Fp-projective periodicity

Leonid Positselski – IM CAS, Prague

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Periodicity Theorems: General Setting

Let R be a ring and

$$0 \longrightarrow M \longrightarrow L \longrightarrow M \longrightarrow 0 \qquad (*)$$

be a short exact sequence of (right) R-modules in which the leftmost and the rightmost modules coincide.

Suppose that the R-module L belongs to a certain class of modules. Can one say something about the R-module M?

Not necessarily. Take $\Lambda = k[\epsilon]/(\epsilon^2)$ to be the algebra of dual numbers over a field k. Then there is a short exact sequence of Λ-modules $0 \longrightarrow k \longrightarrow \Lambda \longrightarrow k \longrightarrow 0$. So $L = \Lambda$ is a projective-injective Λ-module, but the module $M = k$ is neither projective nor injective over Λ.

Still, quite a few positive results are known.

Periodicity Theorems: Known Results

It is known that, for $0 \longrightarrow M_R \longrightarrow L_R \longrightarrow M_R \longrightarrow 0 \ (*)$:

- If M is flat and L is projective, then M is projective. [Benson and Goodearl 2000]
- If L is pure-projective and $(*)$ is pure, then M is pure-projective. [Simson 2002]
- If L is pure-injective and $(*)$ is pure, then M is pure-injective. [Šťovíček 2014]
- In particular, if L is injective and M is fp-injective, then M is injective.
- If L is cotorsion, then M is cotorsion. [Bazzoni, Cortés-Izurdiaga, and Estrada 2017]
- If R is right coherent and L is fp-projective, then M is fp-projective. [Šaroch and Šťovíček 2018]
- Over any R, if L is fp-projective, then M is weakly fp-projective. [Bazzoni, Hrbek, and P. 2022]

Periodicity Theorems Stated for Modules of Cocycles

The same results can be stated as theorems about modules of cocycles in acyclic complexes:

- In any acyclic complex of projective modules with flat modules of cocycles, the modules of cocycles are actually projective (so the complex is contractible). [Neeman 2008]
- In any pure acyclic complex of pure-projective modules, the modules of cocycles are pure-projective (so the complex is contractible). [Šťovíček 2014]
- In any pure acyclic complex of pure-injective modules, the modules of cocycles are pure-injective (so the complex is contractible). [Šťovíček 2014]
- In any acyclic compex of injective modules with fp-injective modules of cocycles, the modules of cocycles are actually injective (so the complex is contractible).

Periodicity Theorems Stated for Modules of Cocycles

- In any acyclic complex of cotorsion modules, the modules of cocycles are cotorsion. [Bazzoni, Cortés-Izurdiaga, and Estrada 2017]
- In any acyclic complex of fp-projective right modules over a right coherent ring, the modules of cocycles are fp-projective. [Šaroch and Šťovíček 2018]
- In any acyclic complex of fp-projective modules (over any ring), the modules of cocycles are weakly fp-projective. [Bazzoni, Hrbek, and P. 2022]

Over a right coherent ring, the classes of fp-projective and weakly fp-projective right modules coincide.

The proof of Šaroch and Šťovíček is a complicated set-theoretic argument by induction on the cardinals. Our proof is a much simpler homological or homotopical argument using Neeman's theorem and the Hill lemma.

Fp-projectivity and Weak Fp-projectivity: the Definitions

An R-module J is called fp-injective if $\mathrm{Ext}^1_R(T, J) = 0$ for all finitely presented R-modules T. An R-module P is called fp-projective if $\mathrm{Ext}^1_R(P, J) = 0$ for all fp-injective R-modules J.

An R-module J is called strongly fp-injective if $\mathrm{Ext}^n_R(T, J) = 0$ for all finitely presented R-modules T and all $n \geq 1$. An R-module P is called weakly fp-projective if $\mathrm{Ext}^1_R(P, J) = 0$ for all strongly fp-injective R-modules J.

So the pair of classes (fp-projective modules, fp-injective modules) is a complete cotorsion pair in $\mathrm{Mod}\text{-}R$. The pair of classes (weakly fp-projective modules, strongly fp-injective modules) is a hereditary complete cotorsion pair in $\mathrm{Mod}\text{-}R$.

The cotorsion pair (fp-projective right R-modules, fp-injective right R-modules) is hereditary if and only if R is a right coherent ring. In this case, the two cotorsion pairs coincide.

Neeman's Orthogonality Theorem

In Neeman's 2008 paper, the periodicity/contractibility theorem stated above is deduced as a corollary of the following orthogonality theorem.

Theorem (Neeman 2008)

Let $P^\bullet$ be a complex of projective R-modules and $F^\bullet$ be an acyclic complex of flat R-modules with flat modules of cocycles. Then any morphism of complexes $P^\bullet \longrightarrow F^\bullet$ is homotopic to zero.

The following theorem, first stated by Šťovíček, is a corollary of Neeman's theorem.

Theorem (Šťovíček 2014)

Let $P^\bullet$ be a complex of pure-projective R-modules and $X^\bullet$ be a pure-acyclic complex of R-modules. Then any morphism of complexes $P^\bullet \longrightarrow X^\bullet$ is homotopic to zero.

Šťovíček's Orthogonality Theorem

Proof.

To deduce Šťovíček's orthogonality theorem for right modules over a ring R, one needs to apply Neeman's orthogonality theorem to right modules over the "ring with many objects" (otherwise known as "nonunital ring with enough idempotents") $\mathcal{T}$ corresponding to the small additive category of finitely presented right R-modules.

The point is that right R-modules are the same things as flat contravariant additive functors $\mathcal{T} = \mathrm{mod}\text{-}R \longrightarrow \mathrm{Ab}$. Pure-acyclic complexes of R-modules correspond to acyclic complexes of flat $\mathcal{T}$-modules with flat $\mathcal{T}$-modules of cocycles; and pure-projective R-modules correspond to projective $\mathcal{T}$-modules. □

Termwise Fp-projective/Fp-injective Cocycles Orthogonality Theorem

Theorem (Bazzoni, Hrbek, and P. 2022)

Let $P^\bullet$ be a complex of fp-projective R-modules and $J^\bullet$ be an acyclic complex of fp-injective R-modules with fp-injective modules of cocycles. Then any morphism of complexes $P^\bullet \longrightarrow J^\bullet$ is homotopic to zero.

Proof.

Wlog one can assume that all terms of $P^\bullet$ are filtered by finitely presented modules. Then, by another theorem of Šťovíček (based on the Hill lemma), the whole complex $P^\bullet$ is filtered by (bounded below) complexes of finitely presented R-modules. Notice that any finitely presented module is pure-projective. On the other hand, any acyclic complex with fp-injective modules of cocycles is pure acyclic (since fp-injective modules are "absolutely pure").

Termwise Fp-projective/Fp-injective Cocycles Orthogonality Theorem

End of proof.

Now we use the Eklof lemma in order to reduce the question to Šťovíček's orthogonality theorem for complexes of pure-projective modules and pure acyclic complexes.

To ensure applicability of the Eklof lemma, one needs to use the following lemma connecting the groups Hom in the (triangulated) homotopy category of complexes $\mathbf{K}(\mathrm{Mod}\text{-}R)$ with the groups Ext^1 in the abelian category of complexes $\mathbf{C}(\mathrm{Mod}\text{-}R)$. For any two complexes of modules $A^\bullet$ and $B^\bullet$ such that $\mathrm{Ext}^1_R(A^n, B^n) = 0$ for all $n \in \mathbb{Z}$, one has

$$\mathrm{Hom}_{\mathbf{K}(\mathrm{Mod}\text{-}R)}(A^\bullet, B^\bullet[1]) \simeq \mathrm{Ext}^1_{\mathbf{C}(\mathrm{Mod}\text{-}R)}(A^\bullet, B^\bullet).$$

In the situation at hand, we are dealing with varying complexes of fp-projective modules $A^\bullet$ and a fixed complex of fp-injective modules $B^\bullet = J^\bullet$, so the assumption of the lemma holds. □

Main Fp-projective/Weakly Fp-projective Periodicity Theorem

Theorem (Bazzoni, Hrbek, and P. 2022)

Let $P^\bullet$ be an acyclic complex of fp-projective R-modules. Then the modules of cocycles of $P^\bullet$ are weakly fp-projective.

Proof.

It suffices to show that, for any strongly fp-injective R-module Y, the complex $\mathrm{Hom}_R(P^\bullet, Y)$ is acyclic.

Let $0 \longrightarrow Y \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots$ be an injective resolution of Y. Denote the whole acyclic complex $Y \longrightarrow J^\bullet$ by $X^\bullet$.

Then $X^\bullet$ is a complex of fp-injective modules with fp-injective modules of cocycles, since the (weakly fp-projective, strongly fp-injective) cotorsion pair is hereditary. By the termwise fp-projective/fp-injective cocycles orthogonality theorem, the complex $\mathrm{Hom}_R(P^\bullet, X^\bullet)$ is acyclic.

Main Fp-projective/Weakly Fp-projective Periodicity Theorem

End of proof.

On the other hand, the complex $\mathrm{Hom}_R(P^\bullet, J^\bullet)$ is acyclic, since $P^\bullet$ is an acyclic complex of modules and $J^\bullet$ is a bounded below complex of injective modules.

Now we know that both the complexes $\mathrm{Hom}_R(P^\bullet, J^\bullet)$ and $\mathrm{Hom}_R(P^\bullet, X^\bullet)$ are acyclic. Since $X^\bullet$ is the complex $X^\bullet = (Y \to J^\bullet)$, it follows that the complex $\mathrm{Hom}_R(P^\bullet, Y)$ is acyclic, as desired. □

Generalized Periodicity Theorems

The following theorem has two parts (a) and (b).

Part (a) is a common generalization of the Benson–Goodearl and Neeman's flat/projective periodicity theorem and the Šaroch–Šťovíček fp-projective periodicity theorem for coherent rings.

Part (b) is a common generalization of the (essentially Šťovíček's) fp-injective/injective periodicity theorem (for coherent rings) and the cotorsion periodicity theorem of Bazzoni, Cortés-Izurdiaga, and Estrada.

Generalized Periodicity Theorems

Theorem (P. 2023)

Let R be a ring and $\mathcal{S}$ be a class of strongly finitely presented (FP$_\infty$) R-modules containing the free R-module R and closed under extensions and syzygies. Let $(\mathcal{A}, \mathcal{B})$ be the (hereditary complete) cotorsion pair generated by $\mathcal{S}$ in $\mathrm{Mod}\text{-}R$.

Put $\mathcal{C} = \varinjlim \mathcal{S} = \varinjlim \mathcal{A}$. Then, by a result of Angeleri Hügel and Trlifaj, there exists a complete cotorsion pair $(\mathcal{C}, \mathcal{D})$ in $\mathrm{Mod}\text{-}R$. One can show that this cotorsion pair is also hereditary.

So $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{B} \supset \mathcal{D}$.

Let $0 \longrightarrow M \longrightarrow L \longrightarrow M \longrightarrow 0$ be a short exact sequence of modules. Then it is claimed that:

(a) *If $L \in \mathcal{A}$ and $M \in \mathcal{C}$, then $M \in \mathcal{A}$.*

(b) *If $L \in \mathcal{D}$ and $M \in \mathcal{B}$, then $M \in \mathcal{D}$.*

Generalized Periodicity Theorems

Brief sketch of proof.

Part (a) is provable by an argument along the lines of the proof of the main fp-projective/weakly fp-projective periodicity theorem above.

The main difference is that, to prove part (a), one needs to do a proof like the one above within the class $\mathcal{C}$ viewed as an exact category (instead of the whole category $\mathrm{Mod}\text{-}R$).

Part (b) follows from a theorem from the paper of Bazzoni, Cortés-Izurdiaga, and Estrada. □

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