Koszul algebras and one-dependent random 0-1 sequences

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Seminar on cohomology in algebra, geometry, physics and statistics (via Zoom)

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$$\mathrm{Ext}_A^i(k, k) = \prod\nolimits_{j \geqslant i} \mathrm{Ext}_A^{i,j}(k, k).$$

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$$\bigoplus_{n=0}^\infty \mathrm{Ext}_A^{n,n}(k,k) \cong (\mathrm{q}\,A)^!.$$

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For a quadratic algebra A, the first nontrivial Koszulity condition appears in the internal degree 4:

$$\mathrm{Tor}^A_{3,4}(k,k) = 0,$$

then there are two conditions in the internal degree 5:

$$\mathrm{Tor}^A_{3,5}(k,k) = 0 = \mathrm{Tor}^A_{4,5}(k,k) = 0, \qquad \text{etc.}$$

A quadratic algebra A is Koszul if and only if the dual quadratic algebra $A^!$ is Koszul, and if and only if the opposite algebra A^{op} is Koszul.

For a Koszul algebra A, one has $\mathrm{Ext}^*_A(k,k) \cong A^!$. Computing the Euler characteristic (of, say, the bar-complex), one obtains a formula connecting the Hilbert series of A and $A^!$:

$$A(z)A^!(-z) = 1.$$

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$$(A + B + C) \cap D = A \cap D + B \cap D + C \cap D,$$
$$(A + B) \cap C \cap D = A \cap C \cap D + B \cap C \cap D.$$

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In particular, for $n = 4$ the triple of subspaces $R \otimes_k V \otimes_k V$, $V \otimes_k R \otimes_k V$, $V \otimes_k V \otimes_k R \subset V^{\otimes 4}$ should be distributive, etc.

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Hilbert series of quadratic algebras

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The algebra A is Koszul if and only if F has no nonzero solutions. So computing the Hilbert series and Tor spaces of quadratic algebras is about as easy as solving Diophantine equations.

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Theorem (Polishchuk–P., 1992–2005)

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The set of Hilbert series of Koszul algebras with m generators over a fixed field k is finite, of the cardinality not exceeding m^{m^4} if $m \geqslant 2$. The set of all Hilbert series of Koszul algebras with m generators over all fields (of all characteristics) is also finite.

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$$a_n \geqslant 0, \qquad a_i a_j - a_{i+j} \geqslant 0,$$
$$a_i a_j a_k - a_{i+j} a_k - a_i a_{j+k} + a_{i+j+k} \geqslant 0$$

for all i, j, $k \geqslant 1$, etc.

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Consider the sequence of subspaces $X_1^{(k+l)}, \ldots, X_{k+l-1}^{(k+l)} \subset V^{\otimes k+l}$

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Algebraically, two-block factor random sequences as above arise from noncommutative quadratic monomial algebras.

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