Covers, Direct Limits, and Pro-Perfect Topological Rings

Leonid Positselski – IM AV ČR

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Joint work with Silvana Bazzoni:

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S. Bazzoni, L. Positselski. Contramodules over pro-perfect topological rings, the covering property in categorical tilting theory, and homological ring epimorphisms. arXiv:1807.10671

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Every covering class in $\mathcal{A} = A\text{-}\mathrm{Mod}$ is closed under filtered colimits.

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The "only if" assertion is the nontrivial part of the conjecture.

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Now let us have a little further discussion of Bass' results before proceeding to explain their generalization to contramodules and the application to the Enochs conjecture.

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- (iv) *all Bass flat left R-modules have projective covers;*
- (v) *all Bass flat left R-modules are projective.*

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The composition of the contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$ with the obvious embedding $\mathfrak{R}[\mathfrak{C}] \longrightarrow \mathfrak{R}[[\mathfrak{C}]]$ defines the underlying left $\mathfrak{R}$-module structure on every left $\mathfrak{R}$-contramodule.

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Contratensor Product & Flat Contramodules

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A left $\mathfrak{R}$-contramodule $\mathfrak{C}$ is called flat if $- \odot_{\mathfrak{R}} \mathfrak{C}$ is an exact functor $\mathrm{Discr}\text{-}\mathfrak{R} \longrightarrow \mathrm{Ab}$. All projective left $\mathfrak{R}$-contramodules are flat.

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The contratensor product of a discrete right $\mathfrak{R}$-module N and a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ is an abelian group $N \odot_{\mathfrak{R}} \mathfrak{C}$ constructed as the cokernel of (the difference of) two natural maps

$$N \otimes_{\mathbb{Z}} \mathfrak{R}[[\mathfrak{C}]] \rightrightarrows N \otimes_{\mathbb{Z}} \mathfrak{C},$$

one of which is induced by the left contraaction map π and the other one by the discrete right action of $\mathfrak{R}$ in N.

A left $\mathfrak{R}$-contramodule $\mathfrak{C}$ is called flat if $- \odot_{\mathfrak{R}} \mathfrak{C}$ is an exact functor $\mathrm{Discr}\text{-}\mathfrak{R} \longrightarrow \mathrm{Ab}$. All projective left $\mathfrak{R}$-contramodules are flat.

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All open subgroups are strongly closed.

Strongly Closed Subgroups

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The assumption of one of the conditions (a), (b), (c) or (d) is needed in order to deduce (v) from (iv).

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(from a joint work with J. Šťovíček)

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Let $\mathcal{F} \subset \mathcal{B}$ be the full subcategory of all objects $F \in \mathcal{B}$ for which the adjunction morphism $F \longrightarrow \Psi(\Phi(F))$ is an isomorphism.

Then the restrictions of the functors Φ and Ψ are mutually inverse equivalences between the categories $\mathcal{E}$ and $\mathcal{F}$,

$$\Psi|_{\mathcal{E}}\colon \mathcal{E} \cong \mathcal{F} :\!\Phi|_{\mathcal{F}}.$$

Telescope Hom Exactness Condition

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One always has $N = \Phi(\mathfrak{B})$. If M satisfies THEC, then one also has $\mathfrak{B} = \Psi(N)$, essentially because the functor Ψ can be computed as $\mathrm{Hom}_A(M,-)$. Hence $N \in \mathcal{E}$ and $\mathfrak{B} \in \mathcal{F}$, and it follows that Ψ takes any $\mathrm{Add}(M)$-cover of N to a projective cover of $\mathfrak{B}$. □

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Main Corollary (the Application to the Enochs conjecture)

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Let A be an associative ring and M be a left A-module satisfying the telescope Hom exactness condition. Let $\mathfrak{R} = \mathrm{Hom}_A(M, M)^{\mathrm{op}}$ be the topological ring of endomorphisms of M. Suppose that $\mathfrak{R}$ belongs to one of the special classes of topological rings (a), (b), (c), or (d) (e.g., $\mathfrak{R}$ is commutative). Assume that every M-Bass left A-module has an $\mathrm{Add}(M)$-cover. Then the class of left A-modules $\mathrm{Add}(M)$ is closed under filtered colimits in $A\text{-}\mathrm{Mod}$.

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Remark

If Main Conjecture were known to be true, one could drop the assumption of one of the conditions (a-d) in the formulation of the Corollary. The THEC assumption would still be needed.