# Koszul algebras and one-dependent random 0-1 sequences

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## Graded algebras

Let $k$ be a fixed ground field. A positively graded algebra $A$ over $k$ is a graded vector space $A = \bigoplus_{n=0}^{\infty} A_n$ with an associative multiplication $A_i \otimes_k A_j \longrightarrow A_{i+j}$ such that $A_0 = k \cdot 1$ is a one-dimensional vector space spanned by the unit element. We will assume all our graded algebras to have finite-dimensional components, $\dim A_n < \infty \ \forall\, n$.

The Hilbert series of a graded algebra $A$ is the formal power series

$$A(z) = \sum\nolimits_{n=0}^{\infty} (\dim_k A_n) z^n.$$

The free graded algebra (or tensor algebra) generated by a vector space $V$ is the graded algebra $T(V)$ with the components $T_n(V) = V^{\otimes n} = V \otimes_k V \otimes_k \cdots \otimes_k V$ ($n$ factors).

# Quadratic algebras

A graded algebra $A$ is said to be generated by $A_1$ if the natural graded algebra map $\pi\colon T(A_1) \longrightarrow A$ is surjective. A graded algebra $A$ is quadratic if the kernel ideal $J$ of the map $\pi$ is generated by elements of degree 2, that is $J = (R) \subset T(A_1)$, where $R = J \cap (A_1 \otimes_k A_1)$.

A quadratic algebra $A$ is determined by its degree-one component $V = A_1$ and an arbitrary vector subspace $R \subset V \otimes_k V$. The degree-$n$ component of the quadratic algebra $A = T(V)/(R)$ is then computable as

$$A_n = V^{\otimes n} \Big/ \sum\nolimits_{i=1}^{n-1} V^{\otimes i-1} \otimes_k R \otimes_k V^{\otimes n-i-1}.$$

The dual quadratic algebra $B = A^!$ has the space of generators $B_1 = V^*$ and the subspace of quadratic relations $R^\perp \subset V^* \otimes_k V^*$, where $R^\perp$ is the orthogonal complement to $R$ with respect to the natural identification $(V \otimes_k V)^* \cong V^* \otimes_k V^*$.

# Graded Tor and Ext

Given a graded algebra $A$ and a graded $A$-module $M$, a projective resolution of the $A$-module $M$ can be chosen so that it consists of graded $A$-modules and has homogeneous differentials. Therefore, for any graded right $A$-module $M$ and graded left $A$-module $N$, the vector space $\mathrm{Tor}^A(N, M)$ is naturally bigraded,

$$\mathrm{Tor}_i^A(N, M) = \bigoplus\nolimits_j \mathrm{Tor}_{i,j}^A(N, M),$$

where $i$ is the usual homological grading of the Tor, while $j$ is the internal grading induced by the grading of $A$, $M$, and $N$. In particular, for a positively graded algebra $A$ one has

$$\mathrm{Tor}_i^A(k, k) = \bigoplus\nolimits_{j \geqslant i} \mathrm{Tor}_{i,j}^A(k, k),$$

where $k$ is endowed with the trivial left and right $A$-module structures. The story of the bigraded Ext is a bit more complicated, but still for the trivial left $A$-module $k$ one has

$$\mathrm{Ext}_A^i(k, k) = \prod\nolimits_{j \geqslant i} \mathrm{Ext}_A^{i,j}(k, k).$$

## Low-degree and diagonal cohomology

There is a natural isomorphism of vector spaces $\operatorname{Ext}_A^{i,j}(k,k) \cong \operatorname{Tor}^A_{i,j}(k,k)^*$ for any positively graded algebra $A$ and integers $i$, $j$.

For any positively graded algebra $A$, the graded vector space of minimal generators of $A$ is isomorphic to $\operatorname{Tor}^A_1(k,k)$, and the graded space of minimal relations in $A$ is $\operatorname{Tor}^A_2(k,k)$. So a graded algebra $A$ is quadratic if and only if $\operatorname{Tor}^A_{1,j}(k,k) = 0$ for all $j > 1$ and $\operatorname{Tor}^A_{2,j}(k,k) = 0$ for all $j > 2$.

For any positively graded algebra $A$, the graded algebra of diagonal cohomology $\bigoplus_{n=0}^\infty \operatorname{Ext}_A^{n,n}(k,k)$ is quadratic. It is computable as follows. Denote by $\mathrm{q}\,A$ the "quadratic part" of $A$, that is the quadratic algebra generated by $A_1$ and defined by those quadratic relations which hold in $A$. So $\mathrm{q}\,A$ is the universal quadratic algebra endowed with a graded algebra morphism $\mathrm{q}\,A \longrightarrow A$. Then one has

$$\bigoplus\nolimits_{n=0}^\infty \operatorname{Ext}_A^{n,n}(k,k) \cong (\mathrm{q}\,A)^!.$$

# Koszul algebras

A positively graded algebra $A$ is called Koszul if $\mathrm{Tor}^A_{i,j}(k,k) = 0$ for all $i \neq j$. In particular, all Koszul algebras are quadratic.

For a quadratic algebra $A$, the first nontrivial Koszulity condition appears in the internal degree 4:

$$\mathrm{Tor}^A_{3,4}(k,k) = 0,$$

then there are two conditions in the internal degree 5:

$$\mathrm{Tor}^A_{3,5}(k,k) = 0 = \mathrm{Tor}^A_{4,5}(k,k) = 0, \qquad \text{etc.}$$

A quadratic algebra $A$ is Koszul if and only if the dual quadratic algebra $A^!$ is Koszul, and if and only if the opposite algebra $A^{\mathrm{op}}$ is Koszul.

For a Koszul algebra $A$, one has $\mathrm{Ext}^*_A(k,k) \cong A^!$. Computing the Euler characteristic (of, say, the bar-complex), one obtains a formula connecting the Hilbert series of $A$ and $A^!$:

$$A(z)A^!(-z) = 1.$$

## Distributive lattices

Let $W$ be a vector space. A lattice of subspaces in $W$ is a set of subspaces closed under finite sums and intersections. A lattice $\Lambda$ of subspaces in $W$ is said to be distributive if for all $X$, $Y$, $Z \in \Lambda$ the distributivity identity

$$(X+Y)\cap Z = X\cap Z + Y\cap Z$$

holds. A collection of subspaces $X_1$, …, $X_{n-1}$ in $W$ is said to be distributive if the lattice of subspaces in $W$ generated by $X_1$, …, $X_{n-1}$ is distributive.

Any pair of subspaces $A$, $B \subset W$ is distributive. A triple of subspaces $A$, $B$, $C \subset W$ is distributive if and only if the equation $(A+B)\cap C = A\cap C + B\cap C$ holds. A quadruple of subspaces $A$, $B$, $C$, $D \subset W$ is distributive if and only if every its proper subset is distributive and two equations hold:

$$(A+B+C)\cap D = A\cap D + B\cap D + C\cap D,$$
$$(A+B)\cap C\cap D = A\cap C\cap D + B\cap C\cap D.$$

## Distributing basises

A collection of subspaces $X_1, \ldots, X_{n-1}$ in a vector space $W$ is distributive if and only if it is a direct sum of collections of subspaces in one-dimensional vector spaces. Equivalently, this means that there exists a basic $\Omega = \{w_\alpha\}$ in $W$ such that $X_i$ is spanned by $X_i \cap \Omega$ for every $i = 1, \ldots, n-1$.

## Koszulity and distributivity

**Theorem (J. Backelin, Ph.D. Thesis, '81)**

*A quadratic algebra $A = T(V)/(R)$, where $R \subset V \otimes_k V$, is Koszul if and only if for every $n \geqslant 1$ the collection of subspaces*

$$X_i^{(n)} = V^{\otimes i-1} \otimes_k R \otimes_k V^{\otimes n-i-1}, \qquad i = 1, \ldots, n-1$$

*is distributive in the vector space $W^{(n)} = V^{\otimes n}$.*

In particular, for $n = 4$ the triple of subspaces $R \otimes_k V \otimes_k V$, $V \otimes_k R \otimes_k V$, $V \otimes_k V \otimes_k R \subset V^{\otimes 4}$ should be distributive, etc.

## Examples: Monomial algebras

Let $S$ be an oriented graph with $m$ vertices, that is a subset $S \subset \{1, \dots, m\}^2$. The noncommutative quadratic monomial algebra $A$ corresponding to $S$ is the quadratic algebra with generators $x_1$, ..., $x_m$ and relations $x_i x_j = 0$ for all $(i, j) \notin S$. Oriented paths of length $n$ (passing through $n$ vertices) in $S$ form a basis of the component $A_n$. The noncommutative quadratic monomial algebras are Koszul.

Let $T$ be an unoriented graph with $m$ vertices, that is a subset $T \subset \{1, \dots, m\}^2/(\mathbb{Z}/2)$. The commutative quadratic monomial algebra $A$ corresponding to $T$ is the quotient algebra of the polynomial algebra $k[x_1, \dots, x_m]$ by the relations $x_i x_j = 0$ for all $\{i, j\} \notin T$. If $T$ contains no loops, then full subgraphs on $n$ vertices in $T$ form a basis of the component $A_n$. The commutative quadratic monomial algebras are Koszul, too.

## Hilbert series of quadratic algebras

The Hilbert series of quadratic algebras can be very complicated. The following family of quadratic algebras $A_\lambda$ with 3 generators $x$, $y$, $z$ and 3 relations

$$\begin{cases} xz = xy, \\ zx = yx, \\ zy = \lambda yz, \quad \lambda \in k^* \end{cases}$$

has an infinite number of different Hilbert series, depending on whether the parameter $\lambda$ is a root of unity and its primitive degree. Indeed, one computes that

$$xy^{n+1}x = xzy^nx = \lambda^n xy^nzx = \lambda^n xy^{n+1}x,$$

hence $xy^{n+1}x = 0$ whenever $\lambda^n \neq 1$. So the "size" of the algebra depends on whether $\lambda$ is a root of unity, and of what degree. The algebra $A_\lambda$ is Koszul if and only if $\lambda$ is not a root of unity.

# Hilbert series of quadratic algebras

Let $F$ be a system of polynomial Diophantine equations in nonnegative integer variables $x_1, \ldots, x_l$ with the coefficients in $k$. Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ be the formal power series with the coefficient $f_n$ equal to the number of solutions of $F$ with $x_1 + \cdots + x_l = n$. Then there exists a quadratic algebra $A$, constructed in a straightforward manner from the system of equations $F$, with the number of generators $m$ and the number of relations $r$ depending on the “size” of $F$ in a simple way, such that

$$A(z)^{-1} = 1 - mz + rz^2 - z^3 f(z).$$

The algebra $A$ has global dimension at most 3, and

$$\dim_k \operatorname{Tor}^A_{3,3+n}(k,k) = f_n.$$

The algebra $A$ is Koszul if and only if $F$ has no nonzero solutions. So computing the Hilbert series and Tor spaces of quadratic algebras is about as easy as solving Diophantine equations.

## Hilbert series of Koszul algebras

The previous examples (due to Fröberg–Gulliksen–Löfwall, 1986) show that it is algorithmically unsolvable to determine whether a given quadratic algebra $A$ is Koszul. But if you already know that $A$ is Koszul, then computing the Hilbert series $A(z)$ becomes a better-behaved problem, at least in some sense.

**Theorem (Polishchuk–P., 1992–2005)**

*The set of Hilbert series of Koszul algebras with $m$ generators over a fixed field $k$ is finite, of the cardinality not exceeding $m^{m^4}$ if $m \geqslant 2$. The set of all Hilbert series of Koszul algebras with $m$ generators over all fields (of all characteristics) is also finite.*

## Vague idea of proof: Deformation theory

The point is that deformations of a Koszul algebra $A$ are controlled by its component $A_3$ of degree 3, because obstructions to flat deformations lie in the Hochschild cohomology space $HH^3(A,A)$. The $A\otimes_k A^{\mathrm{op}}$-module $A$ is Koszul, so the Tor space $\mathrm{Tor}_3^{A\otimes_k A^{\mathrm{op}}}(k,A)$, describing the 3rd component of the minimal projective resolution of this graded module, is concentrated in the internal degree $j=3$.

Quadratic algebras $A$ with $\dim_k A_1 = m$ and $\dim_k A_2 = s$ form an algebraic variety (a Grassmannian) $Q_{m,s}$. Quadratic algebras with fixed $\dim A_3 = u$ form a locally closed subvariety $Q_{m,s,u}\subset Q_{m,s}$. A result going back to Drinfeld (1986) tells that the Koszul algebras in $Q_{m,s,u}$ form a countable intersection of open subvarieties, and their Hilbert series is locally constant there. So the number of Koszul Hilbert series in $Q_{m,s,u}$ does not exceed the number of irreducible components.

## Rationality conjecture

The following conjecture I very much wanted to prove in the beginning of 1990's:

**Conjecture**

*For any Koszul algebra $A$, the Hilbert series $A(z)$ is a rational function, that is, a fraction of two polynomials in $z$.*

This conjecture is confirmed by the facts that the Hilbert series of quadratic monomial algebras are many and varied, but all of them are rational.

Put $a_n = \dim A_n$. There are no algebraic depencies between the numbers $a_n$, but I discovered that they satisfy a huge system of polynomial inequalities, starting as

$$a_n \geqslant 0, \qquad a_i a_j - a_{i+j} \geqslant 0,$$
$$a_i a_j a_k - a_{i+j} a_k - a_i a_{j+k} + a_{i+j+k} \geqslant 0$$

for all $i$, $j$, $k \geqslant 1$, etc.

## Meromorphic continuation problem

Then I decided that the problem had an analytic flavor. I wanted to prove, at least, that the series $A(z)$ defines a meromorphic function in the whole complex plane.

Several years later, I learned that it would be sufficient.

**Theorem (E. Borel, 1894)**

*Let $f(z) = \sum_{n=0}^{\infty} f_n z^n$ be a formal power series with integer coefficients. Assume that $f$ defines a meromorphic function in the circle $|z| < \rho$ of a radius $\rho > 1$. Then $f$ is a fraction of two polynomials.*

In the meantime, we were able to prove the following theorem. Obviously, for any graded algebra $A$ generated by $A_1$, one has $\dim A_n \leqslant (\dim A_1)^n$ for all $n \geqslant 0$. So the power series $A(z)$ is holomorphic for $|z| < 1/m$ if $m = \dim A_1$.

# Meromorphic continuation theorem

**Theorem (Polishchuk–P., 1993–2005)**

*Let $A$ be a Koszul algebra with $\dim_k A_1 = m$. Then the power series $A(z)$ defines a meromorphic function in the circle $|z| < 2/m$. In fact, one has $A(z) \neq -1$ for $|z| < 2/m$, so the power series $(1 + A(z))^{-1}$ is holomorphic for $|z| < 2/m$.*

Why is it so? Introduce a variable $y = mz/2$. Then the claim is that the coefficients $h_n$ of the power series

$$\sum\nolimits_{n=1}^{\infty} h_n y^n = \frac{1 - A(2y/m)}{1 + A(2y/m)}$$

satisfy $-1 \leqslant h_n \leqslant 1$ for all $n \geqslant 1$. The explanation is that the numbers $h_n$ are, essentially, probabilities of certain events related to the algebra $A$. More precisely, the numbers $0 \leqslant (1 + h_n)/2 \leqslant 1$ are probabilities of events.

# Basic Concepts of Probability

Let $\Omega$ be a set. A $\sigma$-algebra of subsets $\mathcal{F}$ in $\Omega$ is a nonempty set of subsets closed under countable unions, countable intersections, and complements. A probability measure on $\mathcal{F}$ is a countably additive measure $P\colon \mathcal{F} \longrightarrow [0,1] \subset \mathbb{R}$ such that $P(\Omega) = 1$. A probability space is a triple $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is a $\sigma$-algebra of subsets in $\Omega$ and $P$ is a probability measure on $\mathcal{F}$. A subset in $\Omega$ is measurable if it belongs to $\mathcal{F}$. Measurable subsets are interpreted as events.

If $T$ is a topological space, then the Borel $\sigma$-algebra of subsets in $T$ is the $\sigma$-algebra generated by the open subsets. Borel subsets are the subsets belonging to the Borel $\sigma$-algebra.

A function $f : \Omega \longrightarrow T$ is measurable if the preimages of open subsets are measurable, or equivalently, the preimages of Borel subsets are measurable.

A random sequence $(\xi_i)_{i\in\mathbb{Z}}$ with values in $T$ is a sequence of measurable functions $\xi_i : \Omega \longrightarrow T$ defined on some probability space $\Omega$.

# Random sequences

When considering a random sequence $(\xi_i)$, we are only interested in probabilities of events associated with this random sequence.

Let $\Omega'$ be another set with a $\sigma$-algebra of subsets $\mathcal{F}'$, and let $R\colon \Omega \longrightarrow \Omega'$ be a map such that the preimages of subsets belonging to $\mathcal{F}'$ under $R$ belong to $\mathcal{F}$. Suppose that the functions $\xi_i\colon \Omega \longrightarrow T$ factor through $R$, leading to functions $\xi'_i\colon \Omega' \longrightarrow T$. Define a probability measure $P'$ on $\mathcal{F}'$ as the push-forward of the measure $P$ on $\mathcal{F}$. Then the random sequence $(\xi'_i)$ on $\Omega'$ is considered to be equivalent to the random sequence $(\xi_i)$ on $\Omega$.

Consider the topological space $T^{\mathbb{Z}}$ with the product topology, and endow with the $\sigma$-algebra of Borel subsets $\mathcal{B}$. Consider the map $\Xi = (\xi_i)_{i\in\mathbb{Z}}\colon \Omega \longrightarrow T^{\mathbb{Z}}$. Then, up to the above equivalence, a random sequence $(\xi_i)_{i\in\mathbb{Z}}$ is described by the push-forward $D = \Xi_*(P)$ of the probability measure $P$ on $\mathcal{F}$ with respect to the map $\Xi$. So $D\colon \mathcal{B} \longrightarrow [0,1]$ is a Borel probability measure on $T^{\mathbb{Z}}$ describing a random sequence $(\xi_i)_{i\in\mathbb{Z}}$ with values in $T$.

## Random sequences

Assume for simplicity that $T$ is a discrete finite set. Denote by $\mathcal{B}_0$ the standard topology base of the product topology on $T^{\mathbb{Z}}$ consisting of the cylinder subsets

$$\{(t_i)_{i\in\mathbb{Z}} \in T^{\mathbb{Z}} \mid (t_{i_1},\dots,t_{i_n}) \in U\}, \quad i_1,\dots,i_n \in \mathbb{Z},\ U \subset T^{\{i_1,\dots,i_n\}}.$$

Then any finitely-additive measure $P_0\colon \mathcal{B}_0 \longrightarrow [0,1]$ with $P_0(T^{\mathbb{Z}}) = 1$ can be uniquely extended to a probability measure $P\colon \mathcal{B} \longrightarrow [0,1]$ on $T^{\mathbb{Z}}$. In other words, in order to define a random sequence $(\xi_i)_{i\in\mathbb{Z}}$, it suffices to specify, in a compatible way, the probabilities of events depending on finite subsets of the variables $\xi_i$ only.

We are interested in random 0-1-sequences, so $T = \{0,1\}$. We will also assume that $(\xi_i)_{i\in\mathbb{Z}}$ is stationary , that is, for every $k \in \mathbb{Z}$, the sequence $(\xi_{k+i})_{i\in\mathbb{Z}}$ is equivalent to $(\xi_i)_{i\in\mathbb{Z}}$:

$$P\{\xi_1 = t_1,\dots,\xi_n = t_n\} = P\{\xi_{k+1} = t_1,\dots,\xi_{k+n} = t_n\}.$$

## Stationary random 0-1 sequences

A stationary random sequence 0-1 sequence $(\xi_i)_{i\in\mathbb{Z}}$ is uniquely determined by the collection of numbers

$$0 \leqslant [t_1, \dots, t_n] = P\{\xi_1 = t_1, \dots, \xi_n = t_n\} \leqslant 1,$$

$n \geqslant 0$, $t_i \in \{0, 1\}$, which has to satisfy the equations $[\ ] = 1$ and

$$\begin{aligned}[0, t_1, \dots, t_n] + [1, t_1, \dots, t_n] &= [t_1, \dots, t_n] \\ &= [t_1, \dots, t_n, 0] + [t_1, \dots, t_n, 1].\end{aligned}$$

Let us introduce the following starred notation:

$$\begin{aligned}&[s_1, \dots, s_{k-1}, *, s_{k+1}, \dots, s_n] \\ &= [s_1, \dots, s_{k-1}, 0, s_{k+1}, \dots, s_n] + [s_1, \dots, s_{k-1}, 1, s_{k+1}, \dots, s_n],\end{aligned}$$

where $s_i \in \{0, 1, *\}$. So the usage of the starred notation means that we fix the values of $\xi_i$ for some positions $1 \leqslant i \leqslant n$, while leaving them to be arbitrary in the remaining positions; and compute the probability.

In the starred notation, the above consistency equation takes the form $[*, t_1, \ldots, t_n] = [t_1, \ldots, t_n] = [t_1, \ldots, t_n, *]$.

## One-dependent random sequences

The famous Markov property in stochastic processes says that "the future is independent of the past if the present is known". One-dependence is the opposite of Markovianness. It says that the future is independent of the past if nothing is known about the present.

This means the equation

$$P\{\xi_{-k} = t_{-k}, \ldots, \xi_{-1} = t_{-1},\ \xi_1 = t_1, \ldots, \xi_l = t_l\} \\ = P\{\xi_{-k} = t_{-k}, \ldots, \xi_{-1} = t_{-1}\} \cdot P\{\xi_1 = t_1, \ldots, \xi_l = t_l\},$$

for probabilities; or, in the bracket and star notation,

$$[t_{-k}, \ldots, t_{-1}, *, t_1, \ldots, t_l] = [t_{-k}, \ldots, t_{-1}] \cdot [t_1, \ldots, t_l].$$

# One-dependent stationary 0-1 sequences

**Theorem**

*A stationary one-dependent random* 0-1 *sequence* $(\xi_i)_{i\in\mathbb{Z}}$ *is uniquely determined by the sequence of real numbers*

$$\alpha_2 = [1],\ \alpha_3 = [11],\ \ldots,\ \alpha_n = [1^{n-1}],\ \ldots.$$

*All the other probabilities like* $[t_1, \ldots, t_n]$ *are computable as polynomials in* $\alpha_2$, $\alpha_3$, ...

Example of computation: $[1010] =$
$[1*1*] - [111*] - [1*11] + [1111] =$
$[1][1] - [111] - [1][11] + [1111] = \alpha_2^2 - \alpha_4 - \alpha_2\alpha_3 + \alpha_5.$

The sequence of real numbers $(0 \leqslant \alpha_n \leqslant 1)_{n\geqslant 2}$ has to satisfy the system of polynomial inequalities

$$[t_1, \ldots, t_n] \geqslant 0 \quad \text{for all } t_1,\ \ldots,\ t_n \in \{0,1\},\ n \geqslant 1.$$

**Theorem (P., 1993–2005)**

*For any Koszul algebra A, there exists a (unique) stationary one-dependent random* 0-1 *sequence* $(\xi_i)_{i\in\mathbb{Z}}$ *with the parameters*

$$[1^{n-1}] = \alpha_n = a_n/a_1^n, \quad \textit{where } a_n = \dim_k A_n.$$

Here is the idea of the construction. For any fin.-dim. vector space $W$ and a subspace $X \subset W$, the fraction $0 \leqslant (\dim X)/(\dim W) \leqslant 1$ is interpreted as the probability of an event.

For any distributive collection of subspaces $X_1, \ldots, X_{n-1} \subset W$, choose a distributing basis $\Omega$ in $W$ and consider it as a probability space with the full $\sigma$-algebra of measurable subsets $\mathcal{F} = 2^\Omega$ and the uniform probability measure $P(\{w\}) = 1/(\dim W)$ for every $w \in \Omega$. Put $\xi_i(w) = 0$ if $w \in X_i$ and $\xi_i(w) = 1$ otherwise, for every $w \in \Omega$ and $1 \leqslant i \leqslant n-1$. We obtain a finite random 0-1 sequence $(\xi_1, \ldots, \xi_{n-1})$.

## Sketch of proof of the theorem.

Given a Koszul algebra $A$, we construct, for every $n \geqslant 1$, a finite random 0-1 sequence $(\xi_1, \dots, \xi_{n-1})$ in such a way that these sequences agree when $n$ varies. This will define the probabilities $[t_1, \dots, t_{n-1}]$ for all $t_i \in \{0, 1\}$ and $n \geqslant 1$, which is sufficient.

Put $V = A_1$ and $W^{(n)} = V^{\otimes n}$. Consider the collection of $n - 1$ subspaces $X_i^{(n)} = V^{\otimes i-1} \otimes_k R \otimes_k V^{\otimes n-i-1} \subset W^{(n)}$, $i = 1, \dots,$ $n - 1$, where $R \subset V \otimes_k V$ is the space of quadratic relations in $A$. The above construction defines the desired random 0-1 sequence $(\xi_1, \dots, \xi_{n-1})$.

It remains to check the equations for consistency and one-dependence: $[t_1, \dots, t_{k-1}, *, t_{k+1}, \dots, t_{k+l-1}] = [t_1, \dots, t_{k-1}] \cdot [t_{k+1}, \dots, t_{k+l-1}]$. These hold, essentially, for the following reason.

## Sketch of proof of the theorem cont'd.

Consider the sequence of subspaces $X_1^{(k+l)}, \dots, X_{k+l-1}^{(k+l)} \subset V^{\otimes k+l}$, and drop $X_k^{(k+l)}$ out of this sequence. Then the vector space $V^{\otimes k+l}$ decomposes as the tensor product $V^{\otimes k} \otimes_k V^{\otimes l}$,

$$V \otimes_k \cdots \otimes_k V = (V \otimes_k \cdots \otimes_k V) \otimes_k (V \otimes_k \cdots \otimes_k V),$$

and the remaining collection of $k+l-2$ subspaces in $V^{\otimes k+l}$ arises from $k-1$ subspaces in $V^{\otimes k}$ and $l-1$ subspaces in $V^{\otimes l}$.

This kind of "tensor independence" of $X_1^{(k+l)}, \dots, X_{k-1}^{(k+l)}$ from $X_{k+1}^{(k+l)}, \dots, X_{k+l-1}^{(k+l)}$ implies the probabilistic independence of $(\xi_1, \dots, \xi_{k-1})$ from $(\xi_{k+1}, \dots, \xi_{k+l-1})$. □

## Remark

For the one-dependent sequence corresponding to a Koszul algebra $A$, one also has $[0^{n-1}] = a_n^!/a_1^n$, where $a_n^! = \dim_k A_n^!$.

## Sketch of proof of the meromorphic continuation theorem.

Let $A$ be a Koszul algebra with $\dim_k A_1 = m$. We want to show that the function $(1 + A(z))^{-1}$ is holomorphic for $|z| < 2/m$.

Recall the variable change $y = mz/2$ and the formula

$$\sum_{n=1}^{\infty} h_n y^n = \frac{1 - A(2y/m)}{1 + A(2y/m)}.$$

We want to show that $-1 \leqslant h_n \leqslant 1$. For this purpose, one computes that $h_n$ is the following probability of a "$\pm 1$ event":

$$(-1)^n h_n = P\left\{\sum_{i=1}^{n-1} \xi_i \text{ is even}\right\} - P\left\{\sum_{i=1}^{n-1} \xi_i \text{ is odd}\right\}.$$

□

# Two-block factor sequences

A simple and straightforward way in which one-dependent random sequences arise in probability theory is called the two-block factor construction. Let

$$\ldots,\ \eta_{-1},\ \eta_0,\ \eta_1,\ \eta_2,\ \ldots\ : \Omega \longrightarrow [0,1]$$

be a sequence of independent random variables, taking values (let us say) in the interval $[0,1] \subset \mathbb{R}$. Since we are interested in stationary sequences, we shall also assume that $\eta_i$ are identically distributed. Let $f\colon [0,1]^2 \longrightarrow \{0,1\}$ be a Borel measurable function of two variables. Put

$$\xi_i = f(\eta_i, \eta_{i+1})\colon \Omega \longrightarrow \{0,1\}, \quad i \in \mathbb{Z}.$$

Then $(\xi_i)_{i\in\mathbb{Z}}$ is a stationary one-dependent random 0-1 sequence, called a "two-block factor".

Algebraically, two-block factor random sequences as above arise from noncommutative quadratic monomial algebras.

# Conclusion

It has been established in the probability literature that the probabilities $\alpha_n = [1^{n-1}]$ for two-block factor sequences satisfy stricter inequalities than for one-dependent sequences generally. This story was also considered as a part of the wider topic of "Bell inequalities" in quantum mechanics.

However, the present understanding of (either of) these systems of inequalities remains very limited.

Likewise, it has been established in the algebraic literature (largely or even exclusively by computer-assisted search and classification) that there exist Hilbert series of Koszul algebras that cannot be obtained as Hilbert series of quadratic monomial algebras.

But the present understanding of the Hilbert series of Koszul algebras remains very limited. The rationality conjecture is still wide open.

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