

∞ -tilting theory

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The title of this talk could be

Doubly infinite tilting theory

or

Infinitely generated Wakamatsu tilting theory.

This means the theory of infinitely generated (co)tilting modules/objects of infinite projective and injective dimension (i. e., infinitely generated n -tilting theory for $n = \infty$).

Joint work with Jan Št'ovíček:

L. Positselski, J. Št'ovíček. The tilting-cotilting correspondence. arXiv:1710.02230

L. Positselski, J. Št'ovíček. ∞ -tilting theory. arXiv:1711.06169

How many abelian categories are there?

More than one would expect.

Associative ring $R \mapsto$ abelian category of R -modules $R\text{-Mod}$.
Has a projective generator and an injective cogenerator.

How does one produce abelian categories having only a projective generator, or only an injective cogenerator?

Let \mathcal{C} be an additive category and $M \in \mathcal{C}$ an object.
How does one produce an abelian category out of this datum?

Naïve approach: $(\mathcal{C}, M) \mapsto S\text{-Mod}$, where $S = \text{End}_{\mathcal{C}}(M)^{\text{op}}$

(The notation means: S acts in $M \in \mathcal{C}$ on the right, and we consider the category of left S -modules).

How many abelian categories are there?

Two dual, less naïve approaches:

Suppose coproducts exist in \mathcal{C} , and denote by $\text{Add}_{\mathcal{C}}(M) \subset \mathcal{C}$ the full subcategory formed by the direct summands of coproducts of copies of M .

Claim: There exists a unique abelian category \mathcal{B} with enough projectives such that $\mathcal{B}_{\text{proj}} \simeq \text{Add}_{\mathcal{C}}(M)$.

Suppose products exist in \mathcal{C} , and denote by $\text{Prod}_{\mathcal{C}}(M) \subset \mathcal{C}$ the full subcategory formed by the direct summands of products of copies of M .

Claim*: There exists a unique abelian category \mathcal{A} with enough injectives such that $\mathcal{A}_{\text{inj}} \simeq \text{Prod}_{\mathcal{C}}(M)$.

How many abelian categories are there?

Construction 1: \mathcal{B} is the category of all coherent (finitely presented) functors $\text{Add}_{\mathcal{C}}(M)^{\text{op}} \rightarrow \text{Ab}$, which is abelian because $\text{Add}_{\mathcal{C}}(M)$ has weak kernels.

Construction 2: Let $\mathbb{T} : \text{Sets} \rightarrow \text{Sets}$ be the functor taking a set X to the set $\text{Hom}_{\mathcal{C}}(M, M^{(X)})$. Then the functor \mathbb{T} is a monad on the category of sets. One can construct \mathcal{B} as the category of all algebras/modules over this monad.

Remark: Generally, the category of algebras/modules over a monad $\mathbb{T} : \text{Sets} \rightarrow \text{Sets}$ is abelian if and only if it is additive. A monad \mathbb{T} is called **additive** if this is the case. Cocomplete abelian categories \mathcal{B} with a fixed projective generator P correspond bijectively to additive monads on the category of sets.

How many abelian categories are there?

Example 1. Let $\mathcal{C} = R\text{-Mod}$, R an associative ring. Then \mathcal{B} is the category of contra**modules** over the topological ring $\mathfrak{G} = \text{End}_R(M)^{\text{op}}$.

The topology on \mathfrak{G} : a base of neighborhoods of zero is formed by the annihilators of finitely-generated submodules in M .

$\mathcal{B} = \mathfrak{G}\text{-Contra}$ is the category of left \mathfrak{G} -modules endowed with the operations of infinite summation with the coefficients — zero-converging families of elements in \mathfrak{G} .

Example 2. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}[p^{-1}]/\mathbb{Z}$. Then $\mathfrak{G} = \mathbb{Z}_p$ is the topological ring of p -adic integers.

$\mathcal{B} = \mathbb{Z}_p\text{-Contra}$ is equivalent to the full subcategory in Ab consisting of all the abelian groups B such that $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[p^{-1}], B) = 0 = \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}[p^{-1}], B)$.

How many abelian categories are there?

Example 3. Let $\mathcal{C} = R\text{-Mod}$, M a pure-injective R -module. Then \mathcal{A} is a Grothendieck abelian category.

[Herzog–Št'ovíček, 2014.]

Notation and terminology: $\mathcal{B} = \sigma_M(\mathcal{C})$ and $\mathcal{A} = \pi_M(\mathcal{C})$.

Assuming that $M \in \mathcal{C}$ is a “tilting object” in one sense or another, one can call \mathcal{B} the abelian category **tilted** from \mathcal{C} at M .

Assuming that $M \in \mathcal{C}$ is a “cotilting object” in some sense, one can call \mathcal{A} the abelian category **cotilted** from \mathcal{C} at M .

When are σ and π two inverse operations?

Let \mathcal{A} be an abelian category with products and an injective cogenerator $J \in \mathcal{A}$. Then coproducts exist and are exact in \mathcal{A} .

An object $T \in \mathcal{A}$ is called **weakly tilting** if $\text{Ext}_{\mathcal{A}}^i(T, T^{(X)}) = 0$ for all sets X and $i > 0$.

Let $T \in \mathcal{A}$ be a weakly tilting object. The full subcategory $\mathcal{E}_{\max}(T) \subset \mathcal{A}$ consists of all the objects $E \in \mathcal{A}$ such that

- $\text{Ext}_{\mathcal{A}}^i(T, E) = 0$ for all $i > 0$;
- there exists an exact sequence

$$\dots \longrightarrow T^{(X_2)} \longrightarrow T^{(X_1)} \longrightarrow T^{(X_0)} \longrightarrow E \longrightarrow 0$$

in \mathcal{A} , which remains exact after applying the functor $\text{Hom}_{\mathcal{A}}(T, -)$.

When are σ and π two inverse operations?

Lemma

The full subcategory $E_{\max}(T) \subset A$ is closed under

- *extensions,*
- *cokernels of monomorphisms,*
- *kernels of those epimorphisms that remain epimorphisms after applying $\text{Hom}_A(T, -)$,*
- *and direct summands.*

The object $T \in A$ is called ∞ -tilting (or Wakamatsu tilting) if $A_{\text{inj}} \subset E_{\max}(T)$.

When are σ and π two inverse operations?

The dual definition:

Let B be an abelian category with coproducts and a projective generator $P \in B$. Then products exist and are exact in B .

An object $W \in B$ is called **weakly cotilting** if $\text{Ext}_B^i(W^X, W) = 0$ for all sets X and $i > 0$.

Let $W \in B$ be a weakly cotilting object. The full subcategory $F_{\max}(W) \subset B$ is constructed in the way dual to the above construction of the full subcategory $E_{\max}(T) \subset A$.

The object $W \in B$ is called **∞ -cotilting** (or **Wakamatsu cotilting**) if $B_{\text{proj}} \subset F_{\max}(W)$.

Tilting-cotilting correspondence

From now on, all our abelian categories will be complete and cocomplete. The abelian category \mathcal{A} is assumed to have an injective cogenerator, and the abelian category \mathcal{B} is assumed to have a projective generator.

Theorem

There is a one-to-one correspondence between abelian categories \mathcal{A} with an injective cogenerator J and an ∞ -tilting object $T \in \mathcal{A}$, and abelian categories \mathcal{B} with a projective generator P and an ∞ -cotilting object $W \in \mathcal{B}$. The correspondence assigns to a category \mathcal{A} the category $\mathcal{B} = \sigma_T(\mathcal{A})$, and to a category \mathcal{B} the category $\mathcal{A} = \pi_W(\mathcal{B})$.

Tilting-cotilting correspondence: the construction

Let \mathcal{A} be an abelian category with an injective cogenerator J and an ∞ -tilting object $T \in \mathcal{A}$.

Put $\mathcal{B} = \sigma_T(\mathcal{A})$, so $\text{Add}_{\mathcal{A}}(T) \simeq \mathcal{B}_{\text{proj}}$. Let $P \in \mathcal{B}_{\text{proj}}$ be the object corresponding to $T \in \text{Add}(T)$; then P is a projective generator of the abelian category \mathcal{B} .

The embedding functor $\mathcal{B}_{\text{proj}} \simeq \text{Add}_{\mathcal{A}}(T) \longrightarrow \mathcal{A}$ can be uniquely extended to a right exact functor $\Phi: \mathcal{B} \longrightarrow \mathcal{A}$.

The embedding functor $\text{Add}_{\mathcal{A}}(T) \simeq \mathcal{B}_{\text{proj}} \longrightarrow \mathcal{B}$ can be extended to a left exact functor $\Psi: \mathcal{A} \longrightarrow \mathcal{B}$ right adjoint to Φ .

Set $W = \Psi(J) \in \mathcal{B}$. □

Tilting-cotilting correspondence

The functors Φ and Ψ restrict to an equivalence of exact categories

$$\Psi: E_{\max}(T) \simeq F_{\max}(W) : \Phi.$$

If the projective dimension $\text{pd}_A(T)$ and the injective dimension $\text{id}_B(W)$ are **both** finite, then they agree, $\text{pd}_A(T) = n = \text{id}_B(W)$.

In this case, the object $T \in A$ is called ***n*-tilting**, and the object $W \in B$ is called ***n*-cotilting**.

Tilting-cotilting correspondence

An ∞ -tilting pair (T, \mathbb{E}) in A consists of an object $T \in A$ and a full subcategory $\mathbb{E} \subset A$ such that

- $A_{\text{inj}} \subset \mathbb{E}$;
- $\text{Add}_A(T) \subset \mathbb{E}$;
- $\text{Ext}_A^i(T, E) = 0$ for all $E \in \mathbb{E}$ and $i > 0$;
- \mathbb{E} is closed under the cokernels of monomorphisms and extensions in A ;
- every $\text{Add}_A(T)$ -precover of an object of \mathbb{E} is an epimorphism in A with the kernel belonging to \mathbb{E} .

An object $T \in A$ is ∞ -tilting if and only if it is a part of some ∞ -tilting pair (T, \mathbb{E}) in A .

In this case, all such ∞ -tilting pairs (T, \mathbb{E}) with the fixed object T form a complete lattice with respect to inclusion of the full subcategories $\mathbb{E} \subset A$.

Tilting-cotilting correspondence

In particular, $(T, E_{\max}(T))$, with the full subcategory $E_{\max}(T) \subset A$ constructed above, is the **maximal** ∞ -tilting pair for an ∞ -tilting object $T \in A$.

There is also the **minimal** ∞ -tilting pair $(T, E_{\min}(T))$.

This means that, for any ∞ -tilting pair (T, E) , one has $E_{\min}(T) \subset E \subset E_{\max}(T)$.

Tilting-cotilting correspondence

The definition of an ∞ -cotilting pair (W, F) in B is dual to that of an ∞ -tilting pair.

There is a one-to-one correspondence between ∞ -tilting pairs (T, E) in abelian categories A and ∞ -cotilting pairs (W, F) in abelian categories B provided by the rules

- $B = \sigma_T(A)$, and conversely, $A = \pi_W(B)$;
- $P = \Psi(T)$ and $W = \Psi(J)$, and conversely, $J = \Phi(W)$ and $T = \Phi(P)$;
- $F = \Psi(E)$, and conversely, $E = \Phi(F)$.

∞ -tilting derived equivalence

The equivalence of derived categories $D(E) \simeq D(F)$ induced by the equivalence of exact categories

$$\Psi: E \simeq F : \Phi$$

can be thought of as an equivalence of exotic derived categories of the abelian categories A and B .

When T is n -tilting and W is n -cotilting, one has

$$D(A) \simeq D(E_{\max}) \simeq D(F_{\max}) \simeq D(B).$$

In the ∞ -tilting (Wakamatsu) situation, it helps to assume that E is closed under coproducts in A and F is closed under products in B . But we start without this assumption.

(Co)tilting t-structures

Let $D^{\geq 0}(E) \subset D(E)$ be the full subcategory consisting of all the complexes $0 \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$ with $E^i \in E$.

Let $D_A^{\leq 0}(E) \subset D(E)$ be the full subcategory consisting of all the complexes E^\bullet with $E^i \in E$ such that $H_A^i(E^\bullet) = 0$ for $i > 0$.

Then $(D_A^{\leq 0}(E), D^{\geq 0}(E))$ is a t-structure on $D(E)$ with the heart A .

Dually one constructs the t-structure $(D^{\leq 0}(F), D_B^{\geq 0}(F))$ on $D(F)$ with the heart B .

Thus we have two t-structures on the triangulated category $D(E) = D(F)$ with the hearts A and B .

Coderived and Contraderived Categories

Let \mathbf{E} be an exact category with exact coproducts. Then the **coderived category** $D^{\text{co}}(\mathbf{E})$ is the triangulated quotient category

$$D^{\text{co}}(\mathbf{E}) = \text{Hot}(\mathbf{E}) / \text{Acycl}^{\text{co}}(\mathbf{E}),$$

where $\text{Acycl}^{\text{co}}(\mathbf{E})$ is the minimal triangulated subcategory in $\text{Hot}(\mathbf{E})$ containing the totalizations of short exact sequences of complexes in \mathbf{E} and closed under coproducts.

Let \mathbf{F} be an exact category with exact products. Then the **contraderived category** $D^{\text{ctr}}(\mathbf{F})$ is the triangulated quotient category

$$D^{\text{ctr}}(\mathbf{F}) = \text{Hot}(\mathbf{F}) / \text{Acycl}^{\text{ctr}}(\mathbf{F}),$$

where $\text{Acycl}^{\text{ctr}}(\mathbf{F})$ is the minimal triangulated subcategory in $\text{Hot}(\mathbf{F})$ containing the totalizations of short exact sequences of complexes in \mathbf{F} and closed under products.

∞ -tilting derived equivalence

Let (T, E) be a tilting pair in an abelian category A and (W, F) be the corresponding cotilting pair in an abelian category B .

Suppose E is closed under coproducts in A and F is closed under products in B .

Then the embedding functors $E \rightarrow A$ and $F \rightarrow B$ induce triangulated equivalences

$$D^{\text{co}}(E) \simeq D^{\text{co}}(A) \quad \text{and} \quad D^{\text{ctr}}(F) \simeq D^{\text{ctr}}(B)$$

(since E is coresolving & closed under coproducts in A , and F is resolving & closed under products in B).

∞ -tilting derived equivalence

Hence the commutative diagram of triangulated functors (in fact, Verdier quotient functors) and a triangulated equivalence

$$\begin{array}{ccc} D^{\text{co}}(A) & & D^{\text{ctr}}(B) \\ \downarrow & & \downarrow \\ D(E) & \xlongequal{\quad} & D(F) \\ \downarrow & & \downarrow \\ D(A) & & D(B) \end{array}$$

∞ -tilting derived equivalence

So, generally speaking, there is a derived equivalence between A and B on some intermediate level between the co/contraderived and the conventional derived categories.

There may be many (co)tilting pairs for a given (co)tilting object in an abelian category. When the exact subcategory $E \subset A$ or $F \subset B$ is enlarged, the derived category $D(E)$ or $D(F)$ gets deflated (comes closer to the conventional derived category).

In other words, the larger the exact subcategory E or F , the smaller the derived category $D(E)$ or $D(F)$.

Example: locally Noetherian category

Let \mathcal{A} be a locally Noetherian Grothendieck abelian category, and let $J \in \mathcal{A}$ be an injective object such that $\text{Add}_{\mathcal{A}}(J) = \mathcal{A}_{\text{inj}}$.

Then $T = J$ is an ∞ -tilting object in \mathcal{A} . $E_{\max}(T) \subset \mathcal{A}$ is the full subcategory of Gorenstein injective objects.

Set $\mathcal{B} = \sigma_T(\mathcal{A})$. Then \mathcal{A}_{inj} is closed under coproducts in \mathcal{A} , and $\mathcal{B}_{\text{proj}}$ is closed under products in \mathcal{B} . The additive categories \mathcal{A}_{inj} and $\mathcal{B}_{\text{proj}}$ are equivalent.

$(T = J, E = \mathcal{A}_{\text{inj}})$ is a minimal ∞ -tilting pair in \mathcal{A} . $(W = P, F = \mathcal{B}_{\text{proj}})$ is the corresponding minimal ∞ -cotilting pair in \mathcal{B} .

The related derived equivalence is

$$D^{\text{co}}(\mathcal{A}) \simeq \text{Hot}(\mathcal{A}_{\text{inj}}) = \text{Hot}(\mathcal{B}_{\text{proj}}) \simeq D^{\text{ctr}}(\mathcal{B}).$$

Example: Gorenstein locally Noetherian category

Let us say that a locally Noetherian Grothendieck category A is n -Gorenstein if the object $T = J$ as above is n -tilting.

(Equivalently, all injective objects in A have projective dimension $\leq n$ and A has a generator of finite injective dimension.)

Suppose A is n -Gorenstein. Then there are triangulated equivalences

$$D(A) \simeq D(E_{\max}) = D(F_{\max}) \simeq D(B).$$

Furthermore, there is a commutative diagram of triangulated equivalences and Verdier quotient functors

$$\begin{array}{ccccc} D(E_{\min}) & = & D^{\text{co}}(A) & \xlongequal{\quad} & D^{\text{ctr}}(B) & = & D(F_{\min}) \\ & & \downarrow & & \downarrow & & \\ D(E_{\max}) & = & D(A) & \xlongequal{\quad} & D(B) & = & D(F_{\max}) \end{array}$$

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