$$

$$\mathfrak{P} \xrightarrow{\epsilon^*} \operatorname{Hom}_A(\mathcal{C}, \mathfrak{P}) \xrightarrow{\pi} \mathfrak{P}.$$

Example: Contramodules over a coring

If $\mathcal{C}$ is a flat right A-module, then the category of left $\mathcal{C}$-comodules $\mathcal{C}\text{-}\mathrm{Comod}$ is a Grothendieck abelian category. If $\mathcal{C}$ is a projective left A-module, then the category of left $\mathcal{C}$-contramodules $\mathcal{C}\text{-}\mathrm{Contra}$ is a locally presentable abelian category with enough projective objects. The direct summands of the left $\mathcal{C}$-contramodules $\mathrm{Hom}_A(\mathcal{C}, F)$, where F ranges over the free/projective left A-modules, are the projective objects in $\mathcal{C}\text{-}\mathrm{Contra}$.

Theorem

Assume that the left global dimension of A is finite.

(a) *If $\mathcal{C}$ is a flat right A-module, then* $\mathrm{D}^{\mathrm{bco}}(\mathcal{C}\text{-}\mathrm{Comod}) = \mathrm{D}^{\mathrm{pco}}(\mathcal{C}\text{-}\mathrm{Comod})$.

(b) *If $\mathcal{C}$ is a projective left A-module, then* $\mathrm{D}^{\mathrm{bctr}}(\mathcal{C}\text{-}\mathrm{Contra}) = \mathrm{D}^{\mathrm{pctr}}(\mathcal{C}\text{-}\mathrm{Contra})$.

(c) *If $\mathcal{C}$ is a projective left and a flat right A-module, then there is a natural triangulated equivalence*

$$\mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-}\mathrm{Comod}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-}\mathrm{Contra}).$$

Example: Contramodules over a coring

Open problem: can one get rid of the global dimension assumption in the above theorem? Some particular cases are known, e.g., when A is Gorenstein.

Presumably, a general solution to this problem would involve defining semiderived categories of $\mathcal{C}$-comodules and $\mathcal{C}$-contramodules, $\mathrm{D^{si}}(\mathcal{C}\text{-Comod})$ and $\mathrm{D^{si}}(\mathcal{C}\text{-Contra})$.
A semiderived category is a quotient category of the homotopy category which behaves as the derived category along a half of the variables and as a coderived or contraderived category along the other half.

In the context of the problem at hand, one would want a semiderived category which is a mixture of the derived category of A-modules and the co/contraderived category in the direction of $\mathcal{C}$ relative to A. So one wants to have
$\mathrm{D^{si}}(\mathcal{C}\text{-Comod}) = \mathrm{D}(A\text{-Mod}) = \mathrm{D^{si}}(\mathcal{C}\text{-Contra})$ when $\mathcal{C} = A$,
while on the other hand $\mathrm{D^{si}}(\mathcal{C}\text{-Comod}) = \mathrm{D^{co}}(\mathcal{C}\text{-Comod})$ and $\mathrm{D^{si}}(\mathcal{C}\text{-Contra}) = \mathrm{D^{ctr}}(\mathcal{C}\text{-Contra})$ when A is a field.

Examples: Corings and Semialgebras

Then one might hope to have a triangulated equivalence $\mathrm{D^{si}}(\mathcal{C}\text{-}\mathrm{Comod}) \simeq \mathrm{D^{si}}(\mathcal{C}\text{-}\mathrm{Contra})$ for any coring $\mathcal{C}$ (assuming only $\mathcal{C}$ is a projective left and a flat right A-module), which would reduce to the identity equivalence $\mathrm{D}(A\text{-}\mathrm{Mod}) = \mathrm{D}(A\text{-}\mathrm{Mod})$ for $\mathcal{C} = A$ and to the well-known equivalence $\mathrm{D^{co}}(\mathcal{C}\text{-}\mathrm{Comod}) = \mathrm{D^{ctr}}(\mathcal{C}\text{-}\mathrm{Contra})$ for a coalgebra $\mathcal{C}$ over a field.

This open problem appears to be unsolvable, but its analogues have solutions in other contexts.

A semialgebra $\mathcal{S}$ is an associative algebra (or "monoid") object in the tensor category of bicomodules over a coalgebra $\mathcal{C}$ over k, with respect to the cotensor product. So one can say that a coring is a coalgebra over a ring/algebra, while a semialgebra is an algebra over a coalgebra. There are two kinds of module categories over semialgebras: the semimodules and the semicontramodules.

Example: Semialgebras

The category of left $\mathcal{S}$-semimodules $\mathcal{S}\text{-}\mathrm{Simod}$ is a Grothendieck abelian category whenever $\mathcal{S}$ is an injective right $\mathcal{C}$-comodule. The category of left $\mathcal{S}$-semicontramodules $\mathcal{S}\text{-}\mathrm{Sicntr}$ is a locally presentable abelian category with enough projective objects whenever $\mathcal{S}$ is an injective left $\mathcal{C}$-comodule.

For semimodules and semicontramodules, the semiderived categories can be defined, meaning mixtures of the co/contraderived category along the variables from $\mathcal{C}$ and the derived category in the direction of $\mathcal{S}$ relative to $\mathcal{C}$. Assuming that $\mathcal{S}$ is an injective left and right $\mathcal{C}$-comodule, one has a natural triangulated equivalence

$$\mathrm{D}^{\mathrm{si}}(\mathcal{S}\text{-}\mathrm{Simod}) \simeq \mathrm{D}^{\mathrm{si}}(\mathcal{S}\text{-}\mathrm{Sicntr}),$$

called the derived semimodule-semicontramodule correspondence.

Semiderived categories

What is the semiderived category? Before answering this question, a more basic question is worth being asked and answered: What is the derived category?

Let R be an associative algebra over a field k. The derived category $\mathrm{D}(R\text{-}\mathrm{Mod})$ is defined by inverting the quasi-isomorphisms in the homotopy category $\mathrm{K}(R\text{-}\mathrm{Mod})$. What is a quasi-isomorphism?

I suggest the following answer: consider the forgetful functor $\mathrm{K}(R\text{-}\mathrm{Mod}) \longrightarrow \mathrm{K}(k\text{-}\mathrm{Vect})$. Let $f\colon C^\bullet \longrightarrow D^\bullet$ be a morphism of complexes of R-modules. The f is a quasi-isomorphism if and only if, viewed as a morphism of complexes of vector spaces, it is a homotopy equivalence.

So the conventional notion of quasi-isomorphism (hence the derived category) presumes a forgetful functor: forget the module structures, stay with the abelian groups/vector spaces, and see if the morphism is a quasi-isomorphism on this level.

Now it is clear how the semiderived category can be defined. To avoid going into details on semimodules, consider the example of a ring homomorphism.

Let $A \longrightarrow R$ be a morphism of associative rings. Then the R/A-semicoderived category of R-modules $\mathrm{D}^{\mathrm{sico}}_A(R\text{-}\mathrm{Mod})$ is defined as the quotient category of $\mathrm{K}(R\text{-}\mathrm{Mod})$ by the thick subcategory of complexes that are coacyclic as complexes of A-modules.

Similarly, the R/A-semicontraderived category of R-modules $\mathrm{D}^{\mathrm{sictr}}_A(R\text{-}\mathrm{Mod})$ is the quotient category of $\mathrm{K}(R\text{-}\mathrm{Mod})$ by the thick subcategory of complexes that are contraaacyclic as complexes of A-modules.

Here one can use co/contraacyclicity in the sense of Becker or Positselski, as one prefers. The semicoderived category is a mixture of the coderived category along A and the derived category in the direction of R relative to A (and similarly for the semicontraderived category).

Example from Semi-Infinite Algebraic Geometry

A complete, separated topological k-vector space V is said to be locally linearly compact if it has has an open subspace $U \subset V$ which is linearly compact, i.e., open subspaces of finite codimension form a base of neighborhoods of zero in U. Equivalently, U is a projective limit of finite-dimensional vector spaces, endowed with the projective limit topology.

Consider the following example. Let $k((t))$ be the topological field of formal Laurent power series in a variable t with the coefficients in k, with the power series topology. We will consider $k((t))$ not as a field, but just as a topological k-vector space. Then $k((t))$ is a locally linearly compact k-vector space. For example, the subspace of formal Taylor power series $k[[t]] \subset k((t))$ is a linearly compact open subspace.

Example from Semi-Infinite Algebraic Geometry

A typical vector in $k((t))$ has the form $\sum_{n\in\mathbb{Z}} x_n t^n$, where $x_n \in k$ for all $n \in \mathbb{Z}$ and $x_n = 0$ for $n \ll 0$. The coefficients x_n, $n \in \mathbb{Z}$ can be viewed as the coordinates on $k((t))$. They are continuous linear functions $x_n\colon k((t)) \longrightarrow k$.

Let us construct the topological ring of continuous polynomial functions $k((t)) \longrightarrow k$. First of all, the ring of continuous polynomial functions $k[[t]] \longrightarrow k$ is simply the discrete ring of polynomials in countably many variables $S = k[x_0, x_1, x_2, \dots]$.

The topological ring of continuous polynomial functions on $k((t))$ is the projective limit

$$\mathfrak{R} = \varprojlim\nolimits_{n\geq 0} k[x_{-n}, x_{-n+1}, \dots, x_0, x_1, x_2, \dots],$$

where the transition map $k[x_{-n-1}, x_{-n}, x_{-n+1}, \dots] \longrightarrow k[x_{-n}, x_{-n+1}, \dots]$ takes x_{-n-1} to 0. The topology on $\mathfrak{R}$ is the topology of projective limit of the discrete rings $k[x_{-n}, x_{-n+1}, \dots]$.

Example from Semi-Infinite Algebraic Geometry

A discrete $\mathfrak{R}$-module M is the same thing as a module over the polynomial ring $R = k[\dots, x_{-2}, x_{-2}, x_0, x_1, \dots]$ such that for every element $b \in M$ there exists $m \geq 0$ for which $bx_{-n} = 0$ for all $n > m$. We are going to define the semiderived category of discrete $\mathfrak{R}$-modules $\mathrm{D}^{\mathrm{si}}(\mathrm{Discr}\text{-}\mathfrak{R})$.

For this purpose, one needs to choose a linearly compact open subspace $U \subset k((t))$; surprisingly, however, the resulting equivalence relation on complexes of discrete $\mathfrak{R}$-modules does not depend on this choice. Take $U = k[[t]] \subset k((t))$. Consider the topological ring $\mathfrak{A}$ of polynomial functions on the discrete vector space $k((t))/k[[t]]$, defined as the projective limit

$$\mathfrak{A} = \varprojlim\nolimits_{n \geq 0} k[x_{-n}, x_{-n+1}, \dots, x_{-1}].$$

Then $\mathfrak{A}$ is a closed subring in $\mathfrak{R}$, so there is a natural continuous ring homomorphism $\mathfrak{A} \longrightarrow \mathfrak{R}$.

Example from Semi-Infinite Algebraic Geometry

The semiderived category $\mathrm{D^{si}}(\mathrm{Discr}\text{-}\mathfrak{R})$ is defined as the quotient category of $\mathrm{K}(\mathrm{Discr}\text{-}\mathfrak{R})$ by the thick subcategory of all complexes that are coacyclic as complexes in $\mathrm{Discr}\text{-}\mathfrak{A}$. (Since $\mathfrak{A}$ is a projective limit of Noetherian rings, there is no difference between Becker's and Positselski's coacyclicity here.)

Usually the R/A-semiderived category depends on a ring homomorphism $A \longrightarrow R$, or on a subring $A \subset R$, but in this particular example there is a surprising invariance property. It is claimed that the semiderived category $\mathrm{D^{si}}(\mathrm{Discr}\text{-}\mathfrak{R})$ is preserved by all continuous linear coordinate changes on the topological vector space $V = k((t))$. For example, if one takes some $m \in \mathbb{Z}$ and $U = t^m k[[t]] \subset k((t))$, i.e., considers

$$\mathfrak{B} = \varprojlim\nolimits_{n\geq 0} k[x_{-n}, x_{-n+1}, \ldots, x_{m-1}],$$

then the $\mathfrak{R}/\mathfrak{A}$-semiderived category of discrete $\mathfrak{R}$-modules coincides with $\mathfrak{R}/\mathfrak{B}$-semiderived category.

Example from Semi-Infinite Algebraic Geometry

Consequently, the infinite-dimensional general linear group $\mathsf{GL}_k^{\mathrm{cont}}(V) = \mathsf{Aut}_k^{\mathrm{cont}}(V)$, that is the group of all continuous bijective k-linear maps $k((t)) \longrightarrow k((t))$ with continuous inverse maps (a kind of $\mathbb{Z} \times \mathbb{Z}$-indexed infinite matrix group), acts in the semiderived category $\mathrm{D^{si}}(\mathrm{Discr}\text{-}\mathfrak{R})$.

One can also define an associative, commutative, and unital tensor category structure on $\mathrm{D^{si}}(\mathrm{Discr}\text{-}\mathfrak{R})$ with an almost the same (slightly more complicated) invariance property with respect to continuous linear coordinate changes. This operation is called the semitensor product of complexes of discrete $\mathfrak{R}$-modules. The semitensor product is a mixture of some kind of cotensor product along the variables from $\mathfrak{A}$ and the derived tensor product along the remaining variables in $\mathfrak{R}$. It produces doubly unbounded complexes from bounded ones (so it is a kind of semi-infinite homology theory). The unit object of this tensor structure is an acyclic complex!

Pseudo-derived categories

"Pseudo-derived categories" is a common name for pseudo-coderived and pseudo-contraderived categories. Let us discuss the pseudo-coderived categories, as the pseudo-contraderived categories are dual.

Let A be an exact category with exact functors of infinite coproduct. A pseudo-coderived category of A is an intermediate triangulated quotient category between the derived category $\mathrm{D}(A)$ and the (totalization) coderived category $\mathrm{D}^{\mathrm{pco}}(\mathrm{A})$. So D is pseudo-coderived category of A if triangulated Verdier quotient functors $\mathrm{D}^{\mathrm{pco}}(\mathrm{A}) \twoheadrightarrow \mathrm{D} \twoheadrightarrow \mathrm{D}(A)$ are given. In particular, for a Grothendieck abelian category A, Becker's coderived category $\mathrm{D}^{\mathrm{bco}}(\mathrm{A})$ is "a pseudo-coderived category" in this sense.

Pseudo-derived categories appear in $n = \infty$ tilting theory (infinitely generated Wakamatsu tilting theory) and in connection with what people call semidualizing complexes of modules or bimodules (I prefer to call them pseudo-dualizing complexes).

Pseudo-coderived categories

The following construction allows to produce pseudo-coderived categories. A full subcategory $\mathsf{E} \subset \mathsf{A}$ is said to be coresolving if

- E is closed under extensions in A;
- E is closed under the cokernels of admissible monomorphisms in A; and
- for any object $A \in \mathsf{A}$ there exist an object $E \in \mathsf{E}$ and an admissible monomorphism $A \longrightarrow E$ in A.

As $\mathsf{E} \subset \mathsf{A}$ is closed under extensions, E inherits an exact category structure from A.

Theorem

Let A be an exact category with exact functors of infinite coproduct, and let $\mathsf{E} \subset \mathsf{A}$ be a coresolving subcategory closed under coproducts. Then the inclusion $\mathsf{E} \longrightarrow \mathsf{A}$ induces a triangulated equivalence $\mathsf{D}^{\mathrm{pco}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathrm{pco}}(\mathsf{A})$.

Pseudo-coderived categories

Consider the commutative diagram of triangulated functors

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{pco}}(\mathrm{E}) & = & \mathrm{D}^{\mathrm{pco}}(\mathrm{A}) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ \mathrm{D}(\mathrm{E}) & \longrightarrow & \mathrm{D}(\mathrm{A}) \end{array}$$

Since $\mathrm{D}^{\mathrm{pco}}(\mathrm{E}) \longrightarrow \mathrm{D}(E)$ and $\mathrm{D}^{\mathrm{pco}}(\mathrm{E}) \longrightarrow \mathrm{D}(A)$ are Verdier quotient functors, it follows that $\mathrm{D}(E) \longrightarrow \mathrm{D}(A)$ is a Verdier quotient functor, too:

$$\mathrm{D}(\mathrm{E}) \twoheadrightarrow \mathrm{D}(\mathrm{A}).$$

Thus $\mathrm{D}(\mathrm{E})$ is a pseudo-coderived category of A.

H. Becker. Models for singularity categories. *Advances in Math.* **254**, p. 187–232, 2014. `arXiv:1205.4473 [math.CT]`

P. Jørgensen. The homotopy category of complexes of projective modules. *Advances in Math.* **193**, #1, p. 223–232, 2005. `arXiv:math.RA/0312088`

B. Keller, W. Lowen, P. Nicolás. On the (non)vanishing of some "derived" categories of curved dg algebras. *Journ. Pure Appl. Algebra* **214**, #7, p. 1271–1284, 2010. `arXiv:0905.3845 [math.KT]`

H. Krause. The stable derived category of a Noetherian scheme. *Compositio Math.* **141**, #5, p. 1128–1162, 2005. `arXiv:math.AG/0403526`

L. Positselski. Homological algebra of semimodules and semicontramodules: Semi-infinite homological algebra of associative algebraic structures. Appendix C in collaboration with D. Rumynin; Appendix D in collaboration with S. Arkhipov. Monografie Matematyczne vol. 70, Birkhäuser/Springer Basel, 2010. xxiv+349 pp. arXiv:0708.3398 [math.CT]

L. Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Memoirs of the American Math. Society* **212**, #996, 2011. vi+133 pp. arXiv:0905.2621 [math.CT]

L. Positselski. Contraherent cosheaves. Electronic preprint arXiv:1209.2995 [math.CT].

L. Positselski. Contramodules. Electronic preprint arXiv:1503.00991 [math.CT].

L. Positselski. Coherent rings, fp-injective modules, dualizing complexes, and covariant Serre–Grothendieck duality. *Selecta Math. (New Ser.)* **23**, #2, p. 1279–1307, 2017. arXiv:1504.00700 [math.CT]

L. Positselski. Pseudo-dualizing complexes and pseudo-derived categories. *Rendiconti Seminario Matematico Univ. Padova* **143**, p. 153–225, 2020. arXiv:1703.04266 [math.CT]

L. Positselski. Abelian right perpendicular subcategories in module categories. Electronic preprint arXiv:1705.04960 [math.CT].

L. Positselski. Pseudo-dualizing complexes of bicomodules and pairs of t-structures. Electronic preprint arXiv:1907.03364 [math.CT].

L. Positselski. Quasi-coherent torsion sheaves, the semiderived category, and the semitensor product. Electronic preprint arXiv:2104.05517 [math.AG].

L. Positselski, J. Šťovíček. The tilting-cotilting correspondence. *Internat. Math. Research Notices* **2021**, #1, p. 189–274, 2021. arXiv:1710.02230 [math.CT]

L. Positselski, J. Šťovíček. ∞-tilting theory. *Pacific Journ. of Math.* **301**, #1, p. 297–334, 2019. arXiv:1711.06169 [math.CT]

L. Positselski, J. Šťovíček. Derived, coderived, and contraderived categories of locally presentable abelian categories. Electronic preprint arXiv:2101.10797 [math.CT].