Covers, Direct Limits, and Pro-Perfect Topological Rings

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Prior Art (Inspiration):

J. Šaroch. Approximations and Mittag-Leffler conditions — the tools. *Israel Journ. of Math.* **226**, p. 737–756, 2018.

L. Angeleri Hügel, J. Šaroch, J. Trlifaj. Approximations and Mittag-Leffler conditions — the applications. *Israel Journ. of Math.* **226**, p. 757–780, 2018.

Results have some similarity, but the assumptions are different; completely different methods.

Background: Precovers and Covers

Let $\mathcal{A}$ be a category and $\mathcal{T} \subset \mathcal{A}$ be a class of objects. A morphism $p\colon T \longrightarrow M$ in $\mathcal{A}$ is said to be a $\mathcal{T}$-precover (of M) if $T \in \mathcal{T}$ and every morphism $p'\colon T' \longrightarrow M$ with $T' \in \mathcal{T}$ factors through p (i.e., there exists a morphism $f\colon T' \longrightarrow T$ such that $p' = pf$).

A morphism $p\colon T \longrightarrow M$ is said to be a $\mathcal{T}$-cover if it is a $\mathcal{T}$-precover and every endomorphism $f\colon T \longrightarrow T$ for which $pf = p$ is an isomorphism (i.e., f is invertible).

A class of objects $\mathcal{T}$ in a category $\mathcal{A}$ is said to be precovering (resp., covering) if every object of $\mathcal{A}$ has a $\mathcal{T}$-precover (resp., cover).

Covers in Module Categories

Theorem (Enochs, 1981)

Let A be an associative ring and $\mathcal{T} \subset \mathcal{A} = A\text{-}\mathrm{Mod}$ be a class of left A-modules. Assume that $\mathcal{T}$ is precovering and closed under filtered colimits. Then $\mathcal{T}$ is covering.

Conjecture (Enochs, late 1990's)

Every covering class in $\mathcal{A} = A\text{-}\mathrm{Mod}$ is closed under filtered colimits.

Covers in Locally Presentable Abelian Categories

Theorem (J. Rosický & L.P., 2015)

Let $\mathcal{A}$ be a locally presentable category and $\mathcal{T} \subset \mathcal{A}$ be a class of objects. Assume that $\mathcal{T}$ is precovering and closed under filtered colimits. Then $\mathcal{T}$ is covering.

Main Conjecture

Let $\mathcal{B}$ be a locally presentable abelian category with enough projective objects. Then the class of all projective objects in $\mathcal{B}$ is covering if and only if it is closed under filtered colimits.

Notice that in an abelian category with enough projective objects the class of projective objects is always precovering (any epimorphism with a projective domain is a precover). Hence the “if” assertion of the conjecture follows from the theorem.

The “only if” assertion is the nontrivial part of the conjecture.

Left Perfect Rings

Main Conjecture is true for the categories of modules over associative rings $\mathcal{B} = R\text{-}\mathrm{Mod}$. This is a classical result:

Theorem (Bass, 1960)

The following conditions are equivalent for an associative ring R:

1. *all flat left R-modules have projective covers;*
2. *all left R-modules have projective covers;*
3. *all flat left R-modules are projective.*

Associative rings R satisfying these equivalent conditions are called left perfect.

Since for any ring R the flat R-modules are precisely the filtered colimits of projective modules, Main Conjecture holds for module categories in view of the equivalence (ii) $\Longleftrightarrow$ (iii).

Overview

The idea of our approach to proving (some particular cases of) the Enochs conjecture is to first prove Main Conjecture for the categories of contramodules over certain topological rings, by partly deducing it from Bass' results about left perfect rings and partly arguing along the lines of Bass' arguments, extending them from the module to the contramodule case.

Then we deduce particular cases of the Enochs conjecture from particular cases of Main Conjecture, using what might be called a "generalized tilting" technology.

Now let us have a little further discussion of Bass' results before proceeding to explain their generalization to contramodules and the application to the Enochs conjecture.

Bass Flat Modules

A Bass flat left R-module B is a filtered colimit of a chain of free left R-modules with one generator, indexed by the ordinal ω of nonnegative integers:

$$B = \varinjlim \,(R \xrightarrow{*a_1} R \xrightarrow{*a_2} R \xrightarrow{*a_2} \cdots),$$

where $a_n \in R$ and $*a\colon R \longrightarrow R$ denotes the operator of right multiplication with a.

Theorem (Bass, 1960 – cont'd)

The following conditions are also equivalent to the above three conditions for an associative ring R:

- (i)$^\flat$ *all Bass flat left R-modules have projective covers;*
- (iii)$^\flat$ *all Bass flat left R-modules are projective.*

T-Nilpotent Ideals

A ring without unit J is said to be left T-nilpotent if for every sequence of elements $a_1, a_2, \ldots \in J$ there exists $m \geqslant 1$ such that $a_1 a_2 \cdots a_m = 0$.

Theorem (Bass, 1960 – fin'd)

The following condition is also equivalent to the above:

- iv. *the Jacobson radical H of the ring R is left T-nilpotent, and the quotient ring R/H is semisimple Artinian.*

The following "T-nilpotent Nakayama lemma" explains the importance of the T-nilpotency condition.

Lemma (Bass, 1960)

- a. *J is left T-nilpotent if and only if for every nonzero right J-module N there exists $0 \neq x \in N$ such that $xJ = 0$.*
- b. *If J is left T-nilpotent, then for any nonzero left J-module M one has $JM \subsetneq M$.*

Contramodules over Topological Rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

For any set X, denote by $\mathfrak{R}[[X]]$ the set of all infinite formal linear combinations $\sum_{x\in X} r_x x$ of elements of X with the coefficients forming a family converging to zero in the topology of $\mathfrak{R}$, i.e., for any neighborhood of zero $\mathfrak{U}\subset\mathfrak{R}$ the set $\{x \mid r_x\notin\mathfrak{U}\}$ must be finite.

It follows from the conditions on the topology of $\mathfrak{R}$ that there is a well-defined "opening of parentheses" map

$$\phi_X\colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$$

performing infinite summations in the conventional sense of the topology of $\mathfrak{R}$ to compute the coefficients. There is also the obvious "point measure" map $\epsilon_X\colon X\longrightarrow\mathfrak{R}[[X]]$. The natural transformations ϕ and ϵ define the structure of a monad on the functor $X\longmapsto\mathfrak{R}[[X]]\colon \mathrm{Sets}\longrightarrow\mathrm{Sets}$.

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

A left contramodule over the topological ring $\mathfrak{R}$ is an algebra/module over the monad $X \longmapsto \mathfrak{R}[[X]]$ on Sets, that is

- a set $\mathfrak{C}$
- endowed with a contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$
- satisfying the contraassociativity equation $\pi \circ \mathfrak{R}[[\pi]] = \pi \circ \phi_{\mathfrak{C}}$

$$\mathfrak{R}[[\mathfrak{R}[[\mathfrak{C}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$$

- and the contraunitality equation $\pi \circ \epsilon_{\mathfrak{C}} = \mathrm{id}_{\mathfrak{C}}$

$$\mathfrak{C} \longrightarrow \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}.$$

The composition of the contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$ with the obvious embedding $\mathfrak{R}[\mathfrak{C}] \longrightarrow \mathfrak{R}[[\mathfrak{C}]]$ defines the underlying left $\mathfrak{R}$-module structure on every left $\mathfrak{R}$-contramodule.

Contramodules over Topological Rings

For any set X, the set $\mathfrak{R}[[X]]$ has a natural left $\mathfrak{R}$-contramodule structure with the contraaction map $\pi = \phi_X \colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$. The left $\mathfrak{R}$-contramodule $\mathfrak{R}[[X]]$ is called the free left $\mathfrak{R}$-contramodule generated by the set X.

The category of left $\mathfrak{R}$-contramodules is abelian with exact functors of infinite products and enough projective objects, which are the direct summands of the free $\mathfrak{R}$-contramodules $\mathfrak{R}[[X]]$. The forgetful functor $\mathfrak{R}\text{-}\mathrm{Contra} \longrightarrow \mathfrak{R}\text{-}\mathrm{Mod}$ is exact and preserves infinite products.

Let λ be the cardinality of a base of neighborhoods of zero in $\mathfrak{R}$. Then the category $\mathfrak{R}\text{-}\mathrm{Contra}$ is λ^+-locally presentable. The free $\mathfrak{R}$-contramodule with one generator $\mathfrak{R} = \mathfrak{R}[[*]]$ is a λ^+-presentable projective generator of $\mathfrak{R}\text{-}\mathrm{Contra}$.

Contratensor Product & Flat Contramodules

A right $\mathfrak{R}$-module N is called discrete if the action map $N \times \mathfrak{R} \longrightarrow N$ is continuous in the given topology of $\mathfrak{R}$ and the discrete topology of N, i.e., if the annihilator of any element of N is an open right ideal in $\mathfrak{R}$.

The contratensor product of a discrete right $\mathfrak{R}$-module N and a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ is an abelian group $N \odot_{\mathfrak{R}} \mathfrak{C}$ constructed as the cokernel of (the difference of) two natural maps

$$N \otimes_{\mathbb{Z}} \mathfrak{R}[[\mathfrak{C}]] \rightrightarrows N \otimes_{\mathbb{Z}} \mathfrak{C},$$

one of which is induced by the left contraaction map π and the other one by the discrete right action of $\mathfrak{R}$ in N.

A left $\mathfrak{R}$-contramodule $\mathfrak{C}$ is called flat if $- \odot_{\mathfrak{R}} \mathfrak{C}$ is an exact functor $\mathrm{Discr}\text{-}\mathfrak{R} \longrightarrow \mathrm{Ab}$. All projective left $\mathfrak{R}$-contramodules are flat.

The class of flat $\mathfrak{R}$-contramodules is closed under filtered colimits in $\mathfrak{R}\text{-}\mathrm{Contra}$, so all filtered colimits of projective $\mathfrak{R}$-contramodules are flat. It is not known whether the converse is true.

Flat Covers in Contramodule Categories (a Digression)

Theorem (J. Rosický & L.P., 2015)

Let $\mathfrak{R}$ be a complete, separated topological ring with a countable base of neighborhoods of zero consisting of open right ideals. Then the class of flat left $\mathfrak{R}$-contramodules is covering in $\mathfrak{R}\text{-}\mathrm{Contra}$.

It is not known whether this theorem remains true without the countability assumption. It would be sufficient to show that the class of flat left $\mathfrak{R}$-contramodules is precovering.

Bass Flat Contramodules

A Bass flat left $\mathfrak{R}$-contramodule $\mathfrak{B}$ is a filtered colimit of a chain of free left $\mathfrak{R}$-contramodules with one generator, indexed by ω:

$$\mathfrak{B} = \varinjlim \left(\mathfrak{R} \xrightarrow{*a_1} \mathfrak{R} \xrightarrow{*a_2} \mathfrak{R} \xrightarrow{*a_2} \cdots\right), \qquad a_n \in \mathfrak{R}.$$

Lemma

Let $\mathfrak{R}$ be complete, separated topological ring with a base of neighborhoods of zero consisting of open right ideals. Assume that all Bass flat left $\mathfrak{R}$-contramodules have projective covers in $\mathfrak{R}\text{-}\mathrm{Contra}$. Let $\mathfrak{J} \subset \mathfrak{R}$ be an open two-sided ideal. Then the discrete quotient ring $\mathfrak{R}/\mathfrak{J}$ is left perfect.

Idea of proof.

Define a reduction functor $\mathfrak{R}\text{-}\mathrm{Contra} \longrightarrow \mathfrak{R}/\mathfrak{J}\text{-}\mathrm{Mod}$, show that it takes projectives to projectives and projective covers to projective covers. Observe that every Bass flat left $\mathfrak{R}/\mathfrak{J}$-module is the image of a Bass flat left $\mathfrak{R}$-contramodule with respect to this functor. $\square$

Topologically T-nilpotent Ideals

A topological ring without unit $\mathfrak{J}$ is said to be topologically left T-nilpotent if for every sequence of elements $a_1, a_2, \ldots \in \mathfrak{J}$ the sequence of elements $a_1, a_1a_2, a_1a_2a_3, \ldots, a_1a_2\cdots a_m, \ldots$ converges to zero in $\mathfrak{J}$.

Lemma ("Topologically left T-nilpotent Nakayama")

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero consisting of open right ideals, and let $\mathfrak{J} \subset \mathfrak{R}$ be a closed two-sided ideal. Then

- a. *$\mathfrak{J}$ is topologically left T-nilpotent if and only if for every nonzero discrete right $\mathfrak{R}$-module N there exists $0 \neq x \in N$ such that $x\mathfrak{J} = 0$.*
- b. *If $\mathfrak{J}$ is topologically left T-nilpotent, then for any nonzero left $\mathfrak{R}$-contramodule $\mathfrak{C}$ the composition $\mathfrak{J}[[\mathfrak{C}]] \longrightarrow \mathfrak{R}[[\mathfrak{C}]] \xrightarrow{\pi} \mathfrak{C}$ is not surjective.*

Strongly Closed Subgroups

The notion of a topological group (with a base of neighborhoods of zero formed by subgroups) seems to be a bit problematic, in that it is impossible to prove certain natural properties, in general. So one has to impose them as assumptions.

Let $\mathfrak{A}$ be a topological group and $\mathfrak{B} \subset \mathfrak{A}$ be a closed subgroup. The subgroup $\mathfrak{B}$ is said to be strongly closed in $\mathfrak{A}$ if the following two conditions hold:

- the quotient group $\mathfrak{A}/\mathfrak{B}$ is complete in the quotient topology;
- for any set X, the induced map $\mathfrak{A}[[X]] \longrightarrow (\mathfrak{A}/\mathfrak{B})[[X]]$ is surjective, that is any X-indexed family of elements converging to zero in $\mathfrak{A}/\mathfrak{B}$ can be lifted to an X-indexed family of elements converging to zero in $\mathfrak{A}$.

All open subgroups are strongly closed. When $\mathfrak{A}$ has a countable base of neighborhoods of zero, all closed subgroups in $\mathfrak{A}$ are strongly closed.

Pro-Perfect Topological Rings

It is not clear what should be meant by a left pro-perfect topological ring. The obvious definition "the topological limit of a filtered diagram of discrete left perfect rings and surjective maps between them" is both too restrictive and too general. The next theorem lists the conditions which allow to prove what we want.

Theorem

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals, and let $\mathfrak{H} \subset \mathfrak{R}$ be a strongly closed two-sided ideal. Suppose that

- *the ideal $\mathfrak{H}$ is topologically left T-nilpotent; and*
- *the quotient ring $\mathfrak{S} = \mathfrak{R}/\mathfrak{H}$ is isomorphic, as a topological ring, to the topological product of a family of discrete simple Artinian rings $(S_\gamma)_{\gamma\in\Gamma}$.*

Then all flat left $\mathfrak{R}$-contramodules are projective, and all left $\mathfrak{R}$-contramodules have projective covers.

Four Special Classes of Topological Rings

To prove our main results, we will have to assume that the (separated and complete) topological associative ring $\mathfrak{R}$ satisfies one of the following conditions: either

- (a) $\mathfrak{R}$ is commutative; or
- (b) $\mathfrak{R}$ has a countable base of neighborhoods of zero consisting of open two-sided ideals; or
- (c) $\mathfrak{R}$ has a base of neighborhoods of zero consisting of open two-sided ideals, and $\mathfrak{R}$ has only a finite number of semisimple Artinian discrete quotient rings.

More generally, $\mathfrak{R}$ may belong to the following wider class of topological rings (d) containing the classes (a-c):

- (d) $\mathfrak{R}$ has a base of neighborhoods of zero consisting of open right ideals, and there is a topologically left T-tilpotent strongly closed two-sided ideal $\mathfrak{K} \subset \mathfrak{R}$ such that the quotient ring $\mathfrak{R}/\mathfrak{K}$ is isomorphic, as a topological ring, to the topological product of a family of topological rings $(\mathfrak{T}_\delta)_{\delta \in \Delta}$, each of which satisfies one of the conditions (a-c).

Main Theorem

Main Theorem

Let $\mathfrak{R}$ be a topological associative ring satisfying one of the conditions (a), (b), (c), or (d). Then the following conditions are equivalent:

1. *all Bass flat left $\mathfrak{R}$-contramodules have projective covers;*
2. *all left $\mathfrak{R}$-contramodules have projective covers;*
3. *all flat left $\mathfrak{R}$-contramodules are projective;*
4. *all the discrete quotient rings of $\mathfrak{R}$ are left perfect;*
5. *$\mathfrak{R}$ has a topologically left T-nilpotent strongly closed two-sided ideal $\mathfrak{H}$ such that $\mathfrak{R}/\mathfrak{H}$ is isomorphic to the topological product of discrete simple Artinian rings.*

The assumption of one of the conditions (a), (b), (c) or (d) is needed in order to deduce (v) from (iv).

Generalized Tilting Theory

(from a joint work with J. Šťovíček)

Let A be an associative ring and M be a left A-module. Denote by $\mathrm{Add}(M)$ the full subcategory in $A\text{-}\mathrm{Mod}$ consisting of all the direct summands of direct sums of copies of M.

Let $\mathfrak{R} = \mathrm{Hom}_A(M, M)^{\mathrm{op}}$ denote the opposite ring to the ring of endomorphisms of the A-module M (so $\mathfrak{R}$ acts in M on the right). Endow $\mathfrak{R}$ with the topology in which annihilators of finitely generated submodules of M form a base of neighborhoods of zero. Then $\mathfrak{R}$ is a complete, separated topological ring in which open right ideals form a base of neighborhoods of zero.

The category $\mathrm{Add}(M)$ is naturally equivalent to the full subcategory of projective objects in $\mathfrak{R}\text{-}\mathrm{Contra}$. This equivalence extends to a pair of adjoint functors $\Psi\colon A\text{-}\mathrm{Mod} \leftrightarrows \mathfrak{R}\text{-}\mathrm{Contra} :\Phi$, with the right adjoint functor Ψ assigning to any left A-module N the abelian group $\mathrm{Hom}_A(M, N)$, which has a natural left $\mathfrak{R}$-contramodule structure.

Abstract Corresponding Classes

(from a paper of A. Frankild and P. Jørgensen, 2002)

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories and

$$\Psi\colon \mathcal{A} \leftrightarrows \mathcal{B} :\!\Phi$$

be a pair of adjoint functors between them.

Let $\mathcal{E} \subset \mathcal{A}$ be the full subcategory of all objects $E \in \mathcal{A}$ for which the adjunction morphism $\Phi(\Psi(E)) \longrightarrow E$ is an isomorphism.

Let $\mathcal{F} \subset \mathcal{B}$ be the full subcategory of all objects $F \in \mathcal{B}$ for which the adjunction morphism $F \longrightarrow \Psi(\Phi(F))$ is an isomorphism.

Then the restrictions of the functors Φ and Ψ are mutually inverse equivalences between the categories $\mathcal{E}$ and $\mathcal{F}$,

$$\Psi|_{\mathcal{E}}\colon \mathcal{E} \cong \mathcal{F} :\!\Phi|_{\mathcal{F}}.$$

Telescope Hom Exactness Condition

Let A be an associative ring and M be a left A-module. Let f_1, f_2, ... be a sequence of A-module morphisms $f_n\colon M \longrightarrow M$.
An M-Bass A-module is a left A-module of the form

$$N = \varinjlim \left(M \xrightarrow{f_1} M \xrightarrow{f_2} M \xrightarrow{f_3} \cdots\right).$$

So there is a natural (telescope) short exact sequence of A-modules

$$0 \longrightarrow \bigoplus_{n=1}^{\infty} M \longrightarrow \bigoplus_{n=1}^{\infty} M \longrightarrow N \longrightarrow 0.$$

We will say that an A-module M satisfies the *telescope Hom exactness condition* (THEC) if, for any f_1, f_2, ..., this short exact sequence remains exact after applying the functor $\mathrm{Hom}_A(M,-)$.

The following classes of A-modules satisfy THEC:

- all Σ-rigid modules M, i.e., left A-modules for which $\mathrm{Ext}^1_A(M, M^{(\omega)}) = 0$;
- all self-pure-projective modules M, i.e., A-modules for which the functor $\mathrm{Hom}_A(M,-)$ preserves exactness of pure exact sequences $0 \longrightarrow K \longrightarrow M^{(\omega)} \longrightarrow L \longrightarrow 0$, $\;K, L \in A\text{-}\mathrm{Mod}$.

Projective Covers of Bass Contramodules

Lemma

Let M be a left A-module satisfying THEC, and let $\mathfrak{R} = \mathrm{Hom}_A(M, M)^{\mathrm{op}}$ be its topological ring of endomorphisms. Assume that every M-Bass left A-module has an $\mathrm{Add}(M)$-cover. Then any Bass flat left $\mathfrak{R}$-contramodule has a projective cover.

Proof.

Consider the pair of adjoint functors $\Psi\colon A\text{-}\mathrm{Mod} \leftrightarrows \mathfrak{R}\text{-}\mathrm{Contra} :\!\Phi$, and let $\mathcal{E} \subset A\text{-}\mathrm{Mod}$ and $\mathcal{F} \subset \mathfrak{R}\text{-}\mathrm{Contra}$ be the two corresponding classes of objects under this adjoint pair (so $\Psi\colon \mathcal{E} \cong \mathcal{F} :\!\Phi$). Fix a sequence of A-module maps $f_1, f_2, \ldots : M \longrightarrow M$, or, which is the same, a sequence of elements $a_1, a_2, \ldots \in \mathfrak{R}$. Let N and $\mathfrak{B}$ be the related M-Bass A-module and Bass flat $\mathfrak{R}$-contramodule. One always has $N = \Phi(\mathfrak{B})$. If M satisfies THEC, then one also has $\mathfrak{B} = \Psi(N)$, essentially because the functor Ψ can be computed as $\mathrm{Hom}_A(M, -)$. Hence $N \in \mathcal{E}$ and $\mathfrak{B} \in \mathcal{F}$, and it follows that Ψ takes any $\mathrm{Add}(M)$-cover of N to a projective cover of $\mathfrak{B}$. $\square$

Filtered Colimits of Modules from $\mathrm{Add}(M)$

Lemma

Let M be a left A-module, and let $\mathfrak{R} = \mathrm{Hom}_A(M, M)^{\mathrm{op}}$ be its topological ring of endomorphisms. Assume that the class of projective left $\mathfrak{R}$-contramodules is closed under filtered colimits in $\mathfrak{R}\text{-}\mathrm{Contra}$ (e.g., all flat left $\mathfrak{R}$-contramodules are projective). Then the class $\mathrm{Add}(M)$ is closed under filtered colimits in $A\text{-}\mathrm{Mod}$.

Proof.

Since the functors Φ and Ψ restrict to mutually inverse equivalences

$$\Psi\colon\ \mathrm{Add}(M) \cong \mathfrak{R}\text{-}\mathrm{Contra}_{\mathrm{proj}}\ :\!\Phi,$$

any filtered diagram in $\mathrm{Add}(M)$ can be obtained by applying Φ to a filtered diagram in $\mathfrak{R}\text{-}\mathrm{Contra}_{\mathrm{proj}}$. It remains to observe that the functor Φ preserves colimits, since it is a left adjoint. $\square$

Main Corollary (the Application to the Enochs conjecture)

Corollary

Let A be an associative ring and M be a left A-module satisfying the telescope Hom exactness condition. Let $\mathfrak{R} = \mathrm{Hom}_A(M, M)^{\mathrm{op}}$ be the topological ring of endomorphisms of M. Suppose that $\mathfrak{R}$ belongs to one of the special classes of topological rings (a), (b), (c), or (d) (e.g., $\mathfrak{R}$ is commutative). Assume that every M-Bass left A-module has an $\mathrm{Add}(M)$-cover. Then the class of left A-modules $\mathrm{Add}(M)$ is closed under filtered colimits in $A\text{-}\mathrm{Mod}$.

Proof.

Follows from Main Theorem and the two previous lemmas. □

Remark

If Main Conjecture were known to be true, one could drop the assumption of one of the conditions (a-d) in the formulation of the Corollary. The THEC assumption would still be needed.