Fp-projective periodicity

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Periodicity Theorems: General Setting

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Not necessarily.

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Still, quite a few positive results are known.

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- Over any R, if L is fp-projective, then M is weakly fp-projective. [Bazzoni, Hrbek, and P. 2022]

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The proof of Šaroch and Šťovíček is a complicated set-theoretic argument by induction on the cardinals. Our proof is a much simpler homological or homotopical argument using Neeman's theorem and the Hill lemma.

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The point is that right R-modules are the same things as flat contravariant additive functors $\mathcal{T} = \mathrm{mod}\text{-}R \longrightarrow \mathrm{Ab}$. Pure-acyclic complexes of R-modules correspond to acyclic complexes of flat $\mathcal{T}$-modules with flat $\mathcal{T}$-modules of cocycles; and pure-projective R-modules correspond to projective $\mathcal{T}$-modules. □

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Put $\mathcal{C} = \varinjlim \mathcal{S} = \varinjlim \mathcal{A}$. Then, by a result of Angeleri Hügel and Trlifaj, there exists a complete cotorsion pair $(\mathcal{C},\mathcal{D})$ in $\mathrm{Mod}\text{-}R$.

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Brief sketch of proof.

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Part (b) follows from a theorem from the paper of Bazzoni, Cortés-Izurdiaga, and Estrada. □

L. Angeleri Hügel, J. Trlifaj. Direct limits of modules of finite projective dimension. *Rings, modules, algebras, and abelian groups*, Proceedings of the Algebra Conference in Venezia, Lecture Notes in Pure and Appl. Math. **236**, Marcel Dekker, New York–Basel, 2004, p. 27–44.

S. Bazzoni, M. Cortés-Izurdiaga, S. Estrada. Periodic modules and acyclic complexes. *Algebras and Represent. Theory* **23**, #5, p. 1861–1883, 2020. `arXiv:1704.06672 [math.RA]`

S. Bazzoni, M. Hrbek, L. Positselski. Fp-projective periodicity. Electronic preprint `arXiv:2212.02300 [math.CT]`.

D. J. Benson, K. R. Goodearl. Periodic flat modules, and flat modules for finite groups. *Pacific Journ. of Math.* **196**, #1, p. 45–67, 2000.

A. Neeman. The homotopy category of flat modules, and Grothendieck duality. *Inventiones Math.* **174**, #2, p. 225–308, 2008.

L. Positselski. Generalized periodicity theorems. Electronic preprint arXiv:2301.00708 [math.RA].

D. Simson. Pure-periodic modules and a structure of pure-projective resolutions. *Pacific Journ. of Math.* **207**, #1, p. 235–256, 2002.

J. Šaroch, J. Šťovíček. Singular compactness and definability for Σ-cotorsion and Gorenstein modules. *Selecta Math. (New Ser.)* **26**, #2, Paper No. 23, 40 pp., 2020. arXiv:1804.09080 [math.RT]

J. Šťovíček. Deconstructibility and the Hill Lemma in Grothendieck categories. *Forum Math.* **25**, #1, p. 193–219, 2013. arXiv:1005.3251 [math.CT]

J. Šťovíček. On purity and applications to coderived and singularity categories. Electronic preprint arXiv:1412.1615 [math.CT].