Exotic derived categories

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The homotopy category of DG-modules $\mathrm{K}(A\text{-}\mathrm{Mod})$

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A curved DG-ring (B, d_B, h_B)

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What can one do with this homotopy category?

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The upside of Becker's approach to coderived and contraderived categories is that powerful set-theoretic methods, such as the small object argument, can be used for their study. The downside of Becker's approach is that it requires existence of enough injective or projective objects.

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So any two DG-algebras over k can be connected by a chain of transformations some of which are quasi-isomorphisms and others are CDG-algebra isomorphisms.

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Notice that any exact category with infinite coproducts and enough injectives has exact coproduct functors, and any exact category with infinite products and enough projectives has exact product functors. In this sense, having exact coproducts is a weaker condition than having enough injectives (which makes the totalization coderived category a more generally applicable construction than the injective coderived category).

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The category of $\mathcal{C}$-contramodules $\mathcal{C}\text{-}\mathrm{Contra}$ is a locally presentable abelian category with enough projective objects. The projective $\mathcal{C}$-contramodules are the direct summands of the free $\mathcal{C}$-contramodules $\mathrm{Hom}_k(\mathcal{C}, V)$. In particular, any product of projective $\mathcal{C}$-contramodules is projective.

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All these triangulated equivalences remain valid for curved DG-coalgebras over fields.

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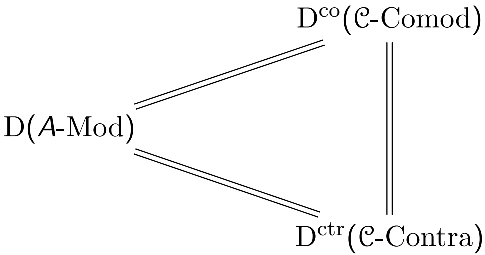

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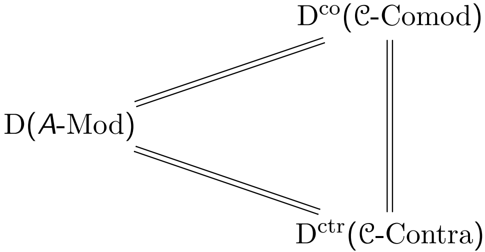

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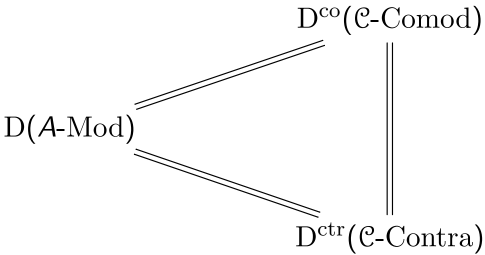

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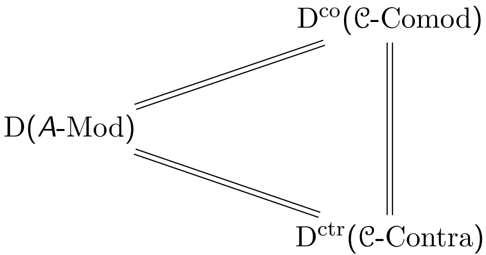

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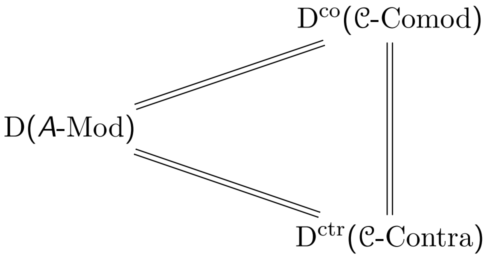

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The vertical equivalence is the derived co-contra correspondence.

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A p-power infinite summation operation on a given abelian group C

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Locally presentable abelian categories with proj. generator

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The idea of describing contramodule categories in terms of infinite summation operations

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Locally presentable abelian categories with proj. generator

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The locally presentable abelian categories with a fixed projective generator

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Let $\mathfrak{R}$ be a complete, separated topological ring in which open right ideals form a base of neighborhoods of zero. The monad $\mathbb{M}_{\mathfrak{R}} \colon \mathrm{Sets} \longrightarrow \mathrm{Sets}$ is constructed in the following way. Let X be a set. Then $\mathbb{M}_{\mathfrak{R}}(X) = \mathfrak{R}[[X]]$ is the set of all infinite formal linear combinations $\sum_{x \in X} r_x x$ of elements $x \in X$ with the coefficients $r_x \in \mathfrak{R}$ such that the family $(r_x)_{x \in X}$ converges to zero in the topology of $\mathfrak{R}$. The latter condition means that, for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$, the set $\{x \in X \mid r_x \notin \mathfrak{U}\}$ is finite.

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The category of left $\mathfrak{R}$-contramodules $\mathfrak{R}\text{-}\mathrm{Contra}$ is abelian and locally presentable with enough projective objects. For a discrete ring R (when the only zero-convergent families of coefficients in R are the finite ones), one has $R\text{-}\mathrm{Contra} = R\text{-}\mathrm{Mod}$.

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Theorem

Example: Contramodules over a coring

If $\mathcal{C}$ is a flat right A-module, then the category of left $\mathcal{C}$-comodules $\mathcal{C}\text{-}\mathrm{Comod}$ is a Grothendieck abelian category. If $\mathcal{C}$ is a projective left A-module, then the category of left $\mathcal{C}$-contramodules $\mathcal{C}\text{-}\mathrm{Contra}$ is a locally presentable abelian category with enough projective objects. The direct summands of the left $\mathcal{C}$-contramodules $\mathrm{Hom}_A(\mathcal{C}, F)$, where F ranges over the free/projective left A-modules, are the projective objects in $\mathcal{C}\text{-}\mathrm{Contra}$.

Theorem

Assume that the left global dimension of A is finite.

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A discrete $\mathfrak{R}$-module M

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Thus $\mathrm{D(E)}$ is a pseudo-coderived category of A.

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