The hat construction, derived categories of the second kind, and Koszul duality

Leonid Positselski – IM CAS, Prague

HART Seminar/Hybrid seminar on derived Koszul duality, Thessaloniki

April 7, 2025

Curved DG-structures

Curved DG-structures

A CDG-ring $B^\blacklozenge = (B, d, h)$ is

Curved DG-structures

A CDG-ring $B^\blacklozenge = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$

Curved DG-structures

A CDG-ring $B^\blacklozenge = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with

Curved DG-structures

A CDG-ring $B^\blacklozenge = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|} a d(b)$ for all a, $b \in B$

Curved DG-structures

A CDG-ring $B^{\blacklozenge} = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|} a d(b)$ for all a, $b \in B$
- and an element $h \in B^2$

Curved DG-structures

A CDG-ring $B^\bullet = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|} a d(b)$ for all a, $b \in B$
- and an element $h \in B^2$ such that

Curved DG-structures

A CDG-ring $B^\blacklozenge = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ for all a, $b \in B$
- and an element $h \in B^2$ such that
- $d^2(b) = [h, b] = hb - bh$ for all $b \in B$

Curved DG-structures

A CDG-ring $B^\bullet = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ for all a, $b \in B$
- and an element $h \in B^2$ such that
- $d^2(b) = [h, b] = hb - bh$ for all $b \in B$
- and $d(h) = 0$.

Curved DG-structures

A CDG-ring $B^\bullet = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ for all a, $b \in B$
- and an element $h \in B^2$ such that
- $d^2(b) = [h, b] = hb - bh$ for all $b \in B$
- and $d(h) = 0$.

h is called the curvature element.

Curved DG-structures

A CDG-ring $B^\blacklozenge = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ for all a, $b \in B$
- and an element $h \in B^2$ such that
- $d^2(b) = [h, b] = hb - bh$ for all $b \in B$
- and $d(h) = 0$.

h is called the curvature element.

An A_∞-algebra is a graded vector space with the operations $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $n = 1, 2, \dots$

Curved DG-structures

A CDG-ring $B^\bullet = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ for all a, $b \in B$
- and an element $h \in B^2$ such that
- $d^2(b) = [h, b] = hb - bh$ for all $b \in B$
- and $d(h) = 0$.

h is called the curvature element.

An A_∞-algebra is a graded vector space with the operations $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $n = 1, 2, \ldots$

A CDG-algebra has $m_0 = h$, $m_1 = d$, and m_2.

Curved DG-structures

A CDG-ring $B^\bullet = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|} a d(b)$ for all a, $b \in B$
- and an element $h \in B^2$ such that
- $d^2(b) = [h, b] = hb - bh$ for all $b \in B$
- and $d(h) = 0$.

h is called the curvature element.

An A_∞-algebra is a graded vector space with the operations $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $n = 1, 2, \ldots$

A CDG-algebra has $m_0 = h$, $m_1 = d$, and m_2.

[Getzler–Jones '90, L.P. '93]

Curved DG-structures

A left CDG-module $M^\bullet = (M, d_M)$ over a CDG-ring (B, d_B, h) is

Curved DG-structures

A left CDG-module $M^\blacklozenge = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with

Curved DG-structures

A left CDG-module $M^\bullet = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M \colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B, \ \ m \in M$

Curved DG-structures

A left CDG-module $M^\bullet = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M \colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B, \;\; m \in M$
- such that $d_M^2(m) = hm$ for all $m \in M$.

Curved DG-structures

A left CDG-module $M^\bullet = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M \colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B, \quad m \in M$
- such that $d_M^2(m) = hm$ for all $m \in M$.

A right CDG-module $N^\bullet = (N, d_N)$ over a CDG-ring (B, d_B, h) is

Curved DG-structures

A left CDG-module $M^\bullet = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M\colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B$, $m \in M$
- such that $d_M^2(m) = hm$ for all $m \in M$.

A right CDG-module $N^\bullet = (N, d_N)$ over a CDG-ring (B, d_B, h) is

- a graded right B-module $N = \bigoplus_{i=-\infty}^{\infty} N^i$ endowed with
- an d_B-derivation $d_N\colon N^i \longrightarrow N^{i+1}$,
  $d_N(nb) = d_N(n)b + (-1)^{|n|} n d_B(b)$ for all $b \in B$, $n \in N$

Curved DG-structures

A left CDG-module $M^\bullet = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M \colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B$, $m \in M$
- such that $d_M^2(m) = hm$ for all $m \in M$.

A right CDG-module $N^\bullet = (N, d_N)$ over a CDG-ring (B, d_B, h) is

- a graded right B-module $N = \bigoplus_{i=-\infty}^{\infty} N^i$ endowed with
- an d_B-derivation $d_N \colon N^i \longrightarrow N^{i+1}$,
  $d_N(nb) = d_N(n)b + (-1)^{|n|} n d_B(b)$ for all $b \in B$, $n \in N$
- such that $d_N^2(n) = -nh$ for all $n \in N$.

Curved DG-structures

A left CDG-module $M^\bullet = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M \colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B, \quad m \in M$
- such that $d_M^2(m) = hm$ for all $m \in M$.

A right CDG-module $N^\bullet = (N, d_N)$ over a CDG-ring (B, d_B, h) is

- a graded right B-module $N = \bigoplus_{i=-\infty}^{\infty} N^i$ endowed with
- an d_B-derivation $d_N \colon N^i \longrightarrow N^{i+1}$,
  $d_N(nb) = d_N(n)b + (-1)^{|n|} n d_B(b)$ for all $b \in B, \quad n \in N$
- such that $d_N^2(n) = -nh$ for all $n \in N$.

A CDG-ring $B^\bullet = (B, d, h)$ is naturally neither a left, nor a right CDG-module over itself

Curved DG-structures

A left CDG-module $M^\bullet = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M\colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B, \quad m \in M$
- such that $d_M^2(m) = hm$ for all $m \in M$.

A right CDG-module $N^\bullet = (N, d_N)$ over a CDG-ring (B, d_B, h) is

- a graded right B-module $N = \bigoplus_{i=-\infty}^{\infty} N^i$ endowed with
- an d_B-derivation $d_N\colon N^i \longrightarrow N^{i+1}$,
  $d_N(nb) = d_N(n)b + (-1)^{|n|} n d_B(b)$ for all $b \in B, \quad n \in N$
- such that $d_N^2(n) = -nh$ for all $n \in N$.

A CDG-ring $B^\bullet = (B, d, h)$ is naturally neither a left, nor a right CDG-module over itself (because the formulas for the square of the differential do not match).

Curved DG-structures

A left CDG-module $M^\blacklozenge = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M \colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B$, $m \in M$
- such that $d_M^2(m) = hm$ for all $m \in M$.

A right CDG-module $N^\blacklozenge = (N, d_N)$ over a CDG-ring (B, d_B, h) is

- a graded right B-module $N = \bigoplus_{i=-\infty}^{\infty} N^i$ endowed with
- an d_B-derivation $d_N \colon N^i \longrightarrow N^{i+1}$,
  $d_N(nb) = d_N(n)b + (-1)^{|n|} n d_B(b)$ for all $b \in B$, $n \in N$
- such that $d_N^2(n) = -nh$ for all $n \in N$.

A CDG-ring $B^\blacklozenge = (B, d, h)$ is naturally neither a left, nor a right CDG-module over itself (because the formulas for the square of the differential do not match). But $B^\blacklozenge$ has a natural structure of CDG-bimodule over itself.

Curved DG-structures occur in connection with

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections: if M is a smooth real manifold or a smooth algebraic variety over a field

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections: if M is a smooth real manifold or a smooth algebraic variety over a field, $\mathcal{E}$ is a vector bundle on M, and $\nabla_{\mathcal{E}}$ is a connection in $\mathcal{E}$

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections: if M is a smooth real manifold or a smooth algebraic variety over a field, $\mathcal{E}$ is a vector bundle on M, and $\nabla_{\mathcal{E}}$ is a connection in $\mathcal{E}$, then the ring $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ of differential forms with coefficients in the bundle of endomorphisms of $\mathcal{E}$ is a CDG-ring

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections: if M is a smooth real manifold or a smooth algebraic variety over a field, $\mathcal{E}$ is a vector bundle on M, and $\nabla_{\mathcal{E}}$ is a connection in $\mathcal{E}$, then the ring $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ of differential forms with coefficients in the bundle of endomorphisms of $\mathcal{E}$ is a CDG-ring with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ and the curvature element $h = h_{\nabla_{\mathcal{E}}} \in \Omega^2(M, \mathcal{E}nd(\mathcal{E}))$

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections: if M is a smooth real manifold or a smooth algebraic variety over a field, $\mathcal{E}$ is a vector bundle on M, and $\nabla_{\mathcal{E}}$ is a connection in $\mathcal{E}$, then the ring $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ of differential forms with coefficients in the bundle of endomorphisms of $\mathcal{E}$ is a CDG-ring with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ and the curvature element $h = h_{\nabla_{\mathcal{E}}} \in \Omega^2(M, \mathcal{E}nd(\mathcal{E}))$, while $(\Omega(M, \mathcal{E}), d_{\nabla_{\mathcal{E}}})$ is a CDG-module over $\Omega(M, \mathcal{E}nd(\mathcal{E}))$;

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections: if M is a smooth real manifold or a smooth algebraic variety over a field, $\mathcal{E}$ is a vector bundle on M, and $\nabla_{\mathcal{E}}$ is a connection in $\mathcal{E}$, then the ring $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ of differential forms with coefficients in the bundle of endomorphisms of $\mathcal{E}$ is a CDG-ring with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ and the curvature element $h = h_{\nabla_{\mathcal{E}}} \in \Omega^2(M, \mathcal{E}nd(\mathcal{E}))$, while $(\Omega(M, \mathcal{E}), d_{\nabla_{\mathcal{E}}})$ is a CDG-module over $\Omega(M, \mathcal{E}nd(\mathcal{E}))$;
- matrix factorizations

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections: if M is a smooth real manifold or a smooth algebraic variety over a field, $\mathcal{E}$ is a vector bundle on M, and $\nabla_{\mathcal{E}}$ is a connection in $\mathcal{E}$, then the ring $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ of differential forms with coefficients in the bundle of endomorphisms of $\mathcal{E}$ is a CDG-ring with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ and the curvature element $h = h_{\nabla_{\mathcal{E}}} \in \Omega^2(M, \mathcal{E}nd(\mathcal{E}))$, while $(\Omega(M, \mathcal{E}), d_{\nabla_{\mathcal{E}}})$ is a CDG-module over $\Omega(M, \mathcal{E}nd(\mathcal{E}))$;
- matrix factorizations, which are the CDG-modules over the $\mathbb{Z}/2$-graded CDG-ring $(B = B^0,\ d = 0,\ h = w)$

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections: if M is a smooth real manifold or a smooth algebraic variety over a field, $\mathcal{E}$ is a vector bundle on M, and $\nabla_{\mathcal{E}}$ is a connection in $\mathcal{E}$, then the ring $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ of differential forms with coefficients in the bundle of endomorphisms of $\mathcal{E}$ is a CDG-ring with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ and the curvature element $h = h_{\nabla_{\mathcal{E}}} \in \Omega^2(M, \mathcal{E}nd(\mathcal{E}))$, while $(\Omega(M, \mathcal{E}), d_{\nabla_{\mathcal{E}}})$ is a CDG-module over $\Omega(M, \mathcal{E}nd(\mathcal{E}))$;
- matrix factorizations, which are the CDG-modules over the $\mathbb{Z}/2$-graded CDG-ring $(B = B^0,\ d = 0,\ h = w)$, where B^0 is an associative ring and $w \in B^0$ is a central element ("the potential").

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ (the supercommutator)

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the change-of-connection element.

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the change-of-connection element. All morphisms of CDG-rings of the form $(\mathrm{id}, a)\colon (B, d_B, h_B) \longrightarrow (B, d'_B, h'_B)$ are isomorphisms

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the change-of-connection element. All morphisms of CDG-rings of the form $(\mathrm{id}, a)\colon (B, d_B, h_B) \longrightarrow (B, d'_B, h'_B)$ are isomorphisms; these are called change-of-connection isomorphisms.

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the change-of-connection element. All morphisms of CDG-rings of the form $(\mathrm{id}, a)\colon (B, d_B, h_B) \longrightarrow (B, d'_B, h'_B)$ are isomorphisms; these are called change-of-connection isomorphisms.

The embedding functor $\mathrm{DG\text{-}rings} \longrightarrow \mathrm{CDG\text{-}rings}$

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the change-of-connection element. All morphisms of CDG-rings of the form $(\mathrm{id}, a)\colon (B, d_B, h_B) \longrightarrow (B, d'_B, h'_B)$ are isomorphisms; these are called change-of-connection isomorphisms.

The embedding functor $\mathrm{DG\text{-}rings} \longrightarrow \mathrm{CDG\text{-}rings}$ is faithful but not fully faithful

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the change-of-connection element. All morphisms of CDG-rings of the form $(\mathrm{id}, a)\colon (B, d_B, h_B) \longrightarrow (B, d'_B, h'_B)$ are isomorphisms; these are called change-of-connection isomorphisms.

The embedding functor $\mathrm{DG\text{-}rings} \longrightarrow \mathrm{CDG\text{-}rings}$ is faithful but not fully faithful: nonisomorphic DG-rings may be isomorphic as CDG-rings.

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the change-of-connection element. All morphisms of CDG-rings of the form $(\mathrm{id}, a)\colon (B, d_B, h_B) \longrightarrow (B, d'_B, h'_B)$ are isomorphisms; these are called change-of-connection isomorphisms.

The embedding functor $\mathrm{DG\text{-}rings} \longrightarrow \mathrm{CDG\text{-}rings}$ is faithful but not fully faithful: nonisomorphic DG-rings may be isomorphic as CDG-rings.

The construction of the DG-category of DG-modules over a DG-ring extends to CDG-rings

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the change-of-connection element. All morphisms of CDG-rings of the form $(\mathrm{id}, a)\colon (B, d_B, h_B) \longrightarrow (B, d'_B, h'_B)$ are isomorphisms; these are called change-of-connection isomorphisms.

The embedding functor $\mathrm{DG\text{-}rings} \longrightarrow \mathrm{CDG\text{-}rings}$ is faithful but not fully faithful: nonisomorphic DG-rings may be isomorphic as CDG-rings.

The construction of the DG-category of DG-modules over a DG-ring extends to CDG-rings: CDG-modules over a CDG-ring form a DG-category.

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the change-of-connection element. All morphisms of CDG-rings of the form $(\mathrm{id}, a)\colon (B, d_B, h_B) \longrightarrow (B, d'_B, h'_B)$ are isomorphisms; these are called change-of-connection isomorphisms.

The embedding functor $\mathrm{DG\text{-}rings} \longrightarrow \mathrm{CDG\text{-}rings}$ is faithful but not fully faithful: nonisomorphic DG-rings may be isomorphic as CDG-rings.

The construction of the DG-category of DG-modules over a DG-ring extends to CDG-rings: CDG-modules over a CDG-ring form a DG-category. (In particular, the DG-categories of DG-modules over CDG-isomorphic DG-rings are isomorphic.)

Curved DG-structures

To be precise, the complex of morphisms $\mathrm{Hom}_B^\bullet(K^\bullet, L^\bullet)$ between CDG-modules (K, d_K) and (L, d_L)

Curved DG-structures

To be precise, the complex of morphisms $\mathrm{Hom}_B^\bullet(K^\blacklozenge, L^\blacklozenge)$ between CDG-modules (K, d_K) and (L, d_L) in the DG-category $B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of left CDG-modules over a CDG-ring $B^\blacklozenge$

To be precise, the complex of morphisms $\mathrm{Hom}_B^\bullet(K^\blacklozenge, L^\blacklozenge)$ between CDG-modules (K, d_K) and (L, d_L) in the DG-category $B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of left CDG-modules over a CDG-ring $B^\blacklozenge$ is defined by the rules:

- $\mathrm{Hom}_B^n(K, L)$ is the group of all homogeneous maps $f\colon K \longrightarrow L$ of degree n

Curved DG-structures

To be precise, the complex of morphisms $\mathrm{Hom}_B^\bullet(K^\bullet, L^\bullet)$ between CDG-modules (K, d_K) and (L, d_L) in the DG-category $B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of left CDG-modules over a CDG-ring $B^\bullet$ is defined by the rules:

- $\mathrm{Hom}_B^n(K, L)$ is the group of all homogeneous maps $f\colon K \longrightarrow L$ of degree n
- commuting with the action of the graded ring B

To be precise, the complex of morphisms $\mathrm{Hom}_B^\bullet(K^\bullet, L^\bullet)$ between CDG-modules (K, d_K) and (L, d_L) in the DG-category $B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of left CDG-modules over a CDG-ring $B^\bullet$ is defined by the rules:

- $\mathrm{Hom}_B^n(K, L)$ is the group of all homogeneous maps $f\colon K \longrightarrow L$ of degree n
- commuting with the action of the graded ring B with the sign rule $f(bk) = (-1)^{n|b|} bf(k)$ for all $b \in B^{|b|}$ and $k \in K$

Curved DG-structures

To be precise, the complex of morphisms $\mathrm{Hom}_B^\bullet(K^\bullet, L^\bullet)$ between CDG-modules (K, d_K) and (L, d_L) in the DG-category $B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of left CDG-modules over a CDG-ring $B^\bullet$ is defined by the rules:

- $\mathrm{Hom}_B^n(K, L)$ is the group of all homogeneous maps $f\colon K \longrightarrow L$ of degree n
- commuting with the action of the graded ring B with the sign rule $f(bk) = (-1)^{n|b|} bf(k)$ for all $b \in B^{|b|}$ and $k \in K$,
- while the differential on $\mathrm{Hom}_B^n(K, L)$ is

Curved DG-structures

To be precise, the complex of morphisms $\mathrm{Hom}_B^\bullet(K^\bullet, L^\bullet)$ between CDG-modules (K, d_K) and (L, d_L) in the DG-category $B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of left CDG-modules over a CDG-ring $B^\bullet$ is defined by the rules:

- $\mathrm{Hom}_B^n(K, L)$ is the group of all homogeneous maps $f\colon K \longrightarrow L$ of degree n
- commuting with the action of the graded ring B with the sign rule $f(bk) = (-1)^{n|b|}bf(k)$ for all $b \in B^{|b|}$ and $k \in K$,
- while the differential on $\mathrm{Hom}_B^n(K, L)$ is, as usually, $d(f)(k) = d_L(f(k)) - (-1)^{|n|}f(d_K(k)) \in \mathrm{Hom}_B^{n+1}(K, L)$.

Curved DG-structures

To be precise, the complex of morphisms $\mathrm{Hom}_B^\bullet(K^\bullet, L^\bullet)$ between CDG-modules (K, d_K) and (L, d_L) in the DG-category $B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of left CDG-modules over a CDG-ring $B^\bullet$ is defined by the rules:

- $\mathrm{Hom}_B^n(K, L)$ is the group of all homogeneous maps $f\colon K \longrightarrow L$ of degree n
- commuting with the action of the graded ring B with the sign rule $f(bk) = (-1)^{n|b|} bf(k)$ for all $b \in B^{|b|}$ and $k \in K$,
- while the differential on $\mathrm{Hom}_B^n(K, L)$ is, as usually, $d(f)(k) = d_L(f(k)) - (-1)^{|n|} f(d_K(k)) \in \mathrm{Hom}_B^{n+1}(K, L)$.

One has $d^2(f) = [d, [d, f]] = [d^2, f] = [h, f] = 0$

Curved DG-structures

To be precise, the complex of morphisms $\mathrm{Hom}_B^\bullet(K^\bullet, L^\bullet)$ between CDG-modules (K, d_K) and (L, d_L) in the DG-category $B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of left CDG-modules over a CDG-ring $B^\bullet$ is defined by the rules:

- $\mathrm{Hom}_B^n(K, L)$ is the group of all homogeneous maps $f\colon K \longrightarrow L$ of degree n
- commuting with the action of the graded ring B with the sign rule $f(bk) = (-1)^{n|b|} bf(k)$ for all $b \in B^{|b|}$ and $k \in K$,
- while the differential on $\mathrm{Hom}_B^n(K, L)$ is, as usually, $d(f)(k) = d_L(f(k)) - (-1)^{|n|} f(d_K(k)) \in \mathrm{Hom}_B^{n+1}(K, L)$.

One has $d^2(f) = [d, [d, f]] = [d^2, f] = [h, f] = 0$, so $\mathrm{Hom}_B^\bullet(K^\bullet, L^\bullet)$ is indeed a complex.

Curved DG-structures

To be precise, the complex of morphisms $\mathrm{Hom}_B^\bullet(K^\bullet, L^\bullet)$ between CDG-modules (K, d_K) and (L, d_L) in the DG-category $B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of left CDG-modules over a CDG-ring $B^\bullet$ is defined by the rules:

- $\mathrm{Hom}_B^n(K, L)$ is the group of all homogeneous maps $f\colon K \longrightarrow L$ of degree n
- commuting with the action of the graded ring B with the sign rule $f(bk) = (-1)^{n|b|} bf(k)$ for all $b \in B^{|b|}$ and $k \in K$,
- while the differential on $\mathrm{Hom}_B^n(K, L)$ is, as usually, $d(f)(k) = d_L(f(k)) - (-1)^{|n|} f(d_K(k)) \in \mathrm{Hom}_B^{n+1}(K, L)$.

One has $d^2(f) = [d, [d, f]] = [d^2, f] = [h, f] = 0$, so $\mathrm{Hom}_B^\bullet(K^\bullet, L^\bullet)$ is indeed a complex.

Replacing a CDG-ring (B, d, h) with an isomorphic CDG-ring (B, d', h') via a connection change $d'(b) = d(b) + [a, b]$ and $h' = h + d(a) + a^2$ with $a \in B^1$,

Curved DG-structures

To be precise, the complex of morphisms $\mathrm{Hom}_B^\bullet(K^\bullet, L^\bullet)$ between CDG-modules (K, d_K) and (L, d_L) in the DG-category $B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of left CDG-modules over a CDG-ring $B^\bullet$ is defined by the rules:

- $\mathrm{Hom}_B^n(K, L)$ is the group of all homogeneous maps $f\colon K \longrightarrow L$ of degree n
- commuting with the action of the graded ring B with the sign rule $f(bk) = (-1)^{n|b|} bf(k)$ for all $b \in B^{|b|}$ and $k \in K$,
- while the differential on $\mathrm{Hom}_B^n(K, L)$ is, as usually, $d(f)(k) = d_L(f(k)) - (-1)^{|n|} f(d_K(k)) \in \mathrm{Hom}_B^{n+1}(K, L)$.

One has $d^2(f) = [d, [d, f]] = [d^2, f] = [h, f] = 0$, so $\mathrm{Hom}_B^\bullet(K^\bullet, L^\bullet)$ is indeed a complex.

Replacing a CDG-ring (B, d, h) with an isomorphic CDG-ring (B, d', h') via a connection change $d'(b) = d(b) + [a, b]$ and $h' = h + d(a) + a^2$ with $a \in B^1$, one transforms the differentials in left CDG-modules M by the rule $d'(m) = d(m) + am$

Curved DG-structures

To be precise, the complex of morphisms $\mathrm{Hom}_B^\bullet(K^\bullet, L^\bullet)$ between CDG-modules (K, d_K) and (L, d_L) in the DG-category $B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of left CDG-modules over a CDG-ring $B^\bullet$ is defined by the rules:

- $\mathrm{Hom}_B^n(K, L)$ is the group of all homogeneous maps $f\colon K \longrightarrow L$ of degree n
- commuting with the action of the graded ring B with the sign rule $f(bk) = (-1)^{n|b|}bf(k)$ for all $b \in B^{|b|}$ and $k \in K$,
- while the differential on $\mathrm{Hom}_B^n(K, L)$ is, as usually, $d(f)(k) = d_L(f(k)) - (-1)^{|n|}f(d_K(k)) \in \mathrm{Hom}_B^{n+1}(K, L)$.

One has $d^2(f) = [d, [d, f]] = [d^2, f] = [h, f] = 0$, so $\mathrm{Hom}_B^\bullet(K^\bullet, L^\bullet)$ is indeed a complex.

Replacing a CDG-ring (B, d, h) with an isomorphic CDG-ring (B, d', h') via a connection change $d'(b) = d(b) + [a, b]$ and $h' = h + d(a) + a^2$ with $a \in B^1$, one transforms the differentials in left CDG-modules M by the rule $d'(m) = d(m) + am$ to establish an isomorphism between the two DG-categories of CDG-modules.

The hat construction

The hat construction

A DG-ring $R^\bullet$ is said to be acyclic if its cohomology ring is the zero ring: $H^*(R^\bullet) = 0$.

The hat construction

A DG-ring $R^\bullet$ is said to be acyclic if its cohomology ring is the zero ring: $H^*(R^\bullet) = 0$. There is a natural equivalence $\mathrm{CDG\text{-}rings} \simeq \mathrm{DG\text{-}rings}^{\mathrm{ac}}$

The hat construction

A DG-ring $R^\bullet$ is said to be acyclic if its cohomology ring is the zero ring: $H^*(R^\bullet) = 0$. There is a natural equivalence $\mathrm{CDG\text{-}rings} \simeq \mathrm{DG\text{-}rings}^{\mathrm{ac}}$ between the category of CDG-rings and the category of acyclic DG-rings.

The hat construction

A DG-ring $R^\bullet$ is said to be acyclic if its cohomology ring is the zero ring: $H^*(R^\bullet) = 0$. There is a natural equivalence $\mathrm{CDG\text{-}rings} \simeq \mathrm{DG\text{-}rings}^{\mathrm{ac}}$ between the category of CDG-rings and the category of acyclic DG-rings.

Given a CDG-ring $B^\blacklozenge = (B, d, h)$, the related acyclic DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ is constructed as follows.

The hat construction

A DG-ring $R^\bullet$ is said to be acyclic if its cohomology ring is the zero ring: $H^*(R^\bullet) = 0$. There is a natural equivalence $\mathrm{CDG\text{-}rings} \simeq \mathrm{DG\text{-}rings}^{\mathrm{ac}}$ between the category of CDG-rings and the category of acyclic DG-rings.

Given a CDG-ring $B^\blacklozenge = (B, d, h)$, the related acyclic DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ is constructed as follows. The graded ring $B[\delta]$ is obtained by adjoining to B a new element δ of cohomological degree 1

The hat construction

A DG-ring $R^\bullet$ is said to be acyclic if its cohomology ring is the zero ring: $H^*(R^\bullet) = 0$. There is a natural equivalence $\mathrm{CDG\text{-}rings} \simeq \mathrm{DG\text{-}rings}^{\mathrm{ac}}$ between the category of CDG-rings and the category of acyclic DG-rings.

Given a CDG-ring $B^\blacklozenge = (B, d, h)$, the related acyclic DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ is constructed as follows. The graded ring $B[\delta]$ is obtained by adjoining to B a new element δ of cohomological degree 1 subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d(b)$ for all $b \in B$;

The hat construction

A DG-ring $R^\bullet$ is said to be acyclic if its cohomology ring is the zero ring: $H^*(R^\bullet) = 0$. There is a natural equivalence $\mathrm{CDG\text{-}rings} \simeq \mathrm{DG\text{-}rings}^{\mathrm{ac}}$ between the category of CDG-rings and the category of acyclic DG-rings.

Given a CDG-ring $B^\blacklozenge = (B, d, h)$, the related acyclic DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ is constructed as follows. The graded ring $B[\delta]$ is obtained by adjoining to B a new element δ of cohomological degree 1 subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d(b)$ for all $b \in B$;
- $\delta^2 = h$.

The hat construction

A DG-ring $R^\bullet$ is said to be acyclic if its cohomology ring is the zero ring: $H^*(R^\bullet) = 0$. There is a natural equivalence $\mathrm{CDG\text{-}rings} \simeq \mathrm{DG\text{-}rings}^{\mathrm{ac}}$ between the category of CDG-rings and the category of acyclic DG-rings.

Given a CDG-ring $B^\blacklozenge = (B, d, h)$, the related acyclic DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ is constructed as follows. The graded ring $B[\delta]$ is obtained by adjoining to B a new element δ of cohomological degree 1 subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d(b)$ for all $b \in B$;
- $\delta^2 = h$.

The differential $\partial = \partial/\partial\delta$ of degree -1 on the graded ring $B[\delta]$

The hat construction

A DG-ring $R^\bullet$ is said to be acyclic if its cohomology ring is the zero ring: $H^*(R^\bullet) = 0$. There is a natural equivalence $\mathrm{CDG\text{-}rings} \simeq \mathrm{DG\text{-}rings}^{\mathrm{ac}}$ between the category of CDG-rings and the category of acyclic DG-rings.

Given a CDG-ring $B^\blacklozenge = (B, d, h)$, the related acyclic DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ is constructed as follows. The graded ring $B[\delta]$ is obtained by adjoining to B a new element δ of cohomological degree 1 subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d(b)$ for all $b \in B$;
- $\delta^2 = h$.

The differential $\partial = \partial/\partial\delta$ of degree -1 on the graded ring $B[\delta]$ is defined by the rules $\partial(\delta) = 1$ and $\partial(b) = 0$ for $b \in B$.

The hat construction

A DG-ring $R^\bullet$ is said to be acyclic if its cohomology ring is the zero ring: $H^*(R^\bullet) = 0$. There is a natural equivalence $\mathrm{CDG\text{-}rings} \simeq \mathrm{DG\text{-}rings}^{\mathrm{ac}}$ between the category of CDG-rings and the category of acyclic DG-rings.

Given a CDG-ring $B^\blacklozenge = (B, d, h)$, the related acyclic DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ is constructed as follows. The graded ring $B[\delta]$ is obtained by adjoining to B a new element δ of cohomological degree 1 subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d(b)$ for all $b \in B$;
- $\delta^2 = h$.

The differential $\partial = \partial/\partial\delta$ of degree -1 on the graded ring $B[\delta]$ is defined by the rules $\partial(\delta) = 1$ and $\partial(b) = 0$ for $b \in B$.

The DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ is obtained by changing the sign of the grading on $(B[\delta], \partial)$

The hat construction

A DG-ring $R^\bullet$ is said to be acyclic if its cohomology ring is the zero ring: $H^*(R^\bullet) = 0$. There is a natural equivalence $\mathrm{CDG\text{-}rings} \simeq \mathrm{DG\text{-}rings}^{\mathrm{ac}}$ between the category of CDG-rings and the category of acyclic DG-rings.

Given a CDG-ring $B^\blacklozenge = (B, d, h)$, the related acyclic DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ is constructed as follows. The graded ring $B[\delta]$ is obtained by adjoining to B a new element δ of cohomological degree 1 subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d(b)$ for all $b \in B$;
- $\delta^2 = h$.

The differential $\partial = \partial/\partial\delta$ of degree -1 on the graded ring $B[\delta]$ is defined by the rules $\partial(\delta) = 1$ and $\partial(b) = 0$ for $b \in B$.

The DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ is obtained by changing the sign of the grading on $(B[\delta], \partial)$; that is, $\widehat{B}^n = B[\delta]^{-n}$ for all $n \in \mathbb{Z}$

The hat construction

A DG-ring $R^\bullet$ is said to be acyclic if its cohomology ring is the zero ring: $H^*(R^\bullet) = 0$. There is a natural equivalence $\mathrm{CDG\text{-}rings} \simeq \mathrm{DG\text{-}rings}^{\mathrm{ac}}$ between the category of CDG-rings and the category of acyclic DG-rings.

Given a CDG-ring $B^\blacklozenge = (B, d, h)$, the related acyclic DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ is constructed as follows. The graded ring $B[\delta]$ is obtained by adjoining to B a new element δ of cohomological degree 1 subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d(b)$ for all $b \in B$;
- $\delta^2 = h$.

The differential $\partial = \partial/\partial\delta$ of degree -1 on the graded ring $B[\delta]$ is defined by the rules $\partial(\delta) = 1$ and $\partial(b) = 0$ for $b \in B$.

The DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ is obtained by changing the sign of the grading on $(B[\delta], \partial)$; that is, $\widehat{B}^n = B[\delta]^{-n}$ for all $n \in \mathbb{Z}$ (to make the differential ∂ increase the degree).

A DG-ring $R^\bullet$ is said to be acyclic if its cohomology ring is the zero ring: $H^*(R^\bullet) = 0$. There is a natural equivalence $\mathrm{CDG\text{-}rings} \simeq \mathrm{DG\text{-}rings}^{\mathrm{ac}}$ between the category of CDG-rings and the category of acyclic DG-rings.

Given a CDG-ring $B^\blacklozenge = (B, d, h)$, the related acyclic DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ is constructed as follows. The graded ring $B[\delta]$ is obtained by adjoining to B a new element δ of cohomological degree 1 subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d(b)$ for all $b \in B$;
- $\delta^2 = h$.

The differential $\partial = \partial/\partial\delta$ of degree -1 on the graded ring $B[\delta]$ is defined by the rules $\partial(\delta) = 1$ and $\partial(b) = 0$ for $b \in B$.

The DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ is obtained by changing the sign of the grading on $(B[\delta], \partial)$; that is, $\widehat{B}^n = B[\delta]^{-n}$ for all $n \in \mathbb{Z}$ (to make the differential ∂ increase the degree). Otherwise the multiplication and the differential stay the same.

The hat construction

Given a morphism of CDG-rings
$(f, a)\colon (B, d_B, h_B) \longrightarrow (A, d_A, h_A)$

The hat construction

Given a morphism of CDG-rings $(f, a)\colon (B, d_B, h_B) \longrightarrow (A, d_A, h_A)$, the induced morphism of acyclic DG-rings $\widehat{f}\colon \widehat{B}^\bullet \longrightarrow \widehat{A}^\bullet$

The hat construction

Given a morphism of CDG-rings $(f, a)\colon (B, d_B, h_B) \longrightarrow (A, d_A, h_A)$, the induced morphism of acyclic DG-rings $\widehat{f}\colon \widehat{B}^\bullet \longrightarrow \widehat{A}^\bullet$ is constructed by the rules $\widehat{f}(b) = f(b)$ for all $b \in B$ and $\widehat{f}(\delta_B) = \delta_A + a$.

The hat construction

Given a morphism of CDG-rings $(f, a)\colon (B, d_B, h_B) \longrightarrow (A, d_A, h_A)$, the induced morphism of acyclic DG-rings $\widehat{f}\colon \widehat{B}^\bullet \longrightarrow \widehat{A}^\bullet$ is constructed by the rules $\widehat{f}(b) = f(b)$ for all $b \in B$ and $\widehat{f}(\delta_B) = \delta_A + a$.

This construction produces the functor $\mathrm{CDG\text{-}rings} \longrightarrow \mathrm{DG\text{-}rings}^{\mathrm{ac}}$.

The hat construction

Given a morphism of CDG-rings $(f,a)\colon (B,d_B,h_B)\longrightarrow (A,d_A,h_A)$, the induced morphism of acyclic DG-rings $\widehat{f}\colon \widehat{B}^\bullet \longrightarrow \widehat{A}^\bullet$ is constructed by the rules $\widehat{f}(b)=f(b)$ for all $b\in B$ and $\widehat{f}(\delta_B)=\delta_A+a$.

This construction produces the functor $\mathrm{CDG\text{-}rings}\longrightarrow \mathrm{DG\text{-}rings}^{\mathrm{ac}}$.

Conversely, given an acyclic DG-ring $R^\bullet=(R,\partial_R)$

The hat construction

Given a morphism of CDG-rings $(f, a)\colon (B, d_B, h_B) \longrightarrow (A, d_A, h_A)$, the induced morphism of acyclic DG-rings $\widehat{f}\colon \widehat{B}^\bullet \longrightarrow \widehat{A}^\bullet$ is constructed by the rules $\widehat{f}(b) = f(b)$ for all $b \in B$ and $\widehat{f}(\delta_B) = \delta_A + a$.

This construction produces the functor $\mathrm{CDG\text{-}rings} \longrightarrow \mathrm{DG\text{-}rings}^{\mathrm{ac}}$.

Conversely, given an acyclic DG-ring $R^\bullet = (R, \partial_R)$, pick an arbitrary element $\delta \in R^{-1}$ such that $\partial_R(\delta) = 1$.

The hat construction

Given a morphism of CDG-rings $(f,a)\colon (B,d_B,h_B) \longrightarrow (A,d_A,h_A)$, the induced morphism of acyclic DG-rings $\widehat{f}\colon \widehat{B}^\bullet \longrightarrow \widehat{A}^\bullet$ is constructed by the rules $\widehat{f}(b) = f(b)$ for all $b \in B$ and $\widehat{f}(\delta_B) = \delta_A + a$.

This construction produces the functor $\mathrm{CDG\text{-}rings} \longrightarrow \mathrm{DG\text{-}rings}^{\mathrm{ac}}$.

Conversely, given an acyclic DG-ring $R^\bullet = (R, \partial_R)$, pick an arbitrary element $\delta \in R^{-1}$ such that $\partial_R(\delta) = 1$. Put

- $B = \ker(\partial_R) \subset R$ with the sign of the grading changed, $B^n = \ker(\partial_R\colon R^{-n} \to R^{-n+1})$;

The hat construction

Given a morphism of CDG-rings $(f,a)\colon (B,d_B,h_B)\longrightarrow (A,d_A,h_A)$, the induced morphism of acyclic DG-rings $\widehat{f}\colon \widehat{B}^\bullet \longrightarrow \widehat{A}^\bullet$ is constructed by the rules $\widehat{f}(b)=f(b)$ for all $b\in B$ and $\widehat{f}(\delta_B)=\delta_A+a$.

This construction produces the functor $\mathrm{CDG\text{-}rings}\longrightarrow \mathrm{DG\text{-}rings}^{\mathrm{ac}}$.

Conversely, given an acyclic DG-ring $R^\bullet=(R,\partial_R)$, pick an arbitrary element $\delta\in R^{-1}$ such that $\partial_R(\delta)=1$. Put

- $B=\ker(\partial_R)\subset R$ with the sign of the grading changed, $B^n=\ker(\partial_R\colon R^{-n}\to R^{-n+1})$;
- $d_B(b)=\delta b-(-1)^{|b|}b\delta\in B$ for $b\in B$;

The hat construction

Given a morphism of CDG-rings $(f,a)\colon (B,d_B,h_B)\longrightarrow (A,d_A,h_A)$, the induced morphism of acyclic DG-rings $\widehat{f}\colon \widehat{B}^\bullet \longrightarrow \widehat{A}^\bullet$ is constructed by the rules $\widehat{f}(b)=f(b)$ for all $b\in B$ and $\widehat{f}(\delta_B)=\delta_A+a$.

This construction produces the functor $\mathrm{CDG\text{-}rings}\longrightarrow \mathrm{DG\text{-}rings}^{\mathrm{ac}}$.

Conversely, given an acyclic DG-ring $R^\bullet=(R,\partial_R)$, pick an arbitrary element $\delta\in R^{-1}$ such that $\partial_R(\delta)=1$. Put

- $B=\ker(\partial_R)\subset R$ with the sign of the grading changed, $B^n=\ker(\partial_R\colon R^{-n}\to R^{-n+1})$;
- $d_B(b)=\delta b-(-1)^{|b|}b\delta\in B$ for $b\in B$;
- $h_B=\delta^2=\delta\cdot\delta\in\ker(\partial_R)^{-2}=B^2$.

The hat construction

Given a morphism of CDG-rings $(f,a)\colon (B,d_B,h_B)\longrightarrow(A,d_A,h_A)$, the induced morphism of acyclic DG-rings $\widehat{f}\colon \widehat{B}^\bullet\longrightarrow\widehat{A}^\bullet$ is constructed by the rules $\widehat{f}(b)=f(b)$ for all $b\in B$ and $\widehat{f}(\delta_B)=\delta_A+a$.

This construction produces the functor $\mathrm{CDG\text{-}rings}\longrightarrow\mathrm{DG\text{-}rings}^{\mathrm{ac}}$.

Conversely, given an acyclic DG-ring $R^\bullet=(R,\partial_R)$, pick an arbitrary element $\delta\in R^{-1}$ such that $\partial_R(\delta)=1$. Put

- $B=\ker(\partial_R)\subset R$ with the sign of the grading changed, $B^n=\ker(\partial_R\colon R^{-n}\to R^{-n+1})$;
- $d_B(b)=\delta b-(-1)^{|b|}b\delta\in B$ for $b\in B$;
- $h_B=\delta^2=\delta\cdot\delta\in\ker(\partial_R)^{-2}=B^2$.

This construction produces the inverse functor $\mathrm{DG\text{-}rings}^{\mathrm{ac}}\longrightarrow\mathrm{CDG\text{-}rings}$.

CDG-rings vs. acyclic DG-rings

CDG-rings vs. acyclic DG-rings

The hat construction is a category equivalence, but recovering a CDG-ring from the related acyclic DG-ring involves an arbitrary choice.

CDG-rings vs. acyclic DG-rings

The hat construction is a category equivalence, but recovering a CDG-ring from the related acyclic DG-ring involves an arbitrary choice.

This is the general situation with curved DG-structures

CDG-rings vs. acyclic DG-rings

The hat construction is a category equivalence, but recovering a CDG-ring from the related acyclic DG-ring involves an arbitrary choice.

This is the general situation with curved DG-structures: constructing a CDG-ring, CDG-coalgebra etc. usually requires making an arbitrary choice.

CDG-rings vs. acyclic DG-rings

The hat construction is a category equivalence, but recovering a CDG-ring from the related acyclic DG-ring involves an arbitrary choice.

This is the general situation with curved DG-structures: constructing a CDG-ring, CDG-coalgebra etc. usually requires making an arbitrary choice. The difference between any two such choices (the result of subtraction)

CDG-rings vs. acyclic DG-rings

The hat construction is a category equivalence, but recovering a CDG-ring from the related acyclic DG-ring involves an arbitrary choice.

This is the general situation with curved DG-structures: constructing a CDG-ring, CDG-coalgebra etc. usually requires making an arbitrary choice. The difference between any two such choices (the result of subtraction) can be used to construct a change-of-connection element

CDG-rings vs. acyclic DG-rings

The hat construction is a category equivalence, but recovering a CDG-ring from the related acyclic DG-ring involves an arbitrary choice.

This is the general situation with curved DG-structures: constructing a CDG-ring, CDG-coalgebra etc. usually requires making an arbitrary choice. The difference between any two such choices (the result of subtraction) can be used to construct a change-of-connection element making the two CDG-rings naturally isomorphic to each other by a change-of-connection isomorphism.

CDG-rings vs. acyclic DG-rings

The hat construction is a category equivalence, but recovering a CDG-ring from the related acyclic DG-ring involves an arbitrary choice.

This is the general situation with curved DG-structures: constructing a CDG-ring, CDG-coalgebra etc. usually requires making an arbitrary choice. The difference between any two such choices (the result of subtraction) can be used to construct a change-of-connection element making the two CDG-rings naturally isomorphic to each other by a change-of-connection isomorphism.

In this sense, acyclic DG-rings (acyclic DG-coagebras etc.) are more invariant objects.

CDG-rings vs. acyclic DG-rings

The hat construction is a category equivalence, but recovering a CDG-ring from the related acyclic DG-ring involves an arbitrary choice.

This is the general situation with curved DG-structures: constructing a CDG-ring, CDG-coalgebra etc. usually requires making an arbitrary choice. The difference between any two such choices (the result of subtraction) can be used to construct a change-of-connection element making the two CDG-rings naturally isomorphic to each other by a change-of-connection isomorphism.

In this sense, acyclic DG-rings (acyclic DG-coagebras etc.) are more invariant objects. But they are also much more counterintuitive, making them harder to work with.

CDG-rings vs. acyclic DG-rings

Another name for acyclic DG-rings is quasi-differential rings.

CDG-rings vs. acyclic DG-rings

Another name for acyclic DG-rings is quasi-differential rings.
The datum of a quasi-differential ring $R^\bullet = (R, \partial)$ with the given graded ring $B = \ker(\partial\colon R \to R)$

CDG-rings vs. acyclic DG-rings

Another name for acyclic DG-rings is quasi-differential rings. The datum of a quasi-differential ring $R^\bullet = (R, \partial)$ with the given graded ring $B = \ker(\partial\colon R \to R)$ is called a quasi-differential structure on the ring B.

CDG-rings vs. acyclic DG-rings

Another name for acyclic DG-rings is quasi-differential rings. The datum of a quasi-differential ring $R^\bullet = (R, \partial)$ with the given graded ring $B = \ker(\partial\colon R \to R)$ is called a quasi-differential structure on the ring B.

This terminology is used in the contexts where one wants to work with an acyclic DG-ring instead of a CDG-ring

CDG-rings vs. acyclic DG-rings

Another name for acyclic DG-rings is quasi-differential rings. The datum of a quasi-differential ring $R^\bullet = (R, \partial)$ with the given graded ring $B = \ker(\partial\colon R \to R)$ is called a quasi-differential structure on the ring B.

This terminology is used in the contexts where one wants to work with an acyclic DG-ring instead of a CDG-ring, in particular when considering the DG-category of modules.

CDG-rings vs. acyclic DG-rings

Another name for acyclic DG-rings is quasi-differential rings. The datum of a quasi-differential ring $R^\bullet = (R, \partial)$ with the given graded ring $B = \ker(\partial\colon R \to R)$ is called a quasi-differential structure on the ring B.

This terminology is used in the contexts where one wants to work with an acyclic DG-ring instead of a CDG-ring, in particular when considering the DG-category of modules.

Let $B^\blacklozenge = (B, d, h)$ be a CDG-ring and $(B[\delta], \partial)$ be the related acyclic DG-ring.

CDG-rings vs. acyclic DG-rings

Another name for acyclic DG-rings is quasi-differential rings. The datum of a quasi-differential ring $R^\bullet = (R, \partial)$ with the given graded ring $B = \ker(\partial\colon R \to R)$ is called a quasi-differential structure on the ring B.

This terminology is used in the contexts where one wants to work with an acyclic DG-ring instead of a CDG-ring, in particular when considering the DG-category of modules.

Let $B^\blacklozenge = (B, d, h)$ be a CDG-ring and $(B[\delta], \partial)$ be the related acyclic DG-ring. How can one construct the DG-category of CDG-modules over $B^\blacklozenge$ in terms of the quasi-differential ring $(B[\delta], \partial)$?

CDG-rings vs. acyclic DG-rings

Another name for acyclic DG-rings is quasi-differential rings. The datum of a quasi-differential ring $R^\bullet = (R, \partial)$ with the given graded ring $B = \ker(\partial\colon R \to R)$ is called a quasi-differential structure on the ring B.

This terminology is used in the contexts where one wants to work with an acyclic DG-ring instead of a CDG-ring, in particular when considering the DG-category of modules.

Let $B^\blacklozenge = (B, d, h)$ be a CDG-ring and $(B[\delta], \partial)$ be the related acyclic DG-ring. How can one construct the DG-category of CDG-modules over $B^\blacklozenge$ in terms of the quasi-differential ring $(B[\delta], \partial)$?

A left CDG-module $M^\blacklozenge = (M, d_M)$ over $B^\blacklozenge$ is the same thing as a graded left $B[\delta]$-module.

CDG-rings vs. acyclic DG-rings

Another name for acyclic DG-rings is quasi-differential rings. The datum of a quasi-differential ring $R^\bullet = (R, \partial)$ with the given graded ring $B = \ker(\partial\colon R \to R)$ is called a quasi-differential structure on the ring B.

This terminology is used in the contexts where one wants to work with an acyclic DG-ring instead of a CDG-ring, in particular when considering the DG-category of modules.

Let $B^\blacklozenge = (B, d, h)$ be a CDG-ring and $(B[\delta], \partial)$ be the related acyclic DG-ring. How can one construct the DG-category of CDG-modules over $B^\blacklozenge$ in terms of the quasi-differential ring $(B[\delta], \partial)$?

A left CDG-module $M^\blacklozenge = (M, d_M)$ over $B^\blacklozenge$ is the same thing as a graded left $B[\delta]$-module. The element δ acts in M by the differential d_M.

CDG-rings vs. acyclic DG-rings

Another name for acyclic DG-rings is quasi-differential rings. The datum of a quasi-differential ring $R^\bullet = (R, \partial)$ with the given graded ring $B = \ker(\partial\colon R \to R)$ is called a quasi-differential structure on the ring B.

This terminology is used in the contexts where one wants to work with an acyclic DG-ring instead of a CDG-ring, in particular when considering the DG-category of modules.

Let $B^\blacklozenge = (B, d, h)$ be a CDG-ring and $(B[\delta], \partial)$ be the related acyclic DG-ring. How can one construct the DG-category of CDG-modules over $B^\blacklozenge$ in terms of the quasi-differential ring $(B[\delta], \partial)$?

A left CDG-module $M^\blacklozenge = (M, d_M)$ over $B^\blacklozenge$ is the same thing as a graded left $B[\delta]$-module. The element δ acts in M by the differential d_M. Notice that there is no differential on M compatible with the differential ∂ on $B[\delta]$.

CDG-rings vs. acyclic DG-rings

Given another left CDG-module $N^{\blacklozenge} = (N, d_N)$ over $B^{\blacklozenge}$

Given another left CDG-module $N^\bullet = (N, d_N)$ over $B^\bullet$, the grading components of the complex of morphisms $\mathrm{Hom}_B(M^\bullet, N^\bullet)$

CDG-rings vs. acyclic DG-rings

Given another left CDG-module $N^\bullet = (N, d_N)$ over $B^\bullet$, the grading components of the complex of morphisms $\mathrm{Hom}_B(M^\bullet, N^\bullet)$ are the groups of homogeneous B-module maps $M \longrightarrow N$ of various degrees.

CDG-rings vs. acyclic DG-rings

Given another left CDG-module $N^{\bullet} = (N, d_N)$ over $B^{\bullet}$, the grading components of the complex of morphisms $\mathrm{Hom}_B(M^{\bullet}, N^{\bullet})$ are the groups of homogeneous B-module maps $M \longrightarrow N$ of various degrees. Notice that these are graded B-module maps rather than graded $B[\delta]$-module maps.

Given another left CDG-module $N^\bullet = (N, d_N)$ over $B^\bullet$, the grading components of the complex of morphisms $\mathrm{Hom}_B(M^\bullet, N^\bullet)$ are the groups of homogeneous B-module maps $M \longrightarrow N$ of various degrees. Notice that these are graded B-module maps rather than graded $B[\delta]$-module maps.

It may be a good exercise to construct the differential d on the complex $\mathrm{Hom}_B(M^\bullet, N^\bullet)$

Given another left CDG-module $N^\bullet = (N, d_N)$ over $B^\bullet$, the grading components of the complex of morphisms $\mathrm{Hom}_B(M^\bullet, N^\bullet)$ are the groups of homogeneous B-module maps $M \longrightarrow N$ of various degrees. Notice that these are graded B-module maps rather than graded $B[\delta]$-module maps.

It may be a good exercise to construct the differential d on the complex $\mathrm{Hom}_B(M^\bullet, N^\bullet)$ working with $M^\bullet$ and $N^\bullet$ as modules over a graded ring R with an acyclic differential ∂_R defined on R

CDG-rings vs. acyclic DG-rings

Given another left CDG-module $N^\bullet = (N, d_N)$ over $B^\bullet$, the grading components of the complex of morphisms $\mathrm{Hom}_B(M^\bullet, N^\bullet)$ are the groups of homogeneous B-module maps $M \longrightarrow N$ of various degrees. Notice that these are graded B-module maps rather than graded $B[\delta]$-module maps.

It may be a good exercise to construct the differential d on the complex $\mathrm{Hom}_B(M^\bullet, N^\bullet)$ working with $M^\bullet$ and $N^\bullet$ as modules over a graded ring R with an acyclic differential ∂_R defined on R such that $B = \ker \partial_R \subset R$.

Given another left CDG-module $N^\bullet = (N, d_N)$ over $B^\bullet$, the grading components of the complex of morphisms $\mathrm{Hom}_B(M^\bullet, N^\bullet)$ are the groups of homogeneous B-module maps $M \longrightarrow N$ of various degrees. Notice that these are graded B-module maps rather than graded $B[\delta]$-module maps.

It may be a good exercise to construct the differential d on the complex $\mathrm{Hom}_B(M^\bullet, N^\bullet)$ working with $M^\bullet$ and $N^\bullet$ as modules over a graded ring R with an acyclic differential ∂_R defined on R such that $B = \ker \partial_R \subset R$. The point of the exercise is to refrain from choosing any specific element $\delta \in R$ for the purposes of the construction

Given another left CDG-module $N^\bullet = (N, d_N)$ over $B^\bullet$, the grading components of the complex of morphisms $\mathrm{Hom}_B(M^\bullet, N^\bullet)$ are the groups of homogeneous B-module maps $M \longrightarrow N$ of various degrees. Notice that these are graded B-module maps rather than graded $B[\delta]$-module maps.

It may be a good exercise to construct the differential d on the complex $\mathrm{Hom}_B(M^\bullet, N^\bullet)$ working with $M^\bullet$ and $N^\bullet$ as modules over a graded ring R with an acyclic differential ∂_R defined on R such that $B = \ker \partial_R \subset R$. The point of the exercise is to refrain from choosing any specific element $\delta \in R$ for the purposes of the construction, but work only with the differential ∂_R (which is more invariant) throughout.

Given another left CDG-module $N^{\bullet} = (N, d_N)$ over $B^{\bullet}$, the grading components of the complex of morphisms $\mathrm{Hom}_B(M^{\bullet}, N^{\bullet})$ are the groups of homogeneous B-module maps $M \longrightarrow N$ of various degrees. Notice that these are graded B-module maps rather than graded $B[\delta]$-module maps.

It may be a good exercise to construct the differential d on the complex $\mathrm{Hom}_B(M^{\bullet}, N^{\bullet})$ working with $M^{\bullet}$ and $N^{\bullet}$ as modules over a graded ring R with an acyclic differential ∂_R defined on R such that $B = \ker \partial_R \subset R$. The point of the exercise is to refrain from choosing any specific element $\delta \in R$ for the purposes of the construction, but work only with the differential ∂_R (which is more invariant) throughout. Then the second part of the exercise is to check that $d^2 = 0$, while still refraining from choosing δ.

CDG-rings vs. acyclic DG-rings

Given another left CDG-module $N^\bullet = (N, d_N)$ over $B^\bullet$, the grading components of the complex of morphisms $\mathrm{Hom}_B(M^\bullet, N^\bullet)$ are the groups of homogeneous B-module maps $M \longrightarrow N$ of various degrees. Notice that these are graded B-module maps rather than graded $B[\delta]$-module maps.

It may be a good exercise to construct the differential d on the complex $\mathrm{Hom}_B(M^\bullet, N^\bullet)$ working with $M^\bullet$ and $N^\bullet$ as modules over a graded ring R with an acyclic differential ∂_R defined on R such that $B = \ker \partial_R \subset R$. The point of the exercise is to refrain from choosing any specific element $\delta \in R$ for the purposes of the construction, but work only with the differential ∂_R (which is more invariant) throughout. Then the second part of the exercise is to check that $d^2 = 0$, while still refraining from choosing δ.

Cf. Section 11.7.1 in the book "Homological algebra of semimodules and semicontramodules" (Birkhäuser, 2010), which is written in a more complicated setting of quasi-differential corings.

The double hat Morita equivalence

The double hat Morita equivalence

Now we start with a CDG-ring $B^\blacklozenge = (B, d_B, h_B)$ and apply the hat construction twice.

The double hat Morita equivalence

Now we start with a CDG-ring $B^\blacklozenge = (B, d_B, h_B)$ and apply the hat construction twice. The result is an acyclic DG-ring $\widehat{\widehat{B}}^\bullet = (\widehat{\widehat{B}}, D)$.

The double hat Morita equivalence

Now we start with a CDG-ring $B^{\blacklozenge} = (B, d_B, h_B)$ and apply the hat construction twice. The result is an acyclic DG-ring $\widehat{\widehat{B}}^{\bullet} = (\widehat{\widehat{B}},\, D)$.

The graded ring $\widehat{\widehat{B}}$ is obtained by adjoining two elements δ of degree 1 and ϵ of degree -1 to the graded ring B

The double hat Morita equivalence

Now we start with a CDG-ring $B^{\blacklozenge} = (B, d_B, h_B)$ and apply the hat construction twice. The result is an acyclic DG-ring $\widehat{\widehat{B}}^{\bullet} = (\widehat{\widehat{B}},\, D)$.

The graded ring $\widehat{\widehat{B}}$ is obtained by adjoining two elements δ of degree 1 and ϵ of degree -1 to the graded ring B, subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d_B(b)$ for all $b \in B$;

The double hat Morita equivalence

Now we start with a CDG-ring $B^\blacklozenge = (B, d_B, h_B)$ and apply the hat construction twice. The result is an acyclic DG-ring $\widehat{\widehat{B}}^\bullet = (\widehat{\widehat{B}}, D)$.

The graded ring $\widehat{\widehat{B}}$ is obtained by adjoining two elements δ of degree 1 and ϵ of degree -1 to the graded ring B, subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d_B(b)$ for all $b \in B$;
- $\epsilon b - (-1)^{|b|} b\epsilon = 0$ for all $b \in B$;

The double hat Morita equivalence

Now we start with a CDG-ring $B^{\blacklozenge} = (B, d_B, h_B)$ and apply the hat construction twice. The result is an acyclic DG-ring $\widehat{\widehat{B}}^{\bullet} = (\widehat{\widehat{B}}, D)$.

The graded ring $\widehat{\widehat{B}}$ is obtained by adjoining two elements δ of degree 1 and ϵ of degree -1 to the graded ring B, subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d_B(b)$ for all $b \in B$;
- $\epsilon b - (-1)^{|b|} b\epsilon = 0$ for all $b \in B$;
- $\delta^2 = h_B$; $\epsilon^2 = 0$;
- $\epsilon\delta + \delta\epsilon = 1$.

The double hat Morita equivalence

Now we start with a CDG-ring $B^{\blacklozenge} = (B, d_B, h_B)$ and apply the hat construction twice. The result is an acyclic DG-ring $\widehat{\widehat{B}}^{\bullet} = (\widehat{\widehat{B}}, D)$.

The graded ring $\widehat{\widehat{B}}$ is obtained by adjoining two elements δ of degree 1 and ϵ of degree -1 to the graded ring B, subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d_B(b)$ for all $b \in B$;
- $\epsilon b - (-1)^{|b|} b\epsilon = 0$ for all $b \in B$;
- $\delta^2 = h_B$; $\epsilon^2 = 0$;
- $\epsilon\delta + \delta\epsilon = 1$.

The differential $D = \partial/\partial\epsilon$ on the graded ring $\widehat{\widehat{B}}$ is defined by the rules $D(b) = 0$, $D(\delta) = 0$, $D(\epsilon) = 1$.

The double hat Morita equivalence

Now we start with a CDG-ring $B^\blacklozenge = (B, d_B, h_B)$ and apply the hat construction twice. The result is an acyclic DG-ring $\widehat{\widehat{B}}^{\bullet} = (\widehat{\widehat{B}}, D)$.

The graded ring $\widehat{\widehat{B}}$ is obtained by adjoining two elements δ of degree 1 and ϵ of degree -1 to the graded ring B, subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d_B(b)$ for all $b \in B$;
- $\epsilon b - (-1)^{|b|} b\epsilon = 0$ for all $b \in B$;
- $\delta^2 = h_B$; $\epsilon^2 = 0$;
- $\epsilon\delta + \delta\epsilon = 1$.

The differential $D = \partial/\partial\epsilon$ on the graded ring $\widehat{\widehat{B}}$ is defined by the rules $D(b) = 0$, $D(\delta) = 0$, $D(\epsilon) = 1$.

Theorem (double hat Morita equivalence theorem)

The double hat Morita equivalence

Now we start with a CDG-ring $B^\blacklozenge = (B, d_B, h_B)$ and apply the hat construction twice. The result is an acyclic DG-ring $\widehat{\widehat{B}}^\bullet = (\widehat{\widehat{B}}, D)$.

The graded ring $\widehat{\widehat{B}}$ is obtained by adjoining two elements δ of degree 1 and ϵ of degree -1 to the graded ring B, subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d_B(b)$ for all $b \in B$;
- $\epsilon b - (-1)^{|b|} b\epsilon = 0$ for all $b \in B$;
- $\delta^2 = h_B$; $\epsilon^2 = 0$;
- $\epsilon\delta + \delta\epsilon = 1$.

The differential $D = \partial/\partial\epsilon$ on the graded ring $\widehat{\widehat{B}}$ is defined by the rules $D(b) = 0$, $D(\delta) = 0$, $D(\epsilon) = 1$.

Theorem (double hat Morita equivalence theorem)

The DG-category $B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of CDG-modules over any CDG-ring $B^\blacklozenge$

The double hat Morita equivalence

Now we start with a CDG-ring $B^\bullet = (B, d_B, h_B)$ and apply the hat construction twice. The result is an acyclic DG-ring $\widehat{\widehat{B}}^\bullet = (\widehat{\widehat{B}}, D)$.

The graded ring $\widehat{\widehat{B}}$ is obtained by adjoining two elements δ of degree 1 and ϵ of degree -1 to the graded ring B, subject to the relations

- $\delta b - (-1)^{|b|} b\delta = d_B(b)$ for all $b \in B$;
- $\epsilon b - (-1)^{|b|} b\epsilon = 0$ for all $b \in B$;
- $\delta^2 = h_B$; $\epsilon^2 = 0$;
- $\epsilon\delta + \delta\epsilon = 1$.

The differential $D = \partial/\partial\epsilon$ on the graded ring $\widehat{\widehat{B}}$ is defined by the rules $D(b) = 0$, $D(\delta) = 0$, $D(\epsilon) = 1$.

Theorem (double hat Morita equivalence theorem)

The DG-category $B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of CDG-modules over any CDG-ring $B^\bullet$ is naturally equivalent to the DG-category $\widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ of DG-modules over the acyclic DG-ring $\widehat{\widehat{B}}^\bullet$.

The double hat Morita equivalence

The DG-functor $B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$ is constructed as follows.

The double hat Morita equivalence

The DG-functor $B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$ is constructed as follows. Given a left CDG-module $M^{\blacklozenge} = (M, d_M)$ over $B^{\blacklozenge}$

The double hat Morita equivalence

The DG-functor $B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$ is constructed as follows. Given a left CDG-module $M^{\blacklozenge} = (M, d_M)$ over $B^{\blacklozenge}$, the underlying graded abelian group of the related DG-module $E^{\bullet} = (E, D_E)$ over $\widehat{\widehat{B}}^{\bullet}$

The double hat Morita equivalence

The DG-functor $B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ is constructed as follows. Given a left CDG-module $M^\blacklozenge = (M, d_M)$ over $B^\blacklozenge$, the underlying graded abelian group of the related DG-module $E^\bullet = (E, D_E)$ over $\widehat{\widehat{B}}^\bullet$ is the direct sum $E = M \oplus M[-1]$.

The double hat Morita equivalence

The DG-functor $B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$ is constructed as follows. Given a left CDG-module $M^{\blacklozenge} = (M, d_M)$ over $B^{\blacklozenge}$, the underlying graded abelian group of the related DG-module $E^{\bullet} = (E, D_E)$ over $\widehat{\widehat{B}}^{\bullet}$ is the direct sum $E = M \oplus M[-1]$. So the elements of the abelian group E^i ($i \in \mathbb{Z}$)

The double hat Morita equivalence

The DG-functor $B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ is constructed as follows. Given a left CDG-module $M^\blacklozenge = (M, d_M)$ over $B^\blacklozenge$, the underlying graded abelian group of the related DG-module $E^\bullet = (E, D_E)$ over $\widehat{\widehat{B}}^\bullet$ is the direct sum $E = M \oplus M[-1]$. So the elements of the abelian group E^i ($i \in \mathbb{Z}$) are pairs (m', m'') with $m' \in M^i$ and $m'' \in M^{i-1}$.

The double hat Morita equivalence

The DG-functor $B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$ is constructed as follows. Given a left CDG-module $M^{\blacklozenge} = (M, d_M)$ over $B^{\blacklozenge}$, the underlying graded abelian group of the related DG-module $E^{\bullet} = (E, D_E)$ over $\widehat{\widehat{B}}^{\bullet}$ is the direct sum $E = M \oplus M[-1]$. So the elements of the abelian group E^i ($i \in \mathbb{Z}$) are pairs (m', m'') with $m' \in M^i$ and $m'' \in M^{i-1}$.

The actions of B, δ, ϵ, and D_E on E are given by the rules

The double hat Morita equivalence

The DG-functor $B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ is constructed as follows. Given a left CDG-module $M^\blacklozenge = (M, d_M)$ over $B^\blacklozenge$, the underlying graded abelian group of the related DG-module $E^\bullet = (E, D_E)$ over $\widehat{\widehat{B}}^\bullet$ is the direct sum $E = M \oplus M[-1]$. So the elements of the abelian group E^i ($i \in \mathbb{Z}$) are pairs (m', m'') with $m' \in M^i$ and $m'' \in M^{i-1}$.

The actions of B, δ, ϵ, and D_E on E are given by the rules

- $b(m', m'') = (bm', (-1)^{|b|} bm'')$ for $b \in B$;

The double hat Morita equivalence

The DG-functor $B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$ is constructed as follows. Given a left CDG-module $M^{\blacklozenge} = (M, d_M)$ over $B^{\blacklozenge}$, the underlying graded abelian group of the related DG-module $E^{\bullet} = (E, D_E)$ over $\widehat{\widehat{B}}^{\bullet}$ is the direct sum $E = M \oplus M[-1]$. So the elements of the abelian group E^i ($i \in \mathbb{Z}$) are pairs (m', m'') with $m' \in M^i$ and $m'' \in M^{i-1}$.

The actions of B, δ, ϵ, and D_E on E are given by the rules

- $b(m', m'') = (bm', (-1)^{|b|} bm'')$ for $b \in B$;
- $\delta(m', m'') = (d_M(m'), m' - d_M(m''))$;

The double hat Morita equivalence

The DG-functor $B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ is constructed as follows. Given a left CDG-module $M^\bullet = (M, d_M)$ over $B^\bullet$, the underlying graded abelian group of the related DG-module $E^\bullet = (E, D_E)$ over $\widehat{\widehat{B}}^\bullet$ is the direct sum $E = M \oplus M[-1]$. So the elements of the abelian group E^i ($i \in \mathbb{Z}$) are pairs (m', m'') with $m' \in M^i$ and $m'' \in M^{i-1}$.

The actions of B, δ, ϵ, and D_E on E are given by the rules

- $b(m', m'') = (bm', (-1)^{|b|} bm'')$ for $b \in B$;
- $\delta(m', m'') = (d_M(m'), m' - d_M(m''))$;
- $\epsilon(m', m'') = (m'', 0)$;
- $D_E(m', m'') = (0, m')$.

The double hat Morita equivalence

The DG-functor $B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$ is constructed as follows. Given a left CDG-module $M^{\blacklozenge} = (M, d_M)$ over $B^{\blacklozenge}$, the underlying graded abelian group of the related DG-module $E^{\bullet} = (E, D_E)$ over $\widehat{\widehat{B}}^{\bullet}$ is the direct sum $E = M \oplus M[-1]$. So the elements of the abelian group E^i $(i \in \mathbb{Z})$ are pairs (m', m'') with $m' \in M^i$ and $m'' \in M^{i-1}$.

The actions of B, δ, ϵ, and D_E on E are given by the rules

- $b(m', m'') = (bm', (-1)^{|b|} bm'')$ for $b \in B$;
- $\delta(m', m'') = (d_M(m'), m' - d_M(m''))$;
- $\epsilon(m', m'') = (m'', 0)$;
- $D_E(m', m'') = (0, m')$.

In particular, the actions of B and δ correspond to the CDG-module structure on the CDG-module $E^{\blacklozenge} = \mathrm{cone}(\mathrm{id}_{M^{\blacklozenge}[-1]})$ over $B^{\blacklozenge}$

The double hat Morita equivalence

The DG-functor $B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}{}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$ is constructed as follows. Given a left CDG-module $M^{\blacklozenge} = (M, d_M)$ over $B^{\blacklozenge}$, the underlying graded abelian group of the related DG-module $E^{\bullet} = (E, D_E)$ over $\widehat{\widehat{B}}{}^{\bullet}$ is the direct sum $E = M \oplus M[-1]$. So the elements of the abelian group E^i $(i \in \mathbb{Z})$ are pairs (m', m'') with $m' \in M^i$ and $m'' \in M^{i-1}$.

The actions of B, δ, ϵ, and D_E on E are given by the rules

- $b(m', m'') = (bm', (-1)^{|b|} bm'')$ for $b \in B$;
- $\delta(m', m'') = (d_M(m'), m' - d_M(m''))$;
- $\epsilon(m', m'') = (m'', 0)$;
- $D_E(m', m'') = (0, m')$.

In particular, the actions of B and δ correspond to the CDG-module structure on the CDG-module $E^{\blacklozenge} = \mathrm{cone}(\mathrm{id}_{M^{\blacklozenge}[-1]})$ over $B^{\blacklozenge}$, with δ acting by the CDG-module differential d_E.

The double hat Morita equivalence

The previous slide defines the action of the DG-functor $B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$ on objects.

The double hat Morita equivalence

The previous slide defines the action of the DG-functor $B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$ on objects. Let us define its action on morphisms.

The double hat Morita equivalence

The previous slide defines the action of the DG-functor $B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ on objects. Let us define its action on morphisms.

Given two left CDG-modules $M^\blacklozenge$ and $N^\blacklozenge$ over $B^\blacklozenge$

The double hat Morita equivalence

The previous slide defines the action of the DG-functor $B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ on objects. Let us define its action on morphisms.

Given two left CDG-modules $M^\blacklozenge$ and $N^\blacklozenge$ over $B^\blacklozenge$, denote the related DG-modules by $E = M \oplus M[-1]$ and $F = N \oplus N[-1]$.

The double hat Morita equivalence

The previous slide defines the action of the DG-functor $B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ on objects. Let us define its action on morphisms.

Given two left CDG-modules $M^\blacklozenge$ and $N^\blacklozenge$ over $B^\blacklozenge$, denote the related DG-modules by $E = M \oplus M[-1]$ and $F = N \oplus N[-1]$. For any (not necessarily closed) morphism $f\colon M \longrightarrow N$ of degree $|f|$

The double hat Morita equivalence

The previous slide defines the action of the DG-functor $B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ on objects. Let us define its action on morphisms.

Given two left CDG-modules $M^\blacklozenge$ and $N^\blacklozenge$ over $B^\blacklozenge$, denote the related DG-modules by $E = M \oplus M[-1]$ and $F = N \oplus N[-1]$. For any (not necessarily closed) morphism $f\colon M \longrightarrow N$ of degree $|f|$, the related morphism $g\colon E \longrightarrow F$ is given by the formula

The double hat Morita equivalence

The previous slide defines the action of the DG-functor $B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ on objects. Let us define its action on morphisms.

Given two left CDG-modules $M^\bullet$ and $N^\bullet$ over $B^\bullet$, denote the related DG-modules by $E = M \oplus M[-1]$ and $F = N \oplus N[-1]$. For any (not necessarily closed) morphism $f\colon M \longrightarrow N$ of degree $|f|$, the related morphism $g\colon E \longrightarrow F$ is given by the formula

$$g(m', m'') = ((-1)^{|f|} f(m') + (df)(m''),\ f(m'')).$$

The double hat Morita equivalence

The previous slide defines the action of the DG-functor $B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$ on objects. Let us define its action on morphisms.

Given two left CDG-modules $M^{\blacklozenge}$ and $N^{\blacklozenge}$ over $B^{\blacklozenge}$, denote the related DG-modules by $E = M \oplus M[-1]$ and $F = N \oplus N[-1]$. For any (not necessarily closed) morphism $f\colon M \longrightarrow N$ of degree $|f|$, the related morphism $g\colon E \longrightarrow F$ is given by the formula

$$g(m', m'') = ((-1)^{|f|} f(m') + (df)(m''),\ f(m'')).$$

To construct the inverse functor, observe that the action of the operators ϵ and D_E on any DG-module (E, D_E) over $\widehat{\widehat{B}}^{\bullet}$

The double hat Morita equivalence

The previous slide defines the action of the DG-functor $B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$ on objects. Let us define its action on morphisms.

Given two left CDG-modules $M^{\blacklozenge}$ and $N^{\blacklozenge}$ over $B^{\blacklozenge}$, denote the related DG-modules by $E = M \oplus M[-1]$ and $F = N \oplus N[-1]$. For any (not necessarily closed) morphism $f\colon M \longrightarrow N$ of degree $|f|$, the related morphism $g\colon E \longrightarrow F$ is given by the formula

$$g(m', m'') = ((-1)^{|f|} f(m') + (df)(m''),\ f(m'')).$$

To construct the inverse functor, observe that the action of the operators ϵ and D_E on any DG-module (E, D_E) over $\widehat{\widehat{B}}^{\bullet}$ defines a representation of a graded ring of 2×2 matrices (with integer entries) in E.

The double hat Morita equivalence

The previous slide defines the action of the DG-functor $B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ on objects. Let us define its action on morphisms.

Given two left CDG-modules $M^\blacklozenge$ and $N^\blacklozenge$ over $B^\blacklozenge$, denote the related DG-modules by $E = M \oplus M[-1]$ and $F = N \oplus N[-1]$. For any (not necessarily closed) morphism $f\colon M \longrightarrow N$ of degree $|f|$, the related morphism $g\colon E \longrightarrow F$ is given by the formula

$$g(m', m'') = ((-1)^{|f|} f(m') + (df)(m''),\ f(m'')).$$

To construct the inverse functor, observe that the action of the operators ϵ and D_E on any DG-module (E, D_E) over $\widehat{\widehat{B}}^\bullet$ defines a representation of a graded ring of 2×2 matrices (with integer entries) in E. By a graded version of Morita theory for 2×2 matrices, such an action induces a direct sum decomposition of the form $E = M \oplus M[-1]$ for some graded abelian group M.

The double hat Morita equivalence

The previous slide defines the action of the DG-functor $B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}} \longrightarrow \widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ on objects. Let us define its action on morphisms.

Given two left CDG-modules $M^\blacklozenge$ and $N^\blacklozenge$ over $B^\blacklozenge$, denote the related DG-modules by $E = M \oplus M[-1]$ and $F = N \oplus N[-1]$. For any (not necessarily closed) morphism $f\colon M \longrightarrow N$ of degree $|f|$, the related morphism $g\colon E \longrightarrow F$ is given by the formula

$$g(m', m'') = ((-1)^{|f|} f(m') + (df)(m''),\ f(m'')).$$

To construct the inverse functor, observe that the action of the operators ϵ and D_E on any DG-module (E, D_E) over $\widehat{\widehat{B}}^\bullet$ defines a representation of a graded ring of 2×2 matrices (with integer entries) in E. By a graded version of Morita theory for 2×2 matrices, such an action induces a direct sum decomposition of the form $E = M \oplus M[-1]$ for some graded abelian group M. Then one can recover the actions of B and d_M on M.

Abelian DG-categories

Abelian DG-categories

For any DG-category $\mathbf{A}$, denote by $\mathrm{Z}^0(\mathbf{A})$ the preadditive category of closed morphisms of degree 0 in $\mathbf{A}$.

For any DG-category $\mathbf{A}$, denote by $\mathrm{Z}^0(\mathbf{A})$ the preadditive category of closed morphisms of degree 0 in $\mathbf{A}$. So the objects of $\mathrm{Z}^0(\mathbf{A})$ are the same as the objects of $\mathbf{A}$

For any DG-category $\mathbf{A}$, denote by $\mathrm{Z}^0(\mathbf{A})$ the preadditive category of closed morphisms of degree 0 in $\mathbf{A}$. So the objects of $\mathrm{Z}^0(\mathbf{A})$ are the same as the objects of $\mathbf{A}$, and $\mathrm{Hom}_{\mathrm{Z}^0(\mathbf{A})}(X, Y)$ is the group of degree-zero cocycles in the complex $\mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y)$.

Abelian DG-categories

For any DG-category $\mathbf{A}$, denote by $\mathrm{Z}^0(\mathbf{A})$ the preadditive category of closed morphisms of degree 0 in $\mathbf{A}$. So the objects of $\mathrm{Z}^0(\mathbf{A})$ are the same as the objects of $\mathbf{A}$, and $\mathrm{Hom}_{\mathrm{Z}^0(\mathbf{A})}(X, Y)$ is the group of degree-zero cocycles in the complex $\mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y)$.

Let $\mathbf{A}$ be a DG-category with a zero object, shifts, and cones

Abelian DG-categories

For any DG-category $\mathbf{A}$, denote by $\mathrm{Z}^0(\mathbf{A})$ the preadditive category of closed morphisms of degree 0 in $\mathbf{A}$. So the objects of $\mathrm{Z}^0(\mathbf{A})$ are the same as the objects of $\mathbf{A}$, and $\mathrm{Hom}_{\mathrm{Z}^0(\mathbf{A})}(X, Y)$ is the group of degree-zero cocycles in the complex $\mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y)$.

Let $\mathbf{A}$ be a DG-category with a zero object, shifts, and cones (a "strongly pretriangulated DG-category").

Abelian DG-categories

For any DG-category $\mathbf{A}$, denote by $\mathsf{Z}^0(\mathbf{A})$ the preadditive category of closed morphisms of degree 0 in $\mathbf{A}$. So the objects of $\mathsf{Z}^0(\mathbf{A})$ are the same as the objects of $\mathbf{A}$, and $\mathrm{Hom}_{\mathsf{Z}^0(\mathbf{A})}(X, Y)$ is the group of degree-zero cocycles in the complex $\mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y)$.

Let $\mathbf{A}$ be a DG-category with a zero object, shifts, and cones (a "strongly pretriangulated DG-category"). Then the DG-category $\mathbf{A}$ is said to be abelian if the additive category $\mathsf{Z}^0(\mathbf{A})$ is abelian.

For any DG-category $\mathbf{A}$, denote by $\mathrm{Z}^0(\mathbf{A})$ the preadditive category of closed morphisms of degree 0 in $\mathbf{A}$. So the objects of $\mathrm{Z}^0(\mathbf{A})$ are the same as the objects of $\mathbf{A}$, and $\mathrm{Hom}_{\mathrm{Z}^0(\mathbf{A})}(X, Y)$ is the group of degree-zero cocycles in the complex $\mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y)$.

Let $\mathbf{A}$ be a DG-category with a zero object, shifts, and cones (a "strongly pretriangulated DG-category"). Then the DG-category $\mathbf{A}$ is said to be abelian if the additive category $\mathrm{Z}^0(\mathbf{A})$ is abelian.

The definition above represents a simplified approach cutting some corners.

Abelian DG-categories

For any DG-category $\mathbf{A}$, denote by $\mathrm{Z}^0(\mathbf{A})$ the preadditive category of closed morphisms of degree 0 in $\mathbf{A}$. So the objects of $\mathrm{Z}^0(\mathbf{A})$ are the same as the objects of $\mathbf{A}$, and $\mathrm{Hom}_{\mathrm{Z}^0(\mathbf{A})}(X, Y)$ is the group of degree-zero cocycles in the complex $\mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y)$.

Let $\mathbf{A}$ be a DG-category with a zero object, shifts, and cones (a "strongly pretriangulated DG-category"). Then the DG-category $\mathbf{A}$ is said to be abelian if the additive category $\mathrm{Z}^0(\mathbf{A})$ is abelian.

The definition above represents a simplified approach cutting some corners. The fully developed approach involves the construction of two underlying abelian categories for an abelian DG-category $\mathbf{A}$

Abelian DG-categories

For any DG-category $\mathbf{A}$, denote by $\mathrm{Z}^0(\mathbf{A})$ the preadditive category of closed morphisms of degree 0 in $\mathbf{A}$. So the objects of $\mathrm{Z}^0(\mathbf{A})$ are the same as the objects of $\mathbf{A}$, and $\mathrm{Hom}_{\mathrm{Z}^0(\mathbf{A})}(X, Y)$ is the group of degree-zero cocycles in the complex $\mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y)$.

Let $\mathbf{A}$ be a DG-category with a zero object, shifts, and cones (a "strongly pretriangulated DG-category"). Then the DG-category $\mathbf{A}$ is said to be abelian if the additive category $\mathrm{Z}^0(\mathbf{A})$ is abelian.

The definition above represents a simplified approach cutting some corners. The fully developed approach involves the construction of two underlying abelian categories for an abelian DG-category $\mathbf{A}$: in addition to the abelian category $\mathrm{Z}^0(\mathbf{A})$ of closed morphisms of degree 0 in $\mathbf{A}$

For any DG-category $\mathbf{A}$, denote by $\mathrm{Z}^0(\mathbf{A})$ the preadditive category of closed morphisms of degree 0 in $\mathbf{A}$. So the objects of $\mathrm{Z}^0(\mathbf{A})$ are the same as the objects of $\mathbf{A}$, and $\mathrm{Hom}_{\mathrm{Z}^0(\mathbf{A})}(X, Y)$ is the group of degree-zero cocycles in the complex $\mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y)$.

Let $\mathbf{A}$ be a DG-category with a zero object, shifts, and cones (a "strongly pretriangulated DG-category"). Then the DG-category $\mathbf{A}$ is said to be abelian if the additive category $\mathrm{Z}^0(\mathbf{A})$ is abelian.

The definition above represents a simplified approach cutting some corners. The fully developed approach involves the construction of two underlying abelian categories for an abelian DG-category $\mathbf{A}$: in addition to the abelian category $\mathrm{Z}^0(\mathbf{A})$ of closed morphisms of degree 0 in $\mathbf{A}$, there is also a naturally constructed abelian category of underlying graded objects $\mathrm{Z}^0(\mathbf{A}^\natural)$ for $\mathbf{A}$.

Abelian DG-categories

For any DG-category $\mathbf{A}$, denote by $\mathrm{Z}^0(\mathbf{A})$ the preadditive category of closed morphisms of degree 0 in $\mathbf{A}$. So the objects of $\mathrm{Z}^0(\mathbf{A})$ are the same as the objects of $\mathbf{A}$, and $\mathrm{Hom}_{\mathrm{Z}^0(\mathbf{A})}(X, Y)$ is the group of degree-zero cocycles in the complex $\mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y)$.

Let $\mathbf{A}$ be a DG-category with a zero object, shifts, and cones (a "strongly pretriangulated DG-category"). Then the DG-category $\mathbf{A}$ is said to be abelian if the additive category $\mathrm{Z}^0(\mathbf{A})$ is abelian.

The definition above represents a simplified approach cutting some corners. The fully developed approach involves the construction of two underlying abelian categories for an abelian DG-category $\mathbf{A}$: in addition to the abelian category $\mathrm{Z}^0(\mathbf{A})$ of closed morphisms of degree 0 in $\mathbf{A}$, there is also a naturally constructed abelian category of underlying graded objects $\mathrm{Z}^0(\mathbf{A}^\natural)$ for $\mathbf{A}$.

For example, if $\mathbf{A} = B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ is the DG-category of CDG-modules over a CDG-ring $B^\blacklozenge$

Abelian DG-categories

For any DG-category $\mathbf{A}$, denote by $\mathrm{Z}^0(\mathbf{A})$ the preadditive category of closed morphisms of degree 0 in $\mathbf{A}$. So the objects of $\mathrm{Z}^0(\mathbf{A})$ are the same as the objects of $\mathbf{A}$, and $\mathrm{Hom}_{\mathrm{Z}^0(\mathbf{A})}(X, Y)$ is the group of degree-zero cocycles in the complex $\mathrm{Hom}^\bullet_{\mathbf{A}}(X, Y)$.

Let $\mathbf{A}$ be a DG-category with a zero object, shifts, and cones (a "strongly pretriangulated DG-category"). Then the DG-category $\mathbf{A}$ is said to be abelian if the additive category $\mathrm{Z}^0(\mathbf{A})$ is abelian.

The definition above represents a simplified approach cutting some corners. The fully developed approach involves the construction of two underlying abelian categories for an abelian DG-category $\mathbf{A}$: in addition to the abelian category $\mathrm{Z}^0(\mathbf{A})$ of closed morphisms of degree 0 in $\mathbf{A}$, there is also a naturally constructed abelian category of underlying graded objects $\mathrm{Z}^0(\mathbf{A}^\natural)$ for $\mathbf{A}$.

For example, if $\mathbf{A} = B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ is the DG-category of CDG-modules over a CDG-ring $B^\bullet$, then the abelian category $\mathrm{Z}^0(\mathbf{A}^\natural)$ is equivalent to the category of graded B-modules.

Here the bécarre construction $\mathbf{A} \longmapsto \mathbf{A}^\natural$

Abelian DG-categories

Here the bécarre construction $\mathbf{A} \longmapsto \mathbf{A}^\natural$ is an "almost involution" on strongly pretriangulated DG-categories.

Abelian DG-categories

Here the bécarre construction $\mathbf{A} \longmapsto \mathbf{A}^\natural$ is an "almost involution" on strongly pretriangulated DG-categories. It is a DG-category version/generalization of the hat construction for (C)DG-rings.

Abelian DG-categories

Here the bécarre construction $\mathbf{A} \longmapsto \mathbf{A}^\natural$ is an "almost involution" on strongly pretriangulated DG-categories. It is a DG-category version/generalization of the hat construction for (C)DG-rings.

For any CDG-ring $B^\bullet$, the bécarre DG-category of the DG-category of CDG-modules $(B^\bullet\text{-Mod}^{\mathrm{cdg}})^\natural$

Abelian DG-categories

Here the bécarre construction $\mathbf{A} \longmapsto \mathbf{A}^\natural$ is an "almost involution" on strongly pretriangulated DG-categories. It is a DG-category version/generalization of the hat construction for (C)DG-rings.

For any CDG-ring $B^\bullet$, the bécarre DG-category of the DG-category of CDG-modules $(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}})^\natural$ is naturally equivalent to the DG-category of DG-modules $\widehat{B}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ over the hat DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$

Here the bécarre construction $\mathbf{A} \longmapsto \mathbf{A}^\natural$ is an "almost involution" on strongly pretriangulated DG-categories. It is a DG-category version/generalization of the hat construction for (C)DG-rings.

For any CDG-ring $B^\blacklozenge$, the bécarre DG-category of the DG-category of CDG-modules $(B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}})^\natural$ is naturally equivalent to the DG-category of DG-modules $\widehat{B}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ over the hat DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$, that is $(B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}})^\natural \simeq \widehat{B}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$.

Abelian DG-categories

Here the bécarre construction $\mathbf{A} \longmapsto \mathbf{A}^\natural$ is an "almost involution" on strongly pretriangulated DG-categories. It is a DG-category version/generalization of the hat construction for (C)DG-rings.

For any CDG-ring $B^\bullet$, the bécarre DG-category of the DG-category of CDG-modules $(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}})^\natural$ is naturally equivalent to the DG-category of DG-modules $\widehat{B}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ over the hat DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$, that is $(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}})^\natural \simeq \widehat{B}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$.

There is also a related concept of an exact DG-category, generalizing the notion of an abelian DG-category.

Abelian DG-categories

Here the bécarre construction $\mathbf{A} \longmapsto \mathbf{A}^\natural$ is an "almost involution" on strongly pretriangulated DG-categories. It is a DG-category version/generalization of the hat construction for (C)DG-rings.

For any CDG-ring $B^\blacklozenge$, the bécarre DG-category of the DG-category of CDG-modules $(B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}})^\natural$ is naturally equivalent to the DG-category of DG-modules $\widehat{B}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ over the hat DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$, that is $(B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}})^\natural \simeq \widehat{B}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$.

There is also a related concept of an exact DG-category, generalizing the notion of an abelian DG-category.

Further details can be found in the preprint and paper

Abelian DG-categories

Here the bécarre construction $\mathbf{A} \longmapsto \mathbf{A}^\natural$ is an "almost involution" on strongly pretriangulated DG-categories. It is a DG-category version/generalization of the hat construction for (C)DG-rings.

For any CDG-ring $B^\blacklozenge$, the bécarre DG-category of the DG-category of CDG-modules $(B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}})^\natural$ is naturally equivalent to the DG-category of DG-modules $\widehat{B}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ over the hat DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$, that is $(B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}})^\natural \simeq \widehat{B}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$.

There is also a related concept of an exact DG-category, generalizing the notion of an abelian DG-category.

Further details can be found in the preprint and paper

- L. Positselski, "Exact DG-categories and fully faithful triangulated inclusion functors", `arXiv:2110.08237` ;

Abelian DG-categories

Here the bécarre construction $\mathbf{A} \longmapsto \mathbf{A}^\natural$ is an "almost involution" on strongly pretriangulated DG-categories. It is a DG-category version/generalization of the hat construction for (C)DG-rings.

For any CDG-ring $B^\blacklozenge$, the bécarre DG-category of the DG-category of CDG-modules $(B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}})^\natural$ is naturally equivalent to the DG-category of DG-modules $\widehat{B}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ over the hat DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$, that is $(B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}})^\natural \simeq \widehat{B}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$.

There is also a related concept of an exact DG-category, generalizing the notion of an abelian DG-category.

Further details can be found in the preprint and paper

- L. Positselski, "Exact DG-categories and fully faithful triangulated inclusion functors", `arXiv:2110.08237`;
- L. Positselski, J. Šťovíček, "Coderived and contraderived categories of locally presentable abelian DG-categories", Math. Zeitschrift **308**, 2024.

Derived categories of the second kind

Derived categories of the second kind

For any DG-category $\mathbf{A}$, denote by $\mathrm{H}^0(\mathbf{A})$ the category of closed morphisms up to cochain homotopy in $\mathbf{A}$

For any DG-category $\mathbf{A}$, denote by $\mathrm{H}^0(\mathbf{A})$ the category of closed morphisms up to cochain homotopy in $\mathbf{A}$, i. e., the quotient category of $\mathrm{Z}^0(\mathbf{A})$ by the ideal of morphisms homotopic to zero.

For any DG-category $\mathbf{A}$, denote by $\mathrm{H}^0(\mathbf{A})$ the category of closed morphisms up to cochain homotopy in $\mathbf{A}$, i. e., the quotient category of $\mathrm{Z}^0(\mathbf{A})$ by the ideal of morphisms homotopic to zero. If $\mathbf{A}$ is a DG-category with a zero object, shifts, and cones, then $\mathrm{H}^0(\mathbf{A})$ is a triangulated category.

Derived categories of the second kind

For any DG-category $\mathbf{A}$, denote by $\mathrm{H}^0(\mathbf{A})$ the category of closed morphisms up to cochain homotopy in $\mathbf{A}$, i. e., the quotient category of $\mathrm{Z}^0(\mathbf{A})$ by the ideal of morphisms homotopic to zero. If $\mathbf{A}$ is a DG-category with a zero object, shifts, and cones, then $\mathrm{H}^0(\mathbf{A})$ is a triangulated category.

Let $\mathbf{A}$ be an abelian DG-category.

Derived categories of the second kind

For any DG-category $\mathbf{A}$, denote by $\mathrm{H}^0(\mathbf{A})$ the category of closed morphisms up to cochain homotopy in $\mathbf{A}$, i. e., the quotient category of $\mathrm{Z}^0(\mathbf{A})$ by the ideal of morphisms homotopic to zero. If $\mathbf{A}$ is a DG-category with a zero object, shifts, and cones, then $\mathrm{H}^0(\mathbf{A})$ is a triangulated category.

Let $\mathbf{A}$ be an abelian DG-category. To any short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ in the abelian category $\mathrm{Z}^0(\mathbf{A})$

Derived categories of the second kind

For any DG-category $\mathbf{A}$, denote by $\mathrm{H}^0(\mathbf{A})$ the category of closed morphisms up to cochain homotopy in $\mathbf{A}$, i. e., the quotient category of $\mathrm{Z}^0(\mathbf{A})$ by the ideal of morphisms homotopic to zero. If $\mathbf{A}$ is a DG-category with a zero object, shifts, and cones, then $\mathrm{H}^0(\mathbf{A})$ is a triangulated category.

Let $\mathbf{A}$ be an abelian DG-category. To any short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ in the abelian category $\mathrm{Z}^0(\mathbf{A})$ one can assign its totalization $\mathrm{Tot}(K \to L \to M)$, which is an object of $\mathbf{A}$.

For any DG-category $\mathbf{A}$, denote by $\mathrm{H}^0(\mathbf{A})$ the category of closed morphisms up to cochain homotopy in $\mathbf{A}$, i. e., the quotient category of $\mathrm{Z}^0(\mathbf{A})$ by the ideal of morphisms homotopic to zero. If $\mathbf{A}$ is a DG-category with a zero object, shifts, and cones, then $\mathrm{H}^0(\mathbf{A})$ is a triangulated category.

Let $\mathbf{A}$ be an abelian DG-category. To any short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ in the abelian category $\mathrm{Z}^0(\mathbf{A})$ one can assign its totalization $\mathrm{Tot}(K \to L \to M)$, which is an object of $\mathbf{A}$. This object can be constructed as an iterated cone, $\mathrm{Tot}(K \to L \to M) = \mathrm{cone}(\mathrm{cone}(K \to L) \to M)$.

Derived categories of the second kind

For any DG-category $\mathbf{A}$, denote by $\mathrm{H}^0(\mathbf{A})$ the category of closed morphisms up to cochain homotopy in $\mathbf{A}$, i. e., the quotient category of $\mathrm{Z}^0(\mathbf{A})$ by the ideal of morphisms homotopic to zero. If $\mathbf{A}$ is a DG-category with a zero object, shifts, and cones, then $\mathrm{H}^0(\mathbf{A})$ is a triangulated category.

Let $\mathbf{A}$ be an abelian DG-category. To any short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ in the abelian category $\mathrm{Z}^0(\mathbf{A})$ one can assign its totalization $\mathrm{Tot}(K \to L \to M)$, which is an object of $\mathbf{A}$. This object can be constructed as an iterated cone, $\mathrm{Tot}(K \to L \to M) = \mathrm{cone}(\mathrm{cone}(K \to L) \to M)$.

The full subcategory of absolutely acyclic objects $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$

Derived categories of the second kind

For any DG-category $\mathbf{A}$, denote by $\mathrm{H}^0(\mathbf{A})$ the category of closed morphisms up to cochain homotopy in $\mathbf{A}$, i. e., the quotient category of $\mathrm{Z}^0(\mathbf{A})$ by the ideal of morphisms homotopic to zero. If $\mathbf{A}$ is a DG-category with a zero object, shifts, and cones, then $\mathrm{H}^0(\mathbf{A})$ is a triangulated category.

Let $\mathbf{A}$ be an abelian DG-category. To any short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ in the abelian category $\mathrm{Z}^0(\mathbf{A})$ one can assign its totalization $\mathrm{Tot}(K \to L \to M)$, which is an object of $\mathbf{A}$. This object can be constructed as an iterated cone, $\mathrm{Tot}(K \to L \to M) = \mathrm{cone}(\mathrm{cone}(K \to L) \to M)$.

The full subcategory of absolutely acyclic objects $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the thick subcategory of $\mathrm{H}^0(\mathbf{A})$ generated by the totalizations of short exact sequences in $\mathrm{Z}^0(\mathbf{A})$.

Derived categories of the second kind

For any DG-category $\mathbf{A}$, denote by $\mathrm{H}^0(\mathbf{A})$ the category of closed morphisms up to cochain homotopy in $\mathbf{A}$, i. e., the quotient category of $\mathrm{Z}^0(\mathbf{A})$ by the ideal of morphisms homotopic to zero. If $\mathbf{A}$ is a DG-category with a zero object, shifts, and cones, then $\mathrm{H}^0(\mathbf{A})$ is a triangulated category.

Let $\mathbf{A}$ be an abelian DG-category. To any short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ in the abelian category $\mathrm{Z}^0(\mathbf{A})$ one can assign its totalization $\mathrm{Tot}(K \to L \to M)$, which is an object of $\mathbf{A}$. This object can be constructed as an iterated cone, $\mathrm{Tot}(K \to L \to M) = \mathrm{cone}(\mathrm{cone}(K \to L) \to M)$.

The full subcategory of absolutely acyclic objects $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the thick subcategory of $\mathrm{H}^0(\mathbf{A})$ generated by the totalizations of short exact sequences in $\mathrm{Z}^0(\mathbf{A})$. Equivalently, viewed as a full subcategory in $\mathrm{Z}^0(\mathbf{A})$

For any DG-category $\mathbf{A}$, denote by $\mathrm{H}^0(\mathbf{A})$ the category of closed morphisms up to cochain homotopy in $\mathbf{A}$, i. e., the quotient category of $\mathrm{Z}^0(\mathbf{A})$ by the ideal of morphisms homotopic to zero. If $\mathbf{A}$ is a DG-category with a zero object, shifts, and cones, then $\mathrm{H}^0(\mathbf{A})$ is a triangulated category.

Let $\mathbf{A}$ be an abelian DG-category. To any short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ in the abelian category $\mathrm{Z}^0(\mathbf{A})$ one can assign its totalization $\mathsf{Tot}(K \to L \to M)$, which is an object of $\mathbf{A}$. This object can be constructed as an iterated cone, $\mathsf{Tot}(K \to L \to M) = \mathsf{cone}(\mathsf{cone}(K \to L) \to M)$.

The full subcategory of absolutely acyclic objects $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the thick subcategory of $\mathrm{H}^0(\mathbf{A})$ generated by the totalizations of short exact sequences in $\mathrm{Z}^0(\mathbf{A})$. Equivalently, viewed as a full subcategory in $\mathrm{Z}^0(\mathbf{A})$, $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A})$ is the minimal full subcategory containing the contractible objects and closed under extensions and direct summands.

Derived categories of the second kind

Assume that infinite coproducts exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$.

Assume that infinite coproducts exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$. Then one can check that coproducts also exist in the triangulated category $\mathrm{H}^0(\mathbf{A})$.

Derived categories of the second kind

Assume that infinite coproducts exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$. Then one can check that coproducts also exist in the triangulated category $\mathrm{H}^0(\mathbf{A})$.

The full subcategory of coacyclic objects $\mathrm{Ac}^{\mathrm{co}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$

Derived categories of the second kind

Assume that infinite coproducts exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$. Then one can check that coproducts also exist in the triangulated category $\mathrm{H}^0(\mathbf{A})$.

The full subcategory of coacyclic objects $\mathrm{Ac}^{\mathrm{co}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the minimal triangulated subcategory in $\mathrm{H}^0(\mathbf{A})$ containing $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A})$ and closed under coproducts.

Assume that infinite coproducts exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$. Then one can check that coproducts also exist in the triangulated category $\mathrm{H}^0(\mathbf{A})$.

The full subcategory of coacyclic objects $\mathrm{Ac}^{\mathrm{co}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the minimal triangulated subcategory in $\mathrm{H}^0(\mathbf{A})$ containing $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A})$ and closed under coproducts.

Dually, if infinite products exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$

Derived categories of the second kind

Assume that infinite coproducts exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$. Then one can check that coproducts also exist in the triangulated category $\mathrm{H}^0(\mathbf{A})$.

The full subcategory of coacyclic objects $\mathrm{Ac}^{\mathrm{co}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the minimal triangulated subcategory in $\mathrm{H}^0(\mathbf{A})$ containing $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A})$ and closed under coproducts.

Dually, if infinite products exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$, then the full subcategory of contraacyclic objects $\mathrm{Ac}^{\mathrm{ctr}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$

Derived categories of the second kind

Assume that infinite coproducts exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$. Then one can check that coproducts also exist in the triangulated category $\mathrm{H}^0(\mathbf{A})$.

The full subcategory of coacyclic objects $\mathrm{Ac}^{\mathrm{co}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the minimal triangulated subcategory in $\mathrm{H}^0(\mathbf{A})$ containing $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A})$ and closed under coproducts.

Dually, if infinite products exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$, then the full subcategory of contraacyclic objects $\mathrm{Ac}^{\mathrm{ctr}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the minimal triangulated subcategory in $\mathrm{H}^0(\mathbf{A})$ containing $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A})$ and closed under products.

Derived categories of the second kind

Assume that infinite coproducts exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$. Then one can check that coproducts also exist in the triangulated category $\mathrm{H}^0(\mathbf{A})$.

The full subcategory of coacyclic objects $\mathrm{Ac}^{\mathrm{co}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the minimal triangulated subcategory in $\mathrm{H}^0(\mathbf{A})$ containing $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A})$ and closed under coproducts.

Dually, if infinite products exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$, then the full subcategory of contraacyclic objects $\mathrm{Ac}^{\mathrm{ctr}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the minimal triangulated subcategory in $\mathrm{H}^0(\mathbf{A})$ containing $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A})$ and closed under products.

The absolute derived, coderived, and contraderived categories of $\mathbf{A}$

Derived categories of the second kind

Assume that infinite coproducts exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$. Then one can check that coproducts also exist in the triangulated category $\mathrm{H}^0(\mathbf{A})$.

The full subcategory of coacyclic objects $\mathrm{Ac}^{\mathrm{co}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the minimal triangulated subcategory in $\mathrm{H}^0(\mathbf{A})$ containing $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A})$ and closed under coproducts.

Dually, if infinite products exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$, then the full subcategory of contraacyclic objects $\mathrm{Ac}^{\mathrm{ctr}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the minimal triangulated subcategory in $\mathrm{H}^0(\mathbf{A})$ containing $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A})$ and closed under products.

The absolute derived, coderived, and contraderived categories of $\mathbf{A}$ are constructed as the triangulated Verdier quotient categories

Derived categories of the second kind

Assume that infinite coproducts exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$. Then one can check that coproducts also exist in the triangulated category $\mathrm{H}^0(\mathbf{A})$.

The full subcategory of coacyclic objects $\mathrm{Ac}^{\mathrm{co}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the minimal triangulated subcategory in $\mathrm{H}^0(\mathbf{A})$ containing $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A})$ and closed under coproducts.

Dually, if infinite products exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$, then the full subcategory of contraacyclic objects $\mathrm{Ac}^{\mathrm{ctr}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the minimal triangulated subcategory in $\mathrm{H}^0(\mathbf{A})$ containing $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A})$ and closed under products.

The absolute derived, coderived, and contraderived categories of $\mathbf{A}$ are constructed as the triangulated Verdier quotient categories

$$\mathrm{D}^{\mathrm{abs}}(\mathbf{A}) = \mathrm{H}^0(\mathbf{A})/\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A}),$$

$$\mathrm{D}^{\mathrm{co}}(\mathbf{A}) = \mathrm{H}^0(\mathbf{A})/\mathrm{Ac}^{\mathrm{co}}(\mathbf{A}), \quad \mathrm{D}^{\mathrm{ctr}}(\mathbf{A}) = \mathrm{H}^0(\mathbf{A})/\mathrm{Ac}^{\mathrm{ctr}}(\mathbf{A})$$

Assume that infinite coproducts exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$. Then one can check that coproducts also exist in the triangulated category $\mathrm{H}^0(\mathbf{A})$.

The full subcategory of coacyclic objects $\mathrm{Ac}^{\mathrm{co}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the minimal triangulated subcategory in $\mathrm{H}^0(\mathbf{A})$ containing $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A})$ and closed under coproducts.

Dually, if infinite products exist in the abelian category $\mathrm{Z}^0(\mathbf{A})$, then the full subcategory of contraacyclic objects $\mathrm{Ac}^{\mathrm{ctr}}(\mathbf{A}) \subset \mathrm{H}^0(\mathbf{A})$ is defined as the minimal triangulated subcategory in $\mathrm{H}^0(\mathbf{A})$ containing $\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A})$ and closed under products.

The absolute derived, coderived, and contraderived categories of $\mathbf{A}$ are constructed as the triangulated Verdier quotient categories

$$\mathrm{D}^{\mathrm{abs}}(\mathbf{A}) = \mathrm{H}^0(\mathbf{A})/\mathrm{Ac}^{\mathrm{abs}}(\mathbf{A}),$$

$$\mathrm{D}^{\mathrm{co}}(\mathbf{A}) = \mathrm{H}^0(\mathbf{A})/\mathrm{Ac}^{\mathrm{co}}(\mathbf{A}), \quad \mathrm{D}^{\mathrm{ctr}}(\mathbf{A}) = \mathrm{H}^0(\mathbf{A})/\mathrm{Ac}^{\mathrm{ctr}}(\mathbf{A})$$

of $\mathrm{H}^0(\mathbf{A})$ by the respective thick subcategories of acyclic objects.

Derived categories of the second kind

Derived categories of the second kind

In particular, for the abelian DG-category $\mathbf{A} = B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of CDG-modules over a CDG-ring $B^\blacklozenge = (B, d, h)$

Derived categories of the second kind

In particular, for the abelian DG-category $\mathbf{A} = B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of CDG-modules over a CDG-ring $B^\bullet = (B, d, h)$, one obtains the absolute derived, coderived, and contraderived category of CDG-modules

Derived categories of the second kind

In particular, for the abelian DG-category $\mathbf{A} = B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of CDG-modules over a CDG-ring $B^\bullet = (B, d, h)$, one obtains the absolute derived, coderived, and contraderived category of CDG-modules

$$\mathrm{D}^{\mathrm{abs}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}), \quad \mathrm{D}^{\mathrm{co}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}), \quad \mathrm{D}^{\mathrm{ctr}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}).$$

Derived categories of the second kind

In particular, for the abelian DG-category $\mathbf{A} = B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of CDG-modules over a CDG-ring $B^\blacklozenge = (B, d, h)$, one obtains the absolute derived, coderived, and contraderived category of CDG-modules

$$\mathrm{D}^{\mathrm{abs}}(B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}}), \quad \mathrm{D}^{\mathrm{co}}(B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}}), \quad \mathrm{D}^{\mathrm{ctr}}(B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}}).$$

Specializing further, for the abelian DG-category $\mathbf{A} = A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ of DG-modules over a DG-ring $A^\bullet = (A, d)$

Derived categories of the second kind

In particular, for the abelian DG-category $\mathbf{A} = B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of CDG-modules over a CDG-ring $B^{\blacklozenge} = (B, d, h)$, one obtains the absolute derived, coderived, and contraderived category of CDG-modules

$$\mathrm{D}^{\mathrm{abs}}(B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}}), \quad \mathrm{D}^{\mathrm{co}}(B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}}), \quad \mathrm{D}^{\mathrm{ctr}}(B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}}).$$

Specializing further, for the abelian DG-category $\mathbf{A} = A^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$ of DG-modules over a DG-ring $A^{\bullet} = (A, d)$, one obtains the absolute derived, coderived, and contraderived category of DG-modules

Derived categories of the second kind

In particular, for the abelian DG-category $\mathbf{A} = B^{\blacklozenge}\text{-Mod}^{\mathrm{cdg}}$ of CDG-modules over a CDG-ring $B^{\blacklozenge} = (B, d, h)$, one obtains the absolute derived, coderived, and contraderived category of CDG-modules

$$\mathrm{D^{abs}}(B^{\blacklozenge}\text{-Mod}^{\mathrm{cdg}}), \quad \mathrm{D^{co}}(B^{\blacklozenge}\text{-Mod}^{\mathrm{cdg}}), \quad \mathrm{D^{ctr}}(B^{\blacklozenge}\text{-Mod}^{\mathrm{cdg}}).$$

Specializing further, for the abelian DG-category $\mathbf{A} = A^{\bullet}\text{-Mod}^{\mathrm{dg}}$ of DG-modules over a DG-ring $A^{\bullet} = (A, d)$, one obtains the absolute derived, coderived, and contraderived category of DG-modules

$$\mathrm{D^{abs}}(A^{\bullet}\text{-Mod}^{\mathrm{dg}}), \quad \mathrm{D^{co}}(A^{\bullet}\text{-Mod}^{\mathrm{dg}}), \quad \mathrm{D^{ctr}}(A^{\bullet}\text{-Mod}^{\mathrm{dg}}).$$

Derived categories of the second kind

In particular, for the abelian DG-category $\mathbf{A} = B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of CDG-modules over a CDG-ring $B^{\blacklozenge} = (B, d, h)$, one obtains the absolute derived, coderived, and contraderived category of CDG-modules

$$\mathrm{D}^{\mathrm{abs}}(B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}}), \quad \mathrm{D}^{\mathrm{co}}(B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}}), \quad \mathrm{D}^{\mathrm{ctr}}(B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}}).$$

Specializing further, for the abelian DG-category $\mathbf{A} = A^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$ of DG-modules over a DG-ring $A^{\bullet} = (A, d)$, one obtains the absolute derived, coderived, and contraderived category of DG-modules

$$\mathrm{D}^{\mathrm{abs}}(A^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}), \quad \mathrm{D}^{\mathrm{co}}(A^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}), \quad \mathrm{D}^{\mathrm{ctr}}(A^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}).$$

Now we can recall the equivalence of DG-categories $B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}} \simeq \widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}}$

Derived categories of the second kind

In particular, for the abelian DG-category $\mathbf{A} = B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ of CDG-modules over a CDG-ring $B^\blacklozenge = (B, d, h)$, one obtains the absolute derived, coderived, and contraderived category of CDG-modules

$$\mathrm{D}^{\mathrm{abs}}(B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}}), \quad \mathrm{D}^{\mathrm{co}}(B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}}), \quad \mathrm{D}^{\mathrm{ctr}}(B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}}).$$

Specializing further, for the abelian DG-category $\mathbf{A} = A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ of DG-modules over a DG-ring $A^\bullet = (A, d)$, one obtains the absolute derived, coderived, and contraderived category of DG-modules

$$\mathrm{D}^{\mathrm{abs}}(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}), \quad \mathrm{D}^{\mathrm{co}}(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}), \quad \mathrm{D}^{\mathrm{ctr}}(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}).$$

Now we can recall the equivalence of DG-categories $B^\blacklozenge\text{-}\mathrm{Mod}^{\mathrm{cdg}} \simeq \widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$, where $\widehat{\widehat{B}}^\bullet = (\widehat{\widehat{B}}, \partial)$ is an acyclic DG-ring produced by the double hat construction applied to $B^\blacklozenge$.

Derived categories of the second kind

As derived categories of the second kind are invariants of an abelian DG-category

Derived categories of the second kind

As derived categories of the second kind are invariants of an abelian DG-category, it follows that the absolute derived/coderived/contraderived categories of an arbitrary CDG-ring

Derived categories of the second kind

As derived categories of the second kind are invariants of an abelian DG-category, it follows that the absolute derived/coderived/contraderived categories of an arbitrary CDG-ring are equivalent to those of some acyclic DG-ring.

Derived categories of the second kind

As derived categories of the second kind are invariants of an abelian DG-category, it follows that the absolute derived/coderived/contraderived categories of an arbitrary CDG-ring are equivalent to those of some acyclic DG-ring.

So we have $\mathrm{D}^{\mathrm{abs}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{abs}}(\widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$,
$\mathrm{D}^{\mathrm{co}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{co}}(\widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$,
$\mathrm{D}^{\mathrm{ctr}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$.

Derived categories of the second kind

As derived categories of the second kind are invariants of an abelian DG-category, it follows that the absolute derived/coderived/contraderived categories of an arbitrary CDG-ring are equivalent to those of some acyclic DG-ring.

So we have $\mathrm{D}^{\mathrm{abs}}(B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{abs}}(\widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}})$,
$\mathrm{D}^{\mathrm{co}}(B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{co}}(\widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}})$,
$\mathrm{D}^{\mathrm{ctr}}(B^{\blacklozenge}\text{-}\mathrm{Mod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\widehat{\widehat{B}}^{\bullet}\text{-}\mathrm{Mod}^{\mathrm{dg}})$.

In particular, to any DG-ring $A^{\bullet} = (A, d)$, the double hat construction assigns an acyclic DG-ring $\widehat{\widehat{A}}^{\bullet} = (\widehat{\widehat{A}}, \partial)$.

Derived categories of the second kind

As derived categories of the second kind are invariants of an abelian DG-category, it follows that the absolute derived/coderived/contraderived categories of an arbitrary CDG-ring are equivalent to those of some acyclic DG-ring.

So we have $\mathrm{D}^{\mathrm{abs}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{abs}}(\widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$,

$\mathrm{D}^{\mathrm{co}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{co}}(\widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$,

$\mathrm{D}^{\mathrm{ctr}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$.

In particular, to any DG-ring $A^\bullet = (A, d)$, the double hat construction assigns an acyclic DG-ring $\widehat{\widehat{A}}^\bullet = (\widehat{\widehat{A}}, \partial)$.

We have $\mathrm{D}^{\mathrm{abs}}(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}) \simeq \mathrm{D}^{\mathrm{abs}}(\widehat{\widehat{A}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$,

$\mathrm{D}^{\mathrm{co}}(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}) \simeq \mathrm{D}^{\mathrm{co}}(\widehat{\widehat{A}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$,

$\mathrm{D}^{\mathrm{ctr}}(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\widehat{\widehat{A}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$.

Derived categories of the second kind

As derived categories of the second kind are invariants of an abelian DG-category, it follows that the absolute derived/coderived/contraderived categories of an arbitrary CDG-ring are equivalent to those of some acyclic DG-ring.

So we have $\mathrm{D}^{\mathrm{abs}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{abs}}(\widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$,

$\mathrm{D}^{\mathrm{co}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{co}}(\widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$,

$\mathrm{D}^{\mathrm{ctr}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\widehat{\widehat{B}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$.

In particular, to any DG-ring $A^\bullet = (A, d)$, the double hat construction assigns an acyclic DG-ring $\widehat{\widehat{A}}^\bullet = (\widehat{\widehat{A}}, \partial)$.

We have $\mathrm{D}^{\mathrm{abs}}(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}) \simeq \mathrm{D}^{\mathrm{abs}}(\widehat{\widehat{A}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$,

$\mathrm{D}^{\mathrm{co}}(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}) \simeq \mathrm{D}^{\mathrm{co}}(\widehat{\widehat{A}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$,

$\mathrm{D}^{\mathrm{ctr}}(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\widehat{\widehat{A}}^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$. Thus it always suffices to consider derived categories of the second kind for acyclic DG-rings!

Derived category of the first kind

Derived category of the first kind

For comparison, the conventional derived category of DG-modules behaves very differently.

Derived category of the first kind

For comparison, the conventional derived category of DG-modules behaves very differently.

Given a DG-ring $A^\bullet = (A, d)$, the derived category $\mathrm{D}(A^\bullet\text{-Mod})$

Derived category of the first kind

For comparison, the conventional derived category of DG-modules behaves very differently.

Given a DG-ring $A^\bullet = (A, d)$, the derived category $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$ is defined as the Verdier quotient category of the homotopy category $\mathrm{H}^0(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$ by the thick subcategory of acyclic DG-modules.

Derived category of the first kind

For comparison, the conventional derived category of DG-modules behaves very differently.

Given a DG-ring $A^\bullet = (A, d)$, the derived category $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$ is defined as the Verdier quotient category of the homotopy category $\mathrm{H}^0(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$ by the thick subcategory of acyclic DG-modules.

In particular, over an acyclic DG-ring $R^\bullet = (R, \partial)$, all DG-modules $E^\bullet$ are also acyclic

Derived category of the first kind

For comparison, the conventional derived category of DG-modules behaves very differently.

Given a DG-ring $A^\bullet = (A, d)$, the derived category $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$ is defined as the Verdier quotient category of the homotopy category $\mathrm{H}^0(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$ by the thick subcategory of acyclic DG-modules.

In particular, over an acyclic DG-ring $R^\bullet = (R, \partial)$, all DG-modules $E^\bullet$ are also acyclic (as $H^*(E^\bullet)$ is a module over the zero ring $H^*(R^\bullet)$, and all such modules are zero).

Derived category of the first kind

For comparison, the conventional derived category of DG-modules behaves very differently.

Given a DG-ring $A^\bullet = (A, d)$, the derived category $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$ is defined as the Verdier quotient category of the homotopy category $\mathrm{H}^0(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$ by the thick subcategory of acyclic DG-modules.

In particular, over an acyclic DG-ring $R^\bullet = (R, \partial)$, all DG-modules $E^\bullet$ are also acyclic (as $H^*(E^\bullet)$ is a module over the zero ring $H^*(R^\bullet)$, and all such modules are zero). So the whole derived category vanishes, $\mathrm{D}(R^\bullet\text{-}\mathrm{Mod}) = 0$.

For comparison, the conventional derived category of DG-modules behaves very differently.

Given a DG-ring $A^\bullet = (A, d)$, the derived category $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$ is defined as the Verdier quotient category of the homotopy category $\mathrm{H}^0(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$ by the thick subcategory of acyclic DG-modules.

In particular, over an acyclic DG-ring $R^\bullet = (R, \partial)$, all DG-modules $E^\bullet$ are also acyclic (as $H^*(E^\bullet)$ is a module over the zero ring $H^*(R^\bullet)$, and all such modules are zero). So the whole derived category vanishes, $\mathrm{D}(R^\bullet\text{-}\mathrm{Mod}) = 0$.

Conversely, if a DG-ring $A^\bullet$ is not acyclic, then $A^\bullet$ is a nonzero object of $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$.

Derived category of the first kind

For comparison, the conventional derived category of DG-modules behaves very differently.

Given a DG-ring $A^\bullet = (A, d)$, the derived category $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$ is defined as the Verdier quotient category of the homotopy category $\mathrm{H}^0(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$ by the thick subcategory of acyclic DG-modules.

In particular, over an acyclic DG-ring $R^\bullet = (R, \partial)$, all DG-modules $E^\bullet$ are also acyclic (as $H^*(E^\bullet)$ is a module over the zero ring $H^*(R^\bullet)$, and all such modules are zero). So the whole derived category vanishes, $\mathrm{D}(R^\bullet\text{-}\mathrm{Mod}) = 0$.

Conversely, if a DG-ring $A^\bullet$ is not acyclic, then $A^\bullet$ is a nonzero object of $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$. So $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod}) \neq 0$ in this case.

Derived category of the first kind

For comparison, the conventional derived category of DG-modules behaves very differently.

Given a DG-ring $A^\bullet = (A, d)$, the derived category $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$ is defined as the Verdier quotient category of the homotopy category $\mathrm{H}^0(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$ by the thick subcategory of acyclic DG-modules.

In particular, over an acyclic DG-ring $R^\bullet = (R, \partial)$, all DG-modules $E^\bullet$ are also acyclic (as $H^*(E^\bullet)$ is a module over the zero ring $H^*(R^\bullet)$, and all such modules are zero). So the whole derived category vanishes, $\mathrm{D}(R^\bullet\text{-}\mathrm{Mod}) = 0$.

Conversely, if a DG-ring $A^\bullet$ is not acyclic, then $A^\bullet$ is a nonzero object of $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$. So $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod}) \neq 0$ in this case.

These arguments show that the derived category $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$ is not an invariant of the DG-category of DG-modules $A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$.

Derived category of the first kind

For comparison, the conventional derived category of DG-modules behaves very differently.

Given a DG-ring $A^\bullet = (A, d)$, the derived category $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$ is defined as the Verdier quotient category of the homotopy category $\mathrm{H}^0(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$ by the thick subcategory of acyclic DG-modules.

In particular, over an acyclic DG-ring $R^\bullet = (R, \partial)$, all DG-modules $E^\bullet$ are also acyclic (as $H^*(E^\bullet)$ is a module over the zero ring $H^*(R^\bullet)$, and all such modules are zero). So the whole derived category vanishes, $\mathrm{D}(R^\bullet\text{-}\mathrm{Mod}) = 0$.

Conversely, if a DG-ring $A^\bullet$ is not acyclic, then $A^\bullet$ is a nonzero object of $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$. So $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod}) \neq 0$ in this case.

These arguments show that the derived category $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$ is not an invariant of the DG-category of DG-modules $A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$. Indeed, $A^\bullet$ is an arbitrary DG-ring and $\widehat{A}^\bullet$ is an acyclic one; still their DG-categories of DG-modules are equivalent.

Derived category of the first kind

The previous slide may look like a paradox, which is resolved as follows.

Derived category of the first kind

The previous slide may look like a paradox, which is resolved as follows.

In order to construct the derived category of DG-modules $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$, one needs to know the cohomology functor H^* on the homotopy category of DG-modules $\mathrm{H}^0(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$.

Derived category of the first kind

The previous slide may look like a paradox, which is resolved as follows.

In order to construct the derived category of DG-modules $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$, one needs to know the cohomology functor H^* on the homotopy category of DG-modules $\mathrm{H}^0(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$. Or one needs to know the object $A^\bullet \in A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$, the free DG-module with one generator

Derived category of the first kind

The previous slide may look like a paradox, which is resolved as follows.

In order to construct the derived category of DG-modules $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$, one needs to know the cohomology functor H^* on the homotopy category of DG-modules $\mathrm{H}^0(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$. Or one needs to know the object $A^\bullet \in A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$, the free DG-module with one generator (which corepresents the cohomology functor on the homotopy category).

Derived category of the first kind

The previous slide may look like a paradox, which is resolved as follows.

In order to construct the derived category of DG-modules $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$, one needs to know the cohomology functor H^* on the homotopy category of DG-modules $\mathrm{H}^0(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$. Or one needs to know the object $A^\bullet \in A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$, the free DG-module with one generator (which corepresents the cohomology functor on the homotopy category).

Knowing the DG-category of DG-modules $A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ as an abstract DG-category is not enough to construct the conventional derived category of DG-modules.

Derived category of the first kind

The previous slide may look like a paradox, which is resolved as follows.

In order to construct the derived category of DG-modules $\mathrm{D}(A^\bullet\text{-}\mathrm{Mod})$, one needs to know the cohomology functor H^* on the homotopy category of DG-modules $\mathrm{H}^0(A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}})$. Or one needs to know the object $A^\bullet \in A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$, the free DG-module with one generator (which corepresents the cohomology functor on the homotopy category).

Knowing the DG-category of DG-modules $A^\bullet\text{-}\mathrm{Mod}^{\mathrm{dg}}$ as an abstract DG-category is not enough to construct the conventional derived category of DG-modules.

But knowing the DG-category of CDG-modules $B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}$ as an abstract DG-category is enough to construct the absolute derived, coderived, and contraderived categories of CDG-modules.

Coalgebras, Comodules, and Contramodules

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a k-vector space endowed with k-linear maps of comultiplication and counit $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a k-vector space endowed with k-linear maps of comultiplication and counit $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$ satisfying the coassociativity and counitality axioms.

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a k-vector space endowed with k-linear maps of comultiplication and counit $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$ satisfying the coassociativity and counitality axioms. These axioms can be obtained by writing down the definition of an associative algebra over a field in the tensor notation and inverting the arrows.

A coassociative coalgebra $\mathcal{C}$ over a field k is a k-vector space endowed with k-linear maps of comultiplication and counit $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$ satisfying the coassociativity and counitality axioms. These axioms can be obtained by writing down the definition of an associative algebra over a field in the tensor notation and inverting the arrows.

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a k-vector space endowed with a k-linear map of left coaction $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ satisfying the coassociativity and counitality axioms.

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a k-vector space endowed with k-linear maps of comultiplication and counit $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$ satisfying the coassociativity and counitality axioms. These axioms can be obtained by writing down the definition of an associative algebra over a field in the tensor notation and inverting the arrows.

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a k-vector space endowed with a k-linear map of left coaction $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ satisfying the coassociativity and counitality axioms.

A left $\mathcal{C}$-contramodule $\mathfrak{P}$ is a k-vector space endowed with a k-linear map of left contraaction $\pi\colon \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a k-vector space endowed with k-linear maps of comultiplication and counit $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$ satisfying the coassociativity and counitality axioms. These axioms can be obtained by writing down the definition of an associative algebra over a field in the tensor notation and inverting the arrows.

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a k-vector space endowed with a k-linear map of left coaction $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ satisfying the coassociativity and counitality axioms.

A left $\mathcal{C}$-contramodule $\mathfrak{P}$ is a k-vector space endowed with a k-linear map of left contraaction $\pi\colon \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ satisfying the contraassociativity and contraunitality axioms. The difference between the left and right contramodules

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a k-vector space endowed with k-linear maps of comultiplication and counit $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$ satisfying the coassociativity and counitality axioms. These axioms can be obtained by writing down the definition of an associative algebra over a field in the tensor notation and inverting the arrows.

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a k-vector space endowed with a k-linear map of left coaction $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ satisfying the coassociativity and counitality axioms.

A left $\mathcal{C}$-contramodule $\mathfrak{P}$ is a k-vector space endowed with a k-linear map of left contraaction $\pi\colon \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ satisfying the contraassociativity and contraunitality axioms. The difference between the left and right contramodules comes from the choice of an isomorphism $\operatorname{Hom}_k(\mathcal{C}, \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P})) \simeq \operatorname{Hom}_k(\mathcal{C} \otimes_k \mathcal{C}, \mathfrak{P})$ in the contraunitality axiom.

Coalgebras, Comodules, and Contramodules

A coassociative coalgebra $\mathcal{C}$ over a field k is a k-vector space endowed with k-linear maps of comultiplication and counit $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\epsilon\colon \mathcal{C} \longrightarrow k$ satisfying the coassociativity and counitality axioms. These axioms can be obtained by writing down the definition of an associative algebra over a field in the tensor notation and inverting the arrows.

A left $\mathcal{C}$-comodule $\mathcal{M}$ is a k-vector space endowed with a k-linear map of left coaction $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ satisfying the coassociativity and counitality axioms.

A left $\mathcal{C}$-contramodule $\mathfrak{P}$ is a k-vector space endowed with a k-linear map of left contraaction $\pi\colon \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ satisfying the contraassociativity and contraunitality axioms. The difference between the left and right contramodules comes from the choice of an isomorphism $\operatorname{Hom}_k(\mathcal{C}, \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P})) \simeq \operatorname{Hom}_k(\mathcal{C} \otimes_k \mathcal{C},\ \mathfrak{P})$ in the contraunitality axiom. For left contramodules, use the one that is a special case of $\operatorname{Hom}_k(U, \operatorname{Hom}_k(V, W)) \simeq \operatorname{Hom}_k(V \otimes_k U,\ W)$.

Graded Coalgebras, Comodules, and Contramodules

For the definitions of graded coalgebras, graded comodules, and graded contramodules

For the definitions of graded coalgebras, graded comodules, and graded contramodules, transfer the above ungraded definitions from the closed monoidal category of ungraded vector spaces

For the definitions of graded coalgebras, graded comodules, and graded contramodules, transfer the above ungraded definitions from the closed monoidal category of ungraded vector spaces to the closed monoidal category of graded ones.

For the definitions of graded coalgebras, graded comodules, and graded contramodules, transfer the above ungraded definitions from the closed monoidal category of ungraded vector spaces to the closed monoidal category of graded ones. Use the natural grading on the tensor product, the graded version of the Hom_k functor, etc.

For the definitions of graded coalgebras, graded comodules, and graded contramodules, transfer the above ungraded definitions from the closed monoidal category of ungraded vector spaces to the closed monoidal category of graded ones. Use the natural grading on the tensor product, the graded version of the Hom_k functor, etc.

The only tricky part of it concerns the functors of forgetting the grading

For the definitions of graded coalgebras, graded comodules, and graded contramodules, transfer the above ungraded definitions from the closed monoidal category of ungraded vector spaces to the closed monoidal category of graded ones. Use the natural grading on the tensor product, the graded version of the Hom_k functor, etc.

The only tricky part of it concerns the functors of forgetting the grading, which produce ungraded coalgebras/comodules/contramodules from graded ones.

For the definitions of graded coalgebras, graded comodules, and graded contramodules, transfer the above ungraded definitions from the closed monoidal category of ungraded vector spaces to the closed monoidal category of graded ones. Use the natural grading on the tensor product, the graded version of the Hom_k functor, etc.

The only tricky part of it concerns the functors of forgetting the grading, which produce ungraded coalgebras/comodules/contramodules from graded ones.

To forget the grading of a graded coalgebra $\mathcal{C}$

For the definitions of graded coalgebras, graded comodules, and graded contramodules, transfer the above ungraded definitions from the closed monoidal category of ungraded vector spaces to the closed monoidal category of graded ones. Use the natural grading on the tensor product, the graded version of the Hom_k functor, etc.

The only tricky part of it concerns the functors of forgetting the grading, which produce ungraded coalgebras/comodules/contramodules from graded ones.

To forget the grading of a graded coalgebra $\mathcal{C}$, use the infinite direct sum functor: $\Sigma\mathcal{C} = \bigoplus_{i\in\mathbb{Z}} \mathcal{C}^i$ is an ungraded coalgebra.

Graded Coalgebras, Comodules, and Contramodules

For the definitions of graded coalgebras, graded comodules, and graded contramodules, transfer the above ungraded definitions from the closed monoidal category of ungraded vector spaces to the closed monoidal category of graded ones. Use the natural grading on the tensor product, the graded version of the Hom_k functor, etc.

The only tricky part of it concerns the functors of forgetting the grading, which produce ungraded coalgebras/comodules/contramodules from graded ones.

To forget the grading of a graded coalgebra $\mathcal{C}$, use the infinite direct sum functor: $\Sigma\mathcal{C} = \bigoplus_{i\in\mathbb{Z}} \mathcal{C}^i$ is an ungraded coalgebra.

To forget the grading of a graded $\mathcal{C}$-comodule $\mathcal{M}$

Graded Coalgebras, Comodules, and Contramodules

For the definitions of graded coalgebras, graded comodules, and graded contramodules, transfer the above ungraded definitions from the closed monoidal category of ungraded vector spaces to the closed monoidal category of graded ones. Use the natural grading on the tensor product, the graded version of the Hom_k functor, etc.

The only tricky part of it concerns the functors of forgetting the grading, which produce ungraded coalgebras/comodules/contramodules from graded ones.

To forget the grading of a graded coalgebra $\mathcal{C}$, use the infinite direct sum functor: $\Sigma\mathcal{C} = \bigoplus_{i\in\mathbb{Z}} \mathcal{C}^i$ is an ungraded coalgebra.

To forget the grading of a graded $\mathcal{C}$-comodule $\mathcal{M}$, use the infinite direct sum functor: $\Sigma\mathcal{M} = \bigoplus_{i\in\mathbb{Z}} \mathcal{M}^i$ is an ungraded $\Sigma\mathcal{C}$-comodule.

Graded Coalgebras, Comodules, and Contramodules

For the definitions of graded coalgebras, graded comodules, and graded contramodules, transfer the above ungraded definitions from the closed monoidal category of ungraded vector spaces to the closed monoidal category of graded ones. Use the natural grading on the tensor product, the graded version of the Hom_k functor, etc.

The only tricky part of it concerns the functors of forgetting the grading, which produce ungraded coalgebras/comodules/contramodules from graded ones.

To forget the grading of a graded coalgebra $\mathcal{C}$, use the infinite direct sum functor: $\Sigma\mathcal{C} = \bigoplus_{i\in\mathbb{Z}} \mathcal{C}^i$ is an ungraded coalgebra.

To forget the grading of a graded $\mathcal{C}$-comodule $\mathcal{M}$, use the infinite direct sum functor: $\Sigma\mathcal{M} = \bigoplus_{i\in\mathbb{Z}} \mathcal{M}^i$ is an ungraded $\Sigma\mathcal{C}$-comodule.

To forget the grading of a graded $\mathcal{C}$-contramodule $\mathfrak{P}$

Graded Coalgebras, Comodules, and Contramodules

For the definitions of graded coalgebras, graded comodules, and graded contramodules, transfer the above ungraded definitions from the closed monoidal category of ungraded vector spaces to the closed monoidal category of graded ones. Use the natural grading on the tensor product, the graded version of the Hom_k functor, etc.

The only tricky part of it concerns the functors of forgetting the grading, which produce ungraded coalgebras/comodules/contramodules from graded ones.

To forget the grading of a graded coalgebra $\mathcal{C}$, use the infinite direct sum functor: $\Sigma\mathcal{C} = \bigoplus_{i\in\mathbb{Z}} \mathcal{C}^i$ is an ungraded coalgebra.

To forget the grading of a graded $\mathcal{C}$-comodule $\mathcal{M}$, use the infinite direct sum functor: $\Sigma\mathcal{M} = \bigoplus_{i\in\mathbb{Z}} \mathcal{M}^i$ is an ungraded $\Sigma\mathcal{C}$-comodule.

To forget the grading of a graded $\mathcal{C}$-contramodule $\mathfrak{P}$, use the infinite direct product functor: $\Pi\mathfrak{P} = \prod_{i\in\mathbb{Z}} \mathfrak{P}^i$ is an ungraded $\Sigma\mathcal{C}$-contramodule.

Curved DG-coalgebras

Curved DG-coalgebras

The definitions of curved DG-coalgebras, curved DG-comodules, and curved DG-contramodules

Curved DG-coalgebras

The definitions of curved DG-coalgebras, curved DG-comodules, and curved DG-contramodules are obtained by dualizing the respective definitions for rings and modules.

Curved DG-coalgebras

The definitions of curved DG-coalgebras, curved DG-comodules, and curved DG-contramodules are obtained by dualizing the respective definitions for rings and modules. In particular, a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ is

Curved DG-coalgebras

The definitions of curved DG-coalgebras, curved DG-comodules, and curved DG-contramodules are obtained by dualizing the respective definitions for rings and modules. In particular, a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ is

- a $\mathbb{Z}$-graded coalgebra $\mathcal{C}$ endowed with
- an odd coderivation $d\colon \mathcal{C}^i \longrightarrow \mathcal{C}^{i+1}$
- and a linear function $h\colon \mathcal{C}^{-2} \longrightarrow k$.

Curved DG-coalgebras

The definitions of curved DG-coalgebras, curved DG-comodules, and curved DG-contramodules are obtained by dualizing the respective definitions for rings and modules. In particular, a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ is

- a $\mathbb{Z}$-graded coalgebra $\mathcal{C}$ endowed with
- an odd coderivation $d\colon \mathcal{C}^i \longrightarrow \mathcal{C}^{i+1}$
- and a linear function $h\colon \mathcal{C}^{-2} \longrightarrow k$.

h is called the curvature linear function. The equations dual to the equations for a CDG-ring are imposed.

Curved DG-coalgebras

The definitions of curved DG-coalgebras, curved DG-comodules, and curved DG-contramodules are obtained by dualizing the respective definitions for rings and modules. In particular, a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ is

- a $\mathbb{Z}$-graded coalgebra $\mathcal{C}$ endowed with
- an odd coderivation $d\colon \mathcal{C}^i \longrightarrow \mathcal{C}^{i+1}$
- and a linear function $h\colon \mathcal{C}^{-2} \longrightarrow k$.

h is called the curvature linear function. The equations dual to the equations for a CDG-ring are imposed.

A morphism of CDG-coalgebras $(\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow (\mathcal{D}, d_\mathcal{D}, h_\mathcal{D})$

Curved DG-coalgebras

The definitions of curved DG-coalgebras, curved DG-comodules, and curved DG-contramodules are obtained by dualizing the respective definitions for rings and modules. In particular, a CDG-coalgebra $\mathcal{C}^\blacklozenge = (\mathcal{C}, d, h)$ is

- a $\mathbb{Z}$-graded coalgebra $\mathcal{C}$ endowed with
- an odd coderivation $d\colon \mathcal{C}^i \longrightarrow \mathcal{C}^{i+1}$
- and a linear function $h\colon \mathcal{C}^{-2} \longrightarrow k$.

h is called the curvature linear function. The equations dual to the equations for a CDG-ring are imposed.

A morphism of CDG-coalgebras $(\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow (\mathcal{D}, d_\mathcal{D}, h_\mathcal{D})$ is a pair (g, b), where

- $g\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a morphism of graded coalgebras
- and $b\colon \mathcal{C}^{-1} \longrightarrow k$ is a linear function.

Curved DG-coalgebras

The definitions of curved DG-coalgebras, curved DG-comodules, and curved DG-contramodules are obtained by dualizing the respective definitions for rings and modules. In particular, a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ is

- a $\mathbb{Z}$-graded coalgebra $\mathcal{C}$ endowed with
- an odd coderivation $d\colon \mathcal{C}^i \longrightarrow \mathcal{C}^{i+1}$
- and a linear function $h\colon \mathcal{C}^{-2} \longrightarrow k$.

h is called the curvature linear function. The equations dual to the equations for a CDG-ring are imposed.

A morphism of CDG-coalgebras $(\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow (\mathcal{D}, d_\mathcal{D}, h_\mathcal{D})$ is a pair (g, b), where

- $g\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a morphism of graded coalgebras
- and $b\colon \mathcal{C}^{-1} \longrightarrow k$ is a linear function.

b is called the change-of-connection linear function. The equations dual to the equations for a morphism of CDG-rings are imposed.

The coalgebra hat construction

The coalgebra hat construction

A DG-coalgebra $\mathcal{D}^\bullet = (\mathcal{D}, d)$ is a CDG-coalgebra with a vanishing curvature linear function.

The coalgebra hat construction

A DG-coalgebra $\mathcal{D}^\bullet = (\mathcal{D}, d)$ is a CDG-coalgebra with a vanishing curvature linear function. A DG-coalgebra $\mathcal{D}^\bullet$ is said to be acyclic if its cohomology coalgebra is the zero coalgebra: $H^*(\mathcal{D}^\bullet) = 0$.

The coalgebra hat construction

A DG-coalgebra $\mathcal{D}^\bullet = (\mathcal{D}, d)$ is a CDG-coalgebra with a vanishing curvature linear function. A DG-coalgebra $\mathcal{D}^\bullet$ is said to be acyclic if its cohomology coalgebra is the zero coalgebra: $H^*(\mathcal{D}^\bullet) = 0$.

There is a natural equivalence $\mathrm{CDG\text{-}coalg}_k \simeq \mathrm{DG\text{-}coalg}_k^{\mathrm{ac}}$

The coalgebra hat construction

A DG-coalgebra $\mathcal{D}^\bullet = (\mathcal{D}, d)$ is a CDG-coalgebra with a vanishing curvature linear function. A DG-coalgebra $\mathcal{D}^\bullet$ is said to be acyclic if its cohomology coalgebra is the zero coalgebra: $H^*(\mathcal{D}^\bullet) = 0$.

There is a natural equivalence $\mathrm{CDG\text{-}coalg}_k \simeq \mathrm{DG\text{-}coalg}_k^{\mathrm{ac}}$ between the category of CDG-coalgebras and the category of acyclic DG-coalgebras over a field k.

The coalgebra hat construction

A DG-coalgebra $\mathcal{D}^\bullet = (\mathcal{D}, d)$ is a CDG-coalgebra with a vanishing curvature linear function. A DG-coalgebra $\mathcal{D}^\bullet$ is said to be acyclic if its cohomology coalgebra is the zero coalgebra: $H^*(\mathcal{D}^\bullet) = 0$.

There is a natural equivalence $\mathrm{CDG\text{-}coalg}_k \simeq \mathrm{DG\text{-}coalg}_k^{\mathrm{ac}}$ between the category of CDG-coalgebras and the category of acyclic DG-coalgebras over a field k. The notation is $\mathcal{C}^\bullet = (\mathcal{C}, d, h) \longmapsto \widehat{\mathcal{C}}^\bullet = (\widehat{\mathcal{C}}, \partial)$.

The coalgebra hat construction

A DG-coalgebra $\mathcal{D}^\bullet = (\mathcal{D}, d)$ is a CDG-coalgebra with a vanishing curvature linear function. A DG-coalgebra $\mathcal{D}^\bullet$ is said to be acyclic if its cohomology coalgebra is the zero coalgebra: $H^*(\mathcal{D}^\bullet) = 0$.

There is a natural equivalence $\mathrm{CDG\text{-}coalg}_k \simeq \mathrm{DG\text{-}coalg}_k^{\mathrm{ac}}$ between the category of CDG-coalgebras and the category of acyclic DG-coalgebras over a field k. The notation is $\mathcal{C}^\bullet = (\mathcal{C}, d, h) \longmapsto \widehat{\mathcal{C}}^\bullet = (\widehat{\mathcal{C}}, \partial)$.

The construction is dual to the CDG-ring case.

The coalgebra hat construction

A DG-coalgebra $\mathcal{D}^\bullet = (\mathcal{D}, d)$ is a CDG-coalgebra with a vanishing curvature linear function. A DG-coalgebra $\mathcal{D}^\bullet$ is said to be acyclic if its cohomology coalgebra is the zero coalgebra: $H^*(\mathcal{D}^\bullet) = 0$.

There is a natural equivalence $\mathrm{CDG\text{-}coalg}_k \simeq \mathrm{DG\text{-}coalg}_k^{\mathrm{ac}}$ between the category of CDG-coalgebras and the category of acyclic DG-coalgebras over a field k. The notation is $\mathcal{C}^\blacklozenge = (\mathcal{C}, d, h) \longmapsto \widehat{\mathcal{C}}^\bullet = (\widehat{\mathcal{C}}, \partial)$.

The construction is dual to the CDG-ring case. Let me just say that, given an acyclic DG-coalgebra $(\mathcal{D}, \partial)$, the underlying graded coalgebra $\mathcal{C}$ of the related CDG-coalgebra $(\mathcal{C}, d, h)$

The coalgebra hat construction

A DG-coalgebra $\mathcal{D}^\bullet = (\mathcal{D}, d)$ is a CDG-coalgebra with a vanishing curvature linear function. A DG-coalgebra $\mathcal{D}^\bullet$ is said to be acyclic if its cohomology coalgebra is the zero coalgebra: $H^*(\mathcal{D}^\bullet) = 0$.

There is a natural equivalence $\mathrm{CDG\text{-}coalg}_k \simeq \mathrm{DG\text{-}coalg}_k^{\mathrm{ac}}$ between the category of CDG-coalgebras and the category of acyclic DG-coalgebras over a field k. The notation is $\mathcal{C}^\blacklozenge = (\mathcal{C}, d, h) \longmapsto \widehat{\mathcal{C}}^\bullet = (\widehat{\mathcal{C}}, \partial)$.

The construction is dual to the CDG-ring case. Let me just say that, given an acyclic DG-coalgebra $(\mathcal{D}, \partial)$, the underlying graded coalgebra $\mathcal{C}$ of the related CDG-coalgebra $(\mathcal{C}, d, h)$ is recovered as the cokernel of the acyclic differential, $\mathcal{C} = \operatorname{coker}(\partial \colon \mathcal{D} \to \mathcal{D})$.

The coalgebra hat construction

A DG-coalgebra $\mathcal{D}^\bullet = (\mathcal{D}, d)$ is a CDG-coalgebra with a vanishing curvature linear function. A DG-coalgebra $\mathcal{D}^\bullet$ is said to be acyclic if its cohomology coalgebra is the zero coalgebra: $H^*(\mathcal{D}^\bullet) = 0$.

There is a natural equivalence $\mathrm{CDG\text{-}coalg}_k \simeq \mathrm{DG\text{-}coalg}_k^{\mathrm{ac}}$ between the category of CDG-coalgebras and the category of acyclic DG-coalgebras over a field k. The notation is $\mathcal{C}^\bullet = (\mathcal{C}, d, h) \longmapsto \widehat{\mathcal{C}}^\bullet = (\widehat{\mathcal{C}}, \partial)$.

The construction is dual to the CDG-ring case. Let me just say that, given an acyclic DG-coalgebra $(\mathcal{D}, \partial)$, the underlying graded coalgebra $\mathcal{C}$ of the related CDG-coalgebra $(\mathcal{C}, d, h)$ is recovered as the cokernel of the acyclic differential, $\mathcal{C} = \mathrm{coker}(\partial\colon \mathcal{D} \to \mathcal{D})$. So $\mathcal{C}$ is a graded quotient coalgebra of $\mathcal{D}$.

CDG-comodules and CDG-contramodules

Let $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ be a CDG-coalgebra over a field k.

CDG-comodules and CDG-contramodules

Let $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ be a CDG-coalgebra over a field k. Then left CDG-comodules $\mathcal{M}^\bullet = (\mathcal{M}, d_\mathcal{M})$ over $\mathcal{C}^\bullet$ form an abelian DG-category $\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}}$.

CDG-comodules and CDG-contramodules

Let $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ be a CDG-coalgebra over a field k. Then left CDG-comodules $\mathcal{M}^\bullet = (\mathcal{M}, d_\mathcal{M})$ over $\mathcal{C}^\bullet$ form an abelian DG-category $\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}}$. Hence one can construct the absolute derived category $\mathrm{D}^{\mathrm{abs}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$ and the coderived category $\mathrm{D}^{\mathrm{co}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$.

CDG-comodules and CDG-contramodules

Let $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ be a CDG-coalgebra over a field k. Then left CDG-comodules $\mathcal{M}^\bullet = (\mathcal{M}, d_\mathcal{M})$ over $\mathcal{C}^\bullet$ form an abelian DG-category $\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}}$. Hence one can construct the absolute derived category $\mathrm{D}^{\mathrm{abs}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$ and the coderived category $\mathrm{D}^{\mathrm{co}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$.

The infinite product functors exist in the abelian category of CDG-comodules $\mathrm{Z}^0(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$, but they are not exact.

CDG-comodules and CDG-contramodules

Let $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ be a CDG-coalgebra over a field k. Then left CDG-comodules $\mathcal{M}^\bullet = (\mathcal{M}, d_\mathcal{M})$ over $\mathcal{C}^\bullet$ form an abelian DG-category $\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}}$. Hence one can construct the absolute derived category $\mathrm{D}^{\mathrm{abs}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$ and the coderived category $\mathrm{D}^{\mathrm{co}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$.

The infinite product functors exist in the abelian category of CDG-comodules $\mathrm{Z}^0(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$, but they are not exact. For this reason, the contraderived category of comodules is not well-behaved and usually not considered.

CDG-comodules and CDG-contramodules

Let $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ be a CDG-coalgebra over a field k. Then left CDG-comodules $\mathcal{M}^\bullet = (\mathcal{M}, d_\mathcal{M})$ over $\mathcal{C}^\bullet$ form an abelian DG-category $\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}}$. Hence one can construct the absolute derived category $\mathrm{D}^{\mathrm{abs}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$ and the coderived category $\mathrm{D}^{\mathrm{co}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$.

The infinite product functors exist in the abelian category of CDG-comodules $\mathrm{Z}^0(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$, but they are not exact. For this reason, the contraderived category of comodules is not well-behaved and usually not considered.

Similarly, left CDG-contramodules $\mathfrak{P}^\bullet = (\mathfrak{P}, d_\mathfrak{P})$ over $\mathcal{C}^\bullet$ form an abelian DG-category $\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}}$.

CDG-comodules and CDG-contramodules

Let $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ be a CDG-coalgebra over a field k. Then left CDG-comodules $\mathcal{M}^\bullet = (\mathcal{M}, d_\mathcal{M})$ over $\mathcal{C}^\bullet$ form an abelian DG-category $\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}}$. Hence one can construct the absolute derived category $\mathrm{D}^{\mathrm{abs}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$ and the coderived category $\mathrm{D}^{\mathrm{co}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$.

The infinite product functors exist in the abelian category of CDG-comodules $\mathrm{Z}^0(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$, but they are not exact. For this reason, the contraderived category of comodules is not well-behaved and usually not considered.

Similarly, left CDG-contramodules $\mathfrak{P}^\bullet = (\mathfrak{P}, d_\mathfrak{P})$ over $\mathcal{C}^\bullet$ form an abelian DG-category $\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}}$. Hence one can construct the absolute derived category $\mathrm{D}^{\mathrm{abs}}(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}})$ and the contraderived category $\mathrm{D}^{\mathrm{ctr}}(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}})$.

CDG-comodules and CDG-contramodules

The infinite coproduct functors exist in the abelian category of CDG-contramodules $\mathrm{Z}^0(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}})$, but they are usually not exact

CDG-comodules and CDG-contramodules

The infinite coproduct functors exist in the abelian category of CDG-contramodules $\mathrm{Z}^0(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}})$, but they are usually not exact (unless the graded coalgebra $\mathcal{C}$ has projective dimension $\leqslant 1$).

CDG-comodules and CDG-contramodules

The infinite coproduct functors exist in the abelian category of CDG-contramodules $\mathrm{Z}^0(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}})$, but they are usually not exact (unless the graded coalgebra $\mathcal{C}$ has projective dimension $\leqslant 1$). For this reason, the coderived category of contramodules is usually not considered.

The infinite coproduct functors exist in the abelian category of CDG-contramodules $Z^0(\mathcal{C}^\bullet\text{-Contra}^{cdg})$, but they are usually not exact (unless the graded coalgebra $\mathcal{C}$ has projective dimension $\leqslant 1$). For this reason, the coderived category of contramodules is usually not considered.

Theorem (Derived co-contra correspondence for CDG-coalgebras)

The infinite coproduct functors exist in the abelian category of CDG-contramodules $Z^0(\mathcal{C}^\bullet\text{-Contra}^{\mathrm{cdg}})$, but they are usually not exact (unless the graded coalgebra $\mathcal{C}$ has projective dimension $\leqslant 1$). For this reason, the coderived category of contramodules is usually not considered.

Theorem (Derived co-contra correspondence for CDG-coalgebras)

For any CDG-coalgebra $\mathcal{C}^\bullet$ over a field k

CDG-comodules and CDG-contramodules

The infinite coproduct functors exist in the abelian category of CDG-contramodules $\mathsf{Z}^0(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}})$, but they are usually not exact (unless the graded coalgebra $\mathcal{C}$ has projective dimension $\leqslant 1$). For this reason, the coderived category of contramodules is usually not considered.

Theorem (Derived co-contra correspondence for CDG-coalgebras)

For any CDG-coalgebra $\mathcal{C}^\bullet$ over a field k, there is a natural triangulated equivalence between the coderived category of left CDG-comodules and the contraderived category of left CDG-contramodules over $\mathcal{C}^\bullet$,

CDG-comodules and CDG-contramodules

The infinite coproduct functors exist in the abelian category of CDG-contramodules $Z^0(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}})$, but they are usually not exact (unless the graded coalgebra $\mathcal{C}$ has projective dimension $\leqslant 1$). For this reason, the coderived category of contramodules is usually not considered.

Theorem (Derived co-contra correspondence for CDG-coalgebras)

For any CDG-coalgebra $\mathcal{C}^\bullet$ over a field k, there is a natural triangulated equivalence between the coderived category of left CDG-comodules and the contraderived category of left CDG-contramodules over $\mathcal{C}^\bullet$,

$$D^{\mathrm{co}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}}) \simeq D^{\mathrm{ctr}}(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}}).$$

CDG-comodules and CDG-contramodules

The infinite coproduct functors exist in the abelian category of CDG-contramodules $\mathrm{Z}^0(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}})$, but they are usually not exact (unless the graded coalgebra $\mathcal{C}$ has projective dimension $\leqslant 1$). For this reason, the coderived category of contramodules is usually not considered.

Theorem (Derived co-contra correspondence for CDG-coalgebras)

For any CDG-coalgebra $\mathcal{C}^\bullet$ over a field k, there is a natural triangulated equivalence between the coderived category of left CDG-comodules and the contraderived category of left CDG-contramodules over $\mathcal{C}^\bullet$,

$$\mathrm{D}^{\mathrm{co}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}}).$$

The pair of mutually inverse equivalences is provided by the derived functors of $\mathcal{C}$-comodule homomorphisms $\mathbb{R}\,\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, -)$

CDG-comodules and CDG-contramodules

The infinite coproduct functors exist in the abelian category of CDG-contramodules $Z^0(\mathcal{C}^\bullet\text{-Contra}^{\mathrm{cdg}})$, but they are usually not exact (unless the graded coalgebra $\mathcal{C}$ has projective dimension $\leqslant 1$). For this reason, the coderived category of contramodules is usually not considered.

Theorem (Derived co-contra correspondence for CDG-coalgebras)

For any CDG-coalgebra $\mathcal{C}^\bullet$ over a field k, there is a natural triangulated equivalence between the coderived category of left CDG-comodules and the contraderived category of left CDG-contramodules over $\mathcal{C}^\bullet$,

$$\mathrm{D}^{\mathrm{co}}(\mathcal{C}^\bullet\text{-Comod}^{\mathrm{cdg}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathcal{C}^\bullet\text{-Contra}^{\mathrm{cdg}}).$$

The pair of mutually inverse equivalences is provided by the derived functors of $\mathcal{C}$-comodule homomorphisms $\mathbb{R}\operatorname{Hom}_{\mathcal{C}}(\mathcal{C}, -)$ and the so-called contratensor product $\mathcal{C} \odot_{\mathcal{C}}^{\mathbb{L}} -$.

Bar construction

Bar construction

The bar-construction is a functor $\mathrm{Bar}\colon \mathrm{DG\text{-}alg}_k^+ \longrightarrow \mathrm{CDG\text{-}coalg}_k$

Bar construction

The bar-construction is a functor $\mathrm{Bar}\colon \mathrm{DG\text{-}alg}_k^+ \longrightarrow \mathrm{CDG\text{-}coalg}_k$ from the category of nonzero DG-algebras to the category of CDG-coalgebras over a field k.

Bar construction

The bar-construction is a functor $\mathrm{Bar}\colon \mathrm{DG\text{-}alg}_k^+ \longrightarrow \mathrm{CDG\text{-}coalg}_k$ from the category of nonzero DG-algebras to the category of CDG-coalgebras over a field k. The full subcategory of nonzero DG-algebras $\mathrm{DG\text{-}alg}_k^+ \subset \mathrm{DG\text{-}alg}_k$ contains acyclic DG-algebras

Bar construction

The bar-construction is a functor Bar: $\mathrm{DG\text{-}alg}_k^+ \longrightarrow \mathrm{CDG\text{-}coalg}_k$ from the category of nonzero DG-algebras to the category of CDG-coalgebras over a field k. The full subcategory of nonzero DG-algebras $\mathrm{DG\text{-}alg}_k^+ \subset \mathrm{DG\text{-}alg}_k$ contains acyclic DG-algebras; it is only the zero DG-algebra $A^\bullet = 0$ with $1 = 0$ that is excluded.

Bar construction

The bar-construction is a functor Bar: $\mathrm{DG\text{-}alg}_k^+ \longrightarrow \mathrm{CDG\text{-}coalg}_k$ from the category of nonzero DG-algebras to the category of CDG-coalgebras over a field k. The full subcategory of nonzero DG-algebras $\mathrm{DG\text{-}alg}_k^+ \subset \mathrm{DG\text{-}alg}_k$ contains acyclic DG-algebras; it is only the zero DG-algebra $A^\bullet = 0$ with $1 = 0$ that is excluded.

Given a DG-algebra $A^\bullet = (A, d_A)$, the related graded coalgebra $\mathrm{Bar}(A)$

Bar construction

The bar-construction is a functor $\mathrm{Bar}\colon \mathrm{DG\text{-}alg}_k^+ \longrightarrow \mathrm{CDG\text{-}coalg}_k$ from the category of nonzero DG-algebras to the category of CDG-coalgebras over a field k. The full subcategory of nonzero DG-algebras $\mathrm{DG\text{-}alg}_k^+ \subset \mathrm{DG\text{-}alg}_k$ contains acyclic DG-algebras; it is only the zero DG-algebra $A^\bullet = 0$ with $1 = 0$ that is excluded.

Given a DG-algebra $A^\bullet = (A, d_A)$, the related graded coalgebra $\mathrm{Bar}(A)$ is the tensor coalgebra $\mathrm{Bar}(A) = \bigoplus_{n=0}^{\infty}(A/(k\cdot 1))[1]^{\otimes n}$

Bar construction

The bar-construction is a functor Bar: $\mathrm{DG\text{-}alg}_k^+ \longrightarrow \mathrm{CDG\text{-}coalg}_k$ from the category of nonzero DG-algebras to the category of CDG-coalgebras over a field k. The full subcategory of nonzero DG-algebras $\mathrm{DG\text{-}alg}_k^+ \subset \mathrm{DG\text{-}alg}_k$ contains acyclic DG-algebras; it is only the zero DG-algebra $A^\bullet = 0$ with $1 = 0$ that is excluded.

Given a DG-algebra $A^\bullet = (A, d_A)$, the related graded coalgebra $\mathrm{Bar}(A)$ is the tensor coalgebra $\mathrm{Bar}(A) = \bigoplus_{n=0}^\infty (A/(k \cdot 1))[1]^{\otimes n}$ cospanned by the quotient vector space of A by the unit line $k \cdot 1 \subset A$, shifted cohomologically by [1].

Bar construction

The bar-construction is a functor Bar: $\mathrm{DG\text{-}alg}_k^+ \longrightarrow \mathrm{CDG\text{-}coalg}_k$ from the category of nonzero DG-algebras to the category of CDG-coalgebras over a field k. The full subcategory of nonzero DG-algebras $\mathrm{DG\text{-}alg}_k^+ \subset \mathrm{DG\text{-}alg}_k$ contains acyclic DG-algebras; it is only the zero DG-algebra $A^\bullet = 0$ with $1 = 0$ that is excluded.

Given a DG-algebra $A^\bullet = (A, d_A)$, the related graded coalgebra $\mathrm{Bar}(A)$ is the tensor coalgebra $\mathrm{Bar}(A) = \bigoplus_{n=0}^{\infty}(A/(k \cdot 1))[1]^{\otimes n}$ cospanned by the quotient vector space of A by the unit line $k \cdot 1 \subset A$, shifted cohomologically by [1].

In order to construct the CDG-coalgebra structure on $\mathrm{Bar}(A)$

Bar construction

The bar-construction is a functor $\mathrm{Bar}\colon \mathrm{DG\text{-}alg}_k^+ \longrightarrow \mathrm{CDG\text{-}coalg}_k$ from the category of nonzero DG-algebras to the category of CDG-coalgebras over a field k. The full subcategory of nonzero DG-algebras $\mathrm{DG\text{-}alg}_k^+ \subset \mathrm{DG\text{-}alg}_k$ contains acyclic DG-algebras; it is only the zero DG-algebra $A^\bullet = 0$ with $1 = 0$ that is excluded.

Given a DG-algebra $A^\bullet = (A, d_A)$, the related graded coalgebra $\mathrm{Bar}(A)$ is the tensor coalgebra $\mathrm{Bar}(A) = \bigoplus_{n=0}^{\infty}(A/(k \cdot 1))[1]^{\otimes n}$ cospanned by the quotient vector space of A by the unit line $k \cdot 1 \subset A$, shifted cohomologically by $[1]$.

In order to construct the CDG-coalgebra structure on $\mathrm{Bar}(A)$, one has to make an arbitrary choice of a complementary graded subspace V to the vector subspace spanned by the unit element $1 \in A^0$.

Bar construction

The bar-construction is a functor $\mathrm{Bar}\colon \mathrm{DG\text{-}alg}_k^+ \longrightarrow \mathrm{CDG\text{-}coalg}_k$ from the category of nonzero DG-algebras to the category of CDG-coalgebras over a field k. The full subcategory of nonzero DG-algebras $\mathrm{DG\text{-}alg}_k^+ \subset \mathrm{DG\text{-}alg}_k$ contains acyclic DG-algebras; it is only the zero DG-algebra $A^\bullet = 0$ with $1 = 0$ that is excluded.

Given a DG-algebra $A^\bullet = (A, d_A)$, the related graded coalgebra $\mathrm{Bar}(A)$ is the tensor coalgebra $\mathrm{Bar}(A) = \bigoplus_{n=0}^{\infty}(A/(k\cdot 1))[1]^{\otimes n}$ cospanned by the quotient vector space of A by the unit line $k\cdot 1 \subset A$, shifted cohomologically by $[1]$.

In order to construct the CDG-coalgebra structure on $\mathrm{Bar}(A)$, one has to make an arbitrary choice of a complementary graded subspace V to the vector subspace spanned by the unit element $1 \in A^0$. Equivalently, one has to choose a homogeneous k-linear retraction $v\colon A \longrightarrow k$ of A onto the subspace $k = k\cdot 1 \subset A$.

Bar construction

Given a graded vector subspace $V \subset A$ such that $A = k \cdot 1 \oplus V$

Bar construction

Given a graded vector subspace $V \subset A$ such that $A = k \cdot 1 \oplus V$, one considers the related components $m_V \colon A \otimes_k A \longrightarrow V$ and $m_k \colon A \otimes_k A \longrightarrow k$ of the multiplication map $m \colon A \otimes_k A \longrightarrow A$

Bar construction

Given a graded vector subspace $V \subset A$ such that $A = k \cdot 1 \oplus V$, one considers the related components $m_V \colon A \otimes_k A \longrightarrow V$ and $m_k \colon A \otimes_k A \longrightarrow k$ of the multiplication map $m \colon A \otimes_k A \longrightarrow A$, as well as the related components $d_V \colon A \longrightarrow V$ and $d_k \colon A \longrightarrow k$ of the differential $d_A \colon A \longrightarrow A$.

Bar construction

Given a graded vector subspace $V \subset A$ such that $A = k \cdot 1 \oplus V$, one considers the related components $m_V \colon A \otimes_k A \longrightarrow V$ and $m_k \colon A \otimes_k A \longrightarrow k$ of the multiplication map $m \colon A \otimes_k A \longrightarrow A$, as well as the related components $d_V \colon A \longrightarrow V$ and $d_k \colon A \longrightarrow k$ of the differential $d_A \colon A \longrightarrow A$.

The maps m_V and d_V are used in the construction of the differential d_{Bar} on the tensor coalgebra $\mathrm{Bar}(A)$.

Bar construction

Given a graded vector subspace $V \subset A$ such that $A = k \cdot 1 \oplus V$, one considers the related components $m_V \colon A \otimes_k A \longrightarrow V$ and $m_k \colon A \otimes_k A \longrightarrow k$ of the multiplication map $m \colon A \otimes_k A \longrightarrow A$, as well as the related components $d_V \colon A \longrightarrow V$ and $d_k \colon A \longrightarrow k$ of the differential $d_A \colon A \longrightarrow A$.

The maps m_V and d_V are used in the construction of the differential d_{Bar} on the tensor coalgebra $\mathrm{Bar}(A)$. The maps m_k and d_k are used in the construction of the curvature linear function $h_{\mathrm{Bar}} \colon \mathrm{Bar}(A)^{-2} \longrightarrow k$.

Bar construction

Given a graded vector subspace $V \subset A$ such that $A = k \cdot 1 \oplus V$, one considers the related components $m_V\colon A \otimes_k A \longrightarrow V$ and $m_k\colon A \otimes_k A \longrightarrow k$ of the multiplication map $m\colon A \otimes_k A \longrightarrow A$, as well as the related components $d_V\colon A \longrightarrow V$ and $d_k\colon A \longrightarrow k$ of the differential $d_A\colon A \longrightarrow A$.

The maps m_V and d_V are used in the construction of the differential d_{Bar} on the tensor coalgebra $\mathrm{Bar}(A)$. The maps m_k and d_k are used in the construction of the curvature linear function $h_{\mathrm{Bar}}\colon \mathrm{Bar}(A)^{-2} \longrightarrow k$. The resulting CDG-coalgebra is denoted by $\mathrm{Bar}_v^{\bullet}(A^\bullet) = (\mathrm{Bar}(A), d_{\mathrm{Bar}}, h_{\mathrm{Bar}})$.

Bar construction

Given a graded vector subspace $V \subset A$ such that $A = k \cdot 1 \oplus V$, one considers the related components $m_V \colon A \otimes_k A \longrightarrow V$ and $m_k \colon A \otimes_k A \longrightarrow k$ of the multiplication map $m \colon A \otimes_k A \longrightarrow A$, as well as the related components $d_V \colon A \longrightarrow V$ and $d_k \colon A \longrightarrow k$ of the differential $d_A \colon A \longrightarrow A$.

The maps m_V and d_V are used in the construction of the differential d_{Bar} on the tensor coalgebra $\mathrm{Bar}(A)$. The maps m_k and d_k are used in the construction of the curvature linear function $h_{\mathrm{Bar}} \colon \mathrm{Bar}(A)^{-2} \longrightarrow k$. The resulting CDG-coalgebra is denoted by $\mathrm{Bar}_v^\bullet(A^\bullet) = (\mathrm{Bar}(A), d_{\mathrm{Bar}}, h_{\mathrm{Bar}})$.

Replacing the retration $v \colon A \longrightarrow k$ by another retraction $v' \colon A \longrightarrow k$

Bar construction

Given a graded vector subspace $V \subset A$ such that $A = k \cdot 1 \oplus V$, one considers the related components $m_V \colon A \otimes_k A \longrightarrow V$ and $m_k \colon A \otimes_k A \longrightarrow k$ of the multiplication map $m \colon A \otimes_k A \longrightarrow A$, as well as the related components $d_V \colon A \longrightarrow V$ and $d_k \colon A \longrightarrow k$ of the differential $d_A \colon A \longrightarrow A$.

The maps m_V and d_V are used in the construction of the differential d_{Bar} on the tensor coalgebra $\mathrm{Bar}(A)$. The maps m_k and d_k are used in the construction of the curvature linear function $h_{\mathrm{Bar}} \colon \mathrm{Bar}(A)^{-2} \longrightarrow k$. The resulting CDG-coalgebra is denoted by $\mathrm{Bar}^{\blacklozenge}_v(A^\bullet) = (\mathrm{Bar}(A), d_{\mathrm{Bar}}, h_{\mathrm{Bar}})$.

Replacing the retration $v \colon A \longrightarrow k$ by another retraction $v' \colon A \longrightarrow k$, one constructs another CDG-coalgebra structure $\mathrm{Bar}^{\blacklozenge}_{v'}(A^\bullet)$ on the same graded coalgebra $\mathrm{Bar}(A)$.

Bar construction

Given a graded vector subspace $V \subset A$ such that $A = k \cdot 1 \oplus V$, one considers the related components $m_V\colon A \otimes_k A \longrightarrow V$ and $m_k\colon A \otimes_k A \longrightarrow k$ of the multiplication map $m\colon A \otimes_k A \longrightarrow A$, as well as the related components $d_V\colon A \longrightarrow V$ and $d_k\colon A \longrightarrow k$ of the differential $d_A\colon A \longrightarrow A$.

The maps m_V and d_V are used in the construction of the differential d_{Bar} on the tensor coalgebra $\mathrm{Bar}(A)$. The maps m_k and d_k are used in the construction of the curvature linear function $h_{\mathrm{Bar}}\colon \mathrm{Bar}(A)^{-2} \longrightarrow k$. The resulting CDG-coalgebra is denoted by $\mathrm{Bar}^\bullet_v(A^\bullet) = (\mathrm{Bar}(A), d_{\mathrm{Bar}}, h_{\mathrm{Bar}})$.

Replacing the retration $v\colon A \longrightarrow k$ by another retraction $v'\colon A \longrightarrow k$, one constructs another CDG-coalgebra structure $\mathrm{Bar}^\bullet_{v'}(A^\bullet)$ on the same graded coalgebra $\mathrm{Bar}(A)$. The difference of the two retractions $v' - v$ is a linear map $A/(k \cdot 1) \longrightarrow k$

Bar construction

Given a graded vector subspace $V \subset A$ such that $A = k \cdot 1 \oplus V$, one considers the related components $m_V \colon A \otimes_k A \longrightarrow V$ and $m_k \colon A \otimes_k A \longrightarrow k$ of the multiplication map $m \colon A \otimes_k A \longrightarrow A$, as well as the related components $d_V \colon A \longrightarrow V$ and $d_k \colon A \longrightarrow k$ of the differential $d_A \colon A \longrightarrow A$.

The maps m_V and d_V are used in the construction of the differential d_{Bar} on the tensor coalgebra $\mathrm{Bar}(A)$. The maps m_k and d_k are used in the construction of the curvature linear function $h_{\mathrm{Bar}} \colon \mathrm{Bar}(A)^{-2} \longrightarrow k$. The resulting CDG-coalgebra is denoted by $\mathrm{Bar}_v^\bullet(A^\bullet) = (\mathrm{Bar}(A), d_{\mathrm{Bar}}, h_{\mathrm{Bar}})$.

Replacing the retration $v \colon A \longrightarrow k$ by another retraction $v' \colon A \longrightarrow k$, one constructs another CDG-coalgebra structure $\mathrm{Bar}_{v'}^\bullet(A^\bullet)$ on the same graded coalgebra $\mathrm{Bar}(A)$. The difference of the two retractions $v' - v$ is a linear map $A/(k \cdot 1) \longrightarrow k$, which can be used to construct a change-of-connection linear function $b \colon \mathrm{Bar}(A)^{-1} \longrightarrow k$

Bar construction

Given a graded vector subspace $V \subset A$ such that $A = k \cdot 1 \oplus V$, one considers the related components $m_V \colon A \otimes_k A \longrightarrow V$ and $m_k \colon A \otimes_k A \longrightarrow k$ of the multiplication map $m \colon A \otimes_k A \longrightarrow A$, as well as the related components $d_V \colon A \longrightarrow V$ and $d_k \colon A \longrightarrow k$ of the differential $d_A \colon A \longrightarrow A$.

The maps m_V and d_V are used in the construction of the differential d_{Bar} on the tensor coalgebra $\mathrm{Bar}(A)$. The maps m_k and d_k are used in the construction of the curvature linear function $h_{\mathrm{Bar}} \colon \mathrm{Bar}(A)^{-2} \longrightarrow k$. The resulting CDG-coalgebra is denoted by $\mathrm{Bar}_v^\bullet(A^\bullet) = (\mathrm{Bar}(A), d_{\mathrm{Bar}}, h_{\mathrm{Bar}})$.

Replacing the retration $v \colon A \longrightarrow k$ by another retraction $v' \colon A \longrightarrow k$, one constructs another CDG-coalgebra structure $\mathrm{Bar}_{v'}^\bullet(A^\bullet)$ on the same graded coalgebra $\mathrm{Bar}(A)$. The difference of the two retractions $v' - v$ is a linear map $A/(k \cdot 1) \longrightarrow k$, which can be used to construct a change-of-connection linear function $b \colon \mathrm{Bar}(A)^{-1} \longrightarrow k$ providing a natural change-of-connection isomorphism of CDG-coalgebras $\mathrm{Bar}_v^\bullet(A^\bullet) \simeq \mathrm{Bar}_{v'}^\bullet(A^\bullet)$.

Cobar construction

Cobar construction

The cobar-construction is a functor
Cob: $\mathrm{CDG\text{-}coalg}_k^+ \longrightarrow \mathrm{CDG\text{-}alg}_k$

Cobar construction

The cobar-construction is a functor
Cob: $\mathrm{CDG\text{-}coalg}_k^+ \longrightarrow \mathrm{CDG\text{-}alg}_k$ from the category of nonzero CDG-coalgebras to the category of CDG-algebras over k.

Cobar construction

The cobar-construction is a functor
Cob: $\mathrm{CDG\text{-}coalg}_k^+ \longrightarrow \mathrm{CDG\text{-}alg}_k$ from the category of nonzero CDG-coalgebras to the category of CDG-algebras over k.

Given a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C})$

Cobar construction

The cobar-construction is a functor $\mathrm{Cob}\colon \mathrm{CDG\text{-}coalg}_k^+ \longrightarrow \mathrm{CDG\text{-}alg}_k$ from the category of nonzero CDG-coalgebras to the category of CDG-algebras over k.

Given a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C})$, the related graded algebra $\mathrm{Cob}(\mathcal{C})$ is the free associative/tensor algebra $\mathrm{Cob}(\mathcal{C}) = \bigoplus_{n=0}^\infty (\ker \epsilon)[-1]^{\otimes n}$

Cobar construction

The cobar-construction is a functor
Cob: $\mathrm{CDG\text{-}coalg}_k^+ \longrightarrow \mathrm{CDG\text{-}alg}_k$ from the category of nonzero CDG-coalgebras to the category of CDG-algebras over k.

Given a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C})$, the related graded algebra $\mathrm{Cob}(\mathcal{C})$ is the free associative/tensor algebra $\mathrm{Cob}(\mathcal{C}) = \bigoplus_{n=0}^\infty (\ker \epsilon)[-1]^{\otimes n}$ spanned by the kernel of the counit map $\ker(\epsilon\colon \mathcal{C} \to k) \subset \mathcal{C}$ shifted cohomologically by $[-1]$.

Cobar construction

The cobar-construction is a functor $\mathrm{Cob}\colon \mathrm{CDG\text{-}coalg}_k^+ \longrightarrow \mathrm{CDG\text{-}alg}_k$ from the category of nonzero CDG-coalgebras to the category of CDG-algebras over k.

Given a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C})$, the related graded algebra $\mathrm{Cob}(\mathcal{C})$ is the free associative/tensor algebra $\mathrm{Cob}(\mathcal{C}) = \bigoplus_{n=0}^\infty (\ker\epsilon)[-1]^{\otimes n}$ spanned by the kernel of the counit map $\ker(\epsilon\colon \mathcal{C} \to k) \subset \mathcal{C}$ shifted cohomologically by $[-1]$.

In order to construct the CDG-algebra structure on $\mathrm{Cob}(\mathcal{C})$

Cobar construction

The cobar-construction is a functor
$\mathrm{Cob}\colon \mathrm{CDG\text{-}coalg}_k^+ \longrightarrow \mathrm{CDG\text{-}alg}_k$ from the category of nonzero CDG-coalgebras to the category of CDG-algebras over k.

Given a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C})$, the related graded algebra $\mathrm{Cob}(\mathcal{C})$ is the free associative/tensor algebra $\mathrm{Cob}(\mathcal{C}) = \bigoplus_{n=0}^\infty (\ker\epsilon)[-1]^{\otimes n}$ spanned by the kernel of the counit map $\ker(\epsilon\colon \mathcal{C} \to k) \subset \mathcal{C}$ shifted cohomologically by $[-1]$.

In order to construct the CDG-algebra structure on $\mathrm{Cob}(\mathcal{C})$, one has to make an arbitrary choice of a homogeneous k-linear section $w\colon k \longrightarrow \mathcal{C}$ of the counit map $\epsilon\colon \mathcal{C} \longrightarrow k$.

Cobar construction

The cobar-construction is a functor $\mathrm{Cob}\colon \mathrm{CDG\text{-}coalg}_k^+ \longrightarrow \mathrm{CDG\text{-}alg}_k$ from the category of nonzero CDG-coalgebras to the category of CDG-algebras over k.

Given a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C})$, the related graded algebra $\mathrm{Cob}(\mathcal{C})$ is the free associative/tensor algebra $\mathrm{Cob}(\mathcal{C}) = \bigoplus_{n=0}^\infty (\ker \epsilon)[-1]^{\otimes n}$ spanned by the kernel of the counit map $\ker(\epsilon\colon \mathcal{C} \to k) \subset \mathcal{C}$ shifted cohomologically by $[-1]$.

In order to construct the CDG-algebra structure on $\mathrm{Cob}(\mathcal{C})$, one has to make an arbitrary choice of a homogeneous k-linear section $w\colon k \longrightarrow \mathcal{C}$ of the counit map $\epsilon\colon \mathcal{C} \longrightarrow k$. Equivalently, one has to choose a decomposition $\mathcal{C} = k \oplus W$ of the graded vector space $\mathcal{C}$ into a direct sum of two homogeneous vector subspaces k and W

Cobar construction

The cobar-construction is a functor $\mathrm{Cob}\colon \mathrm{CDG\text{-}coalg}_k^+ \longrightarrow \mathrm{CDG\text{-}alg}_k$ from the category of nonzero CDG-coalgebras to the category of CDG-algebras over k.

Given a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C})$, the related graded algebra $\mathrm{Cob}(\mathcal{C})$ is the free associative/tensor algebra $\mathrm{Cob}(\mathcal{C}) = \bigoplus_{n=0}^\infty (\ker \epsilon)[-1]^{\otimes n}$ spanned by the kernel of the counit map $\ker(\epsilon\colon \mathcal{C} \to k) \subset \mathcal{C}$ shifted cohomologically by $[-1]$.

In order to construct the CDG-algebra structure on $\mathrm{Cob}(\mathcal{C})$, one has to make an arbitrary choice of a homogeneous k-linear section $w\colon k \longrightarrow \mathcal{C}$ of the counit map $\epsilon\colon \mathcal{C} \longrightarrow k$. Equivalently, one has to choose a decomposition $\mathcal{C} = k \oplus W$ of the graded vector space $\mathcal{C}$ into a direct sum of two homogeneous vector subspaces k and W such that the counit map becomes the projection onto the direct summand k along the direct summand W.

Cobar construction

Given a direct sum decomposition $\mathcal{C} = k \oplus W$ as above

Cobar construction

Given a direct sum decomposition $\mathcal{C} = k \oplus W$ as above, one considers the related components $\mu_W \colon W \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\mu_k \colon k \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ of the comultiplication map $\mu_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$

Cobar construction

Given a direct sum decomposition $\mathcal{C} = k \oplus W$ as above, one considers the related components $\mu_W\colon W \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\mu_k\colon k \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ of the comultiplication map $\mu_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$, as well as the related components $d_W\colon W \longrightarrow \mathcal{C}$ and $d_k\colon k \longrightarrow \mathcal{C}$ of the differential $d_{\mathcal{C}}$

Cobar construction

Given a direct sum decomposition $\mathcal{C} = k \oplus W$ as above, one considers the related components $\mu_W \colon W \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\mu_k \colon k \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ of the comultiplication map $\mu_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$, as well as the related components $d_W \colon W \longrightarrow \mathcal{C}$ and $d_k \colon k \longrightarrow \mathcal{C}$ of the differential $d_{\mathcal{C}}$, and the components $h_W \colon W \longrightarrow k$ and $h_k \colon k \longrightarrow k$ of the curvature linear function $h_{\mathcal{C}}$.

Cobar construction

Given a direct sum decomposition $\mathcal{C} = k \oplus W$ as above, one considers the related components $\mu_W \colon W \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\mu_k \colon k \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ of the comultiplication map $\mu_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$, as well as the related components $d_W \colon W \longrightarrow \mathcal{C}$ and $d_k \colon k \longrightarrow \mathcal{C}$ of the differential $d_{\mathcal{C}}$, and the components $h_W \colon W \longrightarrow k$ and $h_k \colon k \longrightarrow k$ of the curvature linear function $h_{\mathcal{C}}$.

In the usual $\mathbb{Z}$-grading setting, one always has $h_k = 0$ for the grading reasons.

Cobar construction

Given a direct sum decomposition $\mathcal{C} = k \oplus W$ as above, one considers the related components $\mu_W \colon W \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\mu_k \colon k \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ of the comultiplication map $\mu_\mathcal{C} \colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$, as well as the related components $d_W \colon W \longrightarrow \mathcal{C}$ and $d_k \colon k \longrightarrow \mathcal{C}$ of the differential $d_\mathcal{C}$, and the components $h_W \colon W \longrightarrow k$ and $h_k \colon k \longrightarrow k$ of the curvature linear function $h_\mathcal{C}$.

In the usual $\mathbb{Z}$-grading setting, one always has $h_k = 0$ for the grading reasons. But in the $\mathbb{Z}/2\mathbb{Z}$-graded (2-periodic) setting, h_k may be nonzero.

Cobar construction

Given a direct sum decomposition $\mathcal{C} = k \oplus W$ as above, one considers the related components $\mu_W\colon W \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\mu_k\colon k \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ of the comultiplication map $\mu_\mathcal{C}\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$, as well as the related components $d_W\colon W \longrightarrow \mathcal{C}$ and $d_k\colon k \longrightarrow \mathcal{C}$ of the differential $d_\mathcal{C}$, and the components $h_W\colon W \longrightarrow k$ and $h_k\colon k \longrightarrow k$ of the curvature linear function $h_\mathcal{C}$.

In the usual $\mathbb{Z}$-grading setting, one always has $h_k = 0$ for the grading reasons. But in the $\mathbb{Z}/2\mathbb{Z}$-graded (2-periodic) setting, h_k may be nonzero.

The maps μ_W, d_W, and h_W are used in the construction of the differential d_{Cob} on the tensor algebra $\mathrm{Cob}(\mathcal{C})$.

Cobar construction

Given a direct sum decomposition $\mathcal{C} = k \oplus W$ as above, one considers the related components $\mu_W \colon W \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\mu_k \colon k \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ of the comultiplication map $\mu_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$, as well as the related components $d_W \colon W \longrightarrow \mathcal{C}$ and $d_k \colon k \longrightarrow \mathcal{C}$ of the differential $d_{\mathcal{C}}$, and the components $h_W \colon W \longrightarrow k$ and $h_k \colon k \longrightarrow k$ of the curvature linear function $h_{\mathcal{C}}$.

In the usual $\mathbb{Z}$-grading setting, one always has $h_k = 0$ for the grading reasons. But in the $\mathbb{Z}/2\mathbb{Z}$-graded (2-periodic) setting, h_k may be nonzero.

The maps μ_W, d_W, and h_W are used in the construction of the differential d_{Cob} on the tensor algebra $\mathrm{Cob}(\mathcal{C})$. The maps μ_k, d_k, and h_k are used in the construction of the curvature element $h_{\mathrm{Cob}} \in \mathrm{Cob}(\mathcal{C})^2$.

Cobar construction

Given a direct sum decomposition $\mathcal{C} = k \oplus W$ as above, one considers the related components $\mu_W \colon W \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ and $\mu_k \colon k \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$ of the comultiplication map $\mu_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C}$, as well as the related components $d_W \colon W \longrightarrow \mathcal{C}$ and $d_k \colon k \longrightarrow \mathcal{C}$ of the differential $d_{\mathcal{C}}$, and the components $h_W \colon W \longrightarrow k$ and $h_k \colon k \longrightarrow k$ of the curvature linear function $h_{\mathcal{C}}$.

In the usual $\mathbb{Z}$-grading setting, one always has $h_k = 0$ for the grading reasons. But in the $\mathbb{Z}/2\mathbb{Z}$-graded (2-periodic) setting, h_k may be nonzero.

The maps μ_W, d_W, and h_W are used in the construction of the differential d_{Cob} on the tensor algebra $\mathrm{Cob}(\mathcal{C})$. The maps μ_k, d_k, and h_k are used in the construction of the curvature element $h_{\mathrm{Cob}} \in \mathrm{Cob}(\mathcal{C})^2$. The resulting CDG-algebra is denoted by $\mathrm{Cob}_w^\bullet(\mathcal{C}^\bullet) = (\mathrm{Cob}(\mathcal{C}), d_{\mathrm{Cob}}, h_{\mathrm{Cob}})$.

Cobar construction

Replacing the section $w\colon k \longrightarrow \mathcal{C}$ by another section $w'\colon k \longrightarrow \mathcal{C}$

Cobar construction

Replacing the section $w\colon k \longrightarrow \mathcal{C}$ by another section $w'\colon k \longrightarrow \mathcal{C}$, one constructs another CDG-algebra structure $\mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$ on the same graded algebra $\mathrm{Cob}(\mathcal{C})$.

Cobar construction

Replacing the section $w\colon k \longrightarrow \mathcal{C}$ by another section $w'\colon k \longrightarrow \mathcal{C}$, one constructs another CDG-algebra structure $\mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$ on the same graded algebra $\mathrm{Cob}(\mathcal{C})$. The two CDG-algebras are connected by a natural change-of-connection isomorphism $\mathrm{Cob}^\bullet_w(\mathcal{C}^\bullet) \simeq \mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$.

Cobar construction

Replacing the section $w\colon k \longrightarrow \mathcal{C}$ by another section $w'\colon k \longrightarrow \mathcal{C}$, one constructs another CDG-algebra structure $\mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$ on the same graded algebra $\mathrm{Cob}(\mathcal{C})$. The two CDG-algebras are connected by a natural change-of-connection isomorphism $\mathrm{Cob}^\bullet_w(\mathcal{C}^\bullet) \simeq \mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$.

Change-of-connection isomorphisms of CDG-coalgebras $(\mathrm{id}, b)\colon (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow (\mathcal{C}, d'_\mathcal{C}, h'_\mathcal{C})$

Cobar construction

Replacing the section $w\colon k \longrightarrow \mathcal{C}$ by another section $w'\colon k \longrightarrow \mathcal{C}$, one constructs another CDG-algebra structure $\mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$ on the same graded algebra $\mathrm{Cob}(\mathcal{C})$. The two CDG-algebras are connected by a natural change-of-connection isomorphism $\mathrm{Cob}^\bullet_w(\mathcal{C}^\bullet) \simeq \mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$.

Change-of-connection isomorphisms of CDG-coalgebras $(\mathrm{id}, b)\colon (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow (\mathcal{C}, d'_\mathcal{C}, h'_\mathcal{C})$ induce "variable change isomorphisms" of the cobar constructions.

Replacing the section $w\colon k \longrightarrow \mathcal{C}$ by another section $w'\colon k \longrightarrow \mathcal{C}$, one constructs another CDG-algebra structure $\mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$ on the same graded algebra $\mathrm{Cob}(\mathcal{C})$. The two CDG-algebras are connected by a natural change-of-connection isomorphism $\mathrm{Cob}^\bullet_w(\mathcal{C}^\bullet) \simeq \mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$.

Change-of-connection isomorphisms of CDG-coalgebras $(\mathrm{id}, b)\colon (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow (\mathcal{C}, d'_\mathcal{C}, h'_\mathcal{C})$ induce "variable change isomorphisms" of the cobar constructions. Such isomorphisms $\mathrm{Cob}_w(\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow \mathrm{Cob}_w(\mathcal{C}, d'_\mathcal{C}, h'_\mathcal{C})$

Cobar construction

Replacing the section $w\colon k \longrightarrow \mathcal{C}$ by another section $w'\colon k \longrightarrow \mathcal{C}$, one constructs another CDG-algebra structure $\mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$ on the same graded algebra $\mathrm{Cob}(\mathcal{C})$. The two CDG-algebras are connected by a natural change-of-connection isomorphism $\mathrm{Cob}^\bullet_w(\mathcal{C}^\bullet) \simeq \mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$.

Change-of-connection isomorphisms of CDG-coalgebras $(\mathrm{id}, b)\colon (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow (\mathcal{C}, d'_\mathcal{C}, h'_\mathcal{C})$ induce "variable change isomorphisms" of the cobar constructions. Such isomorphisms $\mathrm{Cob}_w(\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow \mathrm{Cob}_w(\mathcal{C}, d'_\mathcal{C}, h'_\mathcal{C})$ involve automorphisms of the free associative algebra $\mathrm{Cob}(\mathcal{C}) = \bigoplus_{n=0}^\infty (\ker \epsilon)[-1]$ adding scalars to elements of the generating vector space $(\ker \epsilon)[-1]$

Cobar construction

Replacing the section $w\colon k \longrightarrow \mathcal{C}$ by another section $w'\colon k \longrightarrow \mathcal{C}$, one constructs another CDG-algebra structure $\mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$ on the same graded algebra $\mathrm{Cob}(\mathcal{C})$. The two CDG-algebras are connected by a natural change-of-connection isomorphism $\mathrm{Cob}^\bullet_w(\mathcal{C}^\bullet) \simeq \mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$.

Change-of-connection isomorphisms of CDG-coalgebras $(\mathrm{id}, b)\colon (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow (\mathcal{C}, d'_\mathcal{C}, h'_\mathcal{C})$ induce "variable change isomorphisms" of the cobar constructions. Such isomorphisms $\mathrm{Cob}_w(\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow \mathrm{Cob}_w(\mathcal{C}, d'_\mathcal{C}, h'_\mathcal{C})$ involve automorphisms of the free associative algebra $\mathrm{Cob}(\mathcal{C}) = \bigoplus_{n=0}^\infty (\ker\epsilon)[-1]$ adding scalars to elements of the generating vector space $(\ker\epsilon)[-1]$, i. e., taking the subspace $(\ker\epsilon)[-1] \subset \mathrm{Cob}(\mathcal{C})$ into the subspace $k \oplus (\ker\epsilon)[-1] = (\ker\epsilon)[-1]^{\otimes 0} \oplus (\ker\epsilon)[-1]^{\otimes 1} \subset \mathrm{Cob}(\mathcal{C})$.

Cobar construction

Replacing the section $w\colon k \longrightarrow \mathcal{C}$ by another section $w'\colon k \longrightarrow \mathcal{C}$, one constructs another CDG-algebra structure $\mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$ on the same graded algebra $\mathrm{Cob}(\mathcal{C})$. The two CDG-algebras are connected by a natural change-of-connection isomorphism $\mathrm{Cob}^\bullet_w(\mathcal{C}^\bullet) \simeq \mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$.

Change-of-connection isomorphisms of CDG-coalgebras $(\mathrm{id}, b)\colon (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow (\mathcal{C}, d'_\mathcal{C}, h'_\mathcal{C})$ induce "variable change isomorphisms" of the cobar constructions. Such isomorphisms $\mathrm{Cob}_w(\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow \mathrm{Cob}_w(\mathcal{C}, d'_\mathcal{C}, h'_\mathcal{C})$ involve automorphisms of the free associative algebra $\mathrm{Cob}(\mathcal{C}) = \bigoplus_{n=0}^\infty (\ker \epsilon)[-1]$ adding scalars to elements of the generating vector space $(\ker \epsilon)[-1]$, i. e., taking the subspace $(\ker \epsilon)[-1] \subset \mathrm{Cob}(\mathcal{C})$ into the subspace $k \oplus (\ker \epsilon)[-1] = (\ker \epsilon)[-1]^{\otimes 0} \oplus (\ker \epsilon)[-1]^{\otimes 1} \subset \mathrm{Cob}(\mathcal{C})$.

Dual-analogous "variable change automorphisms" of the tensor coalgebras do not exist

Cobar construction

Replacing the section $w\colon k \longrightarrow \mathcal{C}$ by another section $w'\colon k \longrightarrow \mathcal{C}$, one constructs another CDG-algebra structure $\mathrm{Cob}^{\bullet}_{w'}(\mathcal{C}^{\bullet})$ on the same graded algebra $\mathrm{Cob}(\mathcal{C})$. The two CDG-algebras are connected by a natural change-of-connection isomorphism $\mathrm{Cob}^{\bullet}_{w}(\mathcal{C}^{\bullet}) \simeq \mathrm{Cob}^{\bullet}_{w'}(\mathcal{C}^{\bullet})$.

Change-of-connection isomorphisms of CDG-coalgebras $(\mathrm{id}, b)\colon (\mathcal{C}, d_{\mathcal{C}}, h_{\mathcal{C}}) \longrightarrow (\mathcal{C}, d'_{\mathcal{C}}, h'_{\mathcal{C}})$ induce "variable change isomorphisms" of the cobar constructions. Such isomorphisms $\mathrm{Cob}_w(\mathcal{C}, d_{\mathcal{C}}, h_{\mathcal{C}}) \longrightarrow \mathrm{Cob}_w(\mathcal{C}, d'_{\mathcal{C}}, h'_{\mathcal{C}})$ involve automorphisms of the free associative algebra $\mathrm{Cob}(\mathcal{C}) = \bigoplus_{n=0}^{\infty}(\ker\epsilon)[-1]$ adding scalars to elements of the generating vector space $(\ker\epsilon)[-1]$, i. e., taking the subspace $(\ker\epsilon)[-1] \subset \mathrm{Cob}(\mathcal{C})$ into the subspace $k \oplus (\ker\epsilon)[-1] = (\ker\epsilon)[-1]^{\otimes 0} \oplus (\ker\epsilon)[-1]^{\otimes 1} \subset \mathrm{Cob}(\mathcal{C})$.

Dual-analogous "variable change automorphisms" of the tensor coalgebras do not exist (as $z \longmapsto z+1$ is not an automorphism of $k[[z]]$).

Cobar construction

Replacing the section $w\colon k \longrightarrow \mathcal{C}$ by another section $w'\colon k \longrightarrow \mathcal{C}$, one constructs another CDG-algebra structure $\mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$ on the same graded algebra $\mathrm{Cob}(\mathcal{C})$. The two CDG-algebras are connected by a natural change-of-connection isomorphism $\mathrm{Cob}^\bullet_w(\mathcal{C}^\bullet) \simeq \mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$.

Change-of-connection isomorphisms of CDG-coalgebras $(\mathrm{id}, b)\colon (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow (\mathcal{C}, d'_\mathcal{C}, h'_\mathcal{C})$ induce "variable change isomorphisms" of the cobar constructions. Such isomorphisms $\mathrm{Cob}_w(\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow \mathrm{Cob}_w(\mathcal{C}, d'_\mathcal{C}, h'_\mathcal{C})$ involve automorphisms of the free associative algebra $\mathrm{Cob}(\mathcal{C}) = \bigoplus_{n=0}^\infty (\ker \epsilon)[-1]$ adding scalars to elements of the generating vector space $(\ker \epsilon)[-1]$, i. e., taking the subspace $(\ker \epsilon)[-1] \subset \mathrm{Cob}(\mathcal{C})$ into the subspace $k \oplus (\ker \epsilon)[-1] = (\ker \epsilon)[-1]^{\otimes 0} \oplus (\ker \epsilon)[-1]^{\otimes 1} \subset \mathrm{Cob}(\mathcal{C})$.

Dual-analogous "variable change automorphisms" of the tensor coalgebras do not exist (as $z \longmapsto z+1$ is not an automorphism of $k[[z]]$). For this reason, the bar construction Bar_v is only a functor on the category of DG-algebras

Cobar construction

Replacing the section $w\colon k \longrightarrow \mathcal{C}$ by another section $w'\colon k \longrightarrow \mathcal{C}$, one constructs another CDG-algebra structure $\mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$ on the same graded algebra $\mathrm{Cob}(\mathcal{C})$. The two CDG-algebras are connected by a natural change-of-connection isomorphism $\mathrm{Cob}^\bullet_w(\mathcal{C}^\bullet) \simeq \mathrm{Cob}^\bullet_{w'}(\mathcal{C}^\bullet)$.

Change-of-connection isomorphisms of CDG-coalgebras $(\mathrm{id}, b)\colon (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow (\mathcal{C}, d'_\mathcal{C}, h'_\mathcal{C})$ induce "variable change isomorphisms" of the cobar constructions. Such isomorphisms $\mathrm{Cob}_w(\mathcal{C}, d_\mathcal{C}, h_\mathcal{C}) \longrightarrow \mathrm{Cob}_w(\mathcal{C}, d'_\mathcal{C}, h'_\mathcal{C})$ involve automorphisms of the free associative algebra $\mathrm{Cob}(\mathcal{C}) = \bigoplus_{n=0}^\infty (\ker\epsilon)[-1]$ adding scalars to elements of the generating vector space $(\ker\epsilon)[-1]$, i. e., taking the subspace $(\ker\epsilon)[-1] \subset \mathrm{Cob}(\mathcal{C})$ into the subspace $k \oplus (\ker\epsilon)[-1] = (\ker\epsilon)[-1]^{\otimes 0} \oplus (\ker\epsilon)[-1]^{\otimes 1} \subset \mathrm{Cob}(\mathcal{C})$.

Dual-analogous "variable change automorphisms" of the tensor coalgebras do not exist (as $z \longmapsto z+1$ is not an automorphism of $k[[z]]$). For this reason, the bar construction Bar_v is only a functor on the category of DG-algebras, and not of CDG-algebras.

Conilpotent CDG-coalgebras

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$.

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C}, \gamma)$ is called conilpotent

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C}, \gamma)$ is called conilpotent if for every element $c \in \mathcal{C}$ there exists an integer $n \geqslant 0$

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C}, \gamma)$ is called conilpotent if for every element $c \in \mathcal{C}$ there exists an integer $n \geqslant 0$ such that c is annihilated by the iterated comultiplication map $\mathcal{C} \longrightarrow \mathcal{C}^{\otimes n+1} \longrightarrow (\mathcal{C}/\gamma(k))^{\otimes n+1}$

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C}, \gamma)$ is called conilpotent if for every element $c \in \mathcal{C}$ there exists an integer $n \geqslant 0$ such that c is annihilated by the iterated comultiplication map $\mathcal{C} \longrightarrow \mathcal{C}^{\otimes n+1} \longrightarrow (\mathcal{C}/\gamma(k))^{\otimes n+1}$ (i. e., the image of c in $(\mathcal{C}/\gamma(k))^{\otimes n+1}$ vanishes).

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C}, \gamma)$ is called conilpotent if for every element $c \in \mathcal{C}$ there exists an integer $n \geqslant 0$ such that c is annihilated by the iterated comultiplication map $\mathcal{C} \longrightarrow \mathcal{C}^{\otimes n+1} \longrightarrow (\mathcal{C}/\gamma(k))^{\otimes n+1}$ (i. e., the image of c in $(\mathcal{C}/\gamma(k))^{\otimes n+1}$ vanishes).

A coaugmentation of a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C}, \gamma)$ is called conilpotent if for every element $c \in \mathcal{C}$ there exists an integer $n \geqslant 0$ such that c is annihilated by the iterated comultiplication map $\mathcal{C} \longrightarrow \mathcal{C}^{\otimes n+1} \longrightarrow (\mathcal{C}/\gamma(k))^{\otimes n+1}$ (i. e., the image of c in $(\mathcal{C}/\gamma(k))^{\otimes n+1}$ vanishes).

A coaugmentation of a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ is a morphism of CDG-coalgebras $(\gamma, 0)\colon (k, 0, 0) \longrightarrow (\mathcal{C}, d, h)$.

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C}, \gamma)$ is called conilpotent if for every element $c \in \mathcal{C}$ there exists an integer $n \geqslant 0$ such that c is annihilated by the iterated comultiplication map $\mathcal{C} \longrightarrow \mathcal{C}^{\otimes n+1} \longrightarrow (\mathcal{C}/\gamma(k))^{\otimes n+1}$ (i. e., the image of c in $(\mathcal{C}/\gamma(k))^{\otimes n+1}$ vanishes).

A coaugmentation of a CDG-coalgebra $\mathcal{C}^\blacklozenge = (\mathcal{C}, d, h)$ is a morphism of CDG-coalgebras $(\gamma, 0)\colon (k, 0, 0) \longrightarrow (\mathcal{C}, d, h)$. A coaugmented CDG-coalgebra $(\mathcal{C}^\blacklozenge, \gamma)$ is called conilpotent

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C},\gamma)$ is called conilpotent if for every element $c \in \mathcal{C}$ there exists an integer $n \geqslant 0$ such that c is annihilated by the iterated comultiplication map $\mathcal{C} \longrightarrow \mathcal{C}^{\otimes n+1} \longrightarrow (\mathcal{C}/\gamma(k))^{\otimes n+1}$ (i. e., the image of c in $(\mathcal{C}/\gamma(k))^{\otimes n+1}$ vanishes).

A coaugmentation of a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ is a morphism of CDG-coalgebras $(\gamma, 0)\colon (k, 0, 0) \longrightarrow (\mathcal{C}, d, h)$. A coaugmented CDG-coalgebra $(\mathcal{C}^\bullet, \gamma)$ is called conilpotent if the graded coalgebra $\mathcal{C}$ with the coaugmentation γ is conilpotent.

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C},\gamma)$ is called conilpotent if for every element $c \in \mathcal{C}$ there exists an integer $n \geqslant 0$ such that c is annihilated by the iterated comultiplication map $\mathcal{C} \longrightarrow \mathcal{C}^{\otimes n+1} \longrightarrow (\mathcal{C}/\gamma(k))^{\otimes n+1}$ (i. e., the image of c in $(\mathcal{C}/\gamma(k))^{\otimes n+1}$ vanishes).

A coaugmentation of a CDG-coalgebra $\mathcal{C}^{\bullet} = (\mathcal{C}, d, h)$ is a morphism of CDG-coalgebras $(\gamma, 0)\colon (k, 0, 0) \longrightarrow (\mathcal{C}, d, h)$. A coaugmented CDG-coalgebra $(\mathcal{C}^{\bullet}, \gamma)$ is called conilpotent if the graded coalgebra $\mathcal{C}$ with the coaugmentation γ is conilpotent.

Under Koszul duality, lack of a chosen (co)augmentation on one side corresponds to a nonzero curvature on the other side.

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C}, \gamma)$ is called conilpotent if for every element $c \in \mathcal{C}$ there exists an integer $n \geqslant 0$ such that c is annihilated by the iterated comultiplication map $\mathcal{C} \longrightarrow \mathcal{C}^{\otimes n+1} \longrightarrow (\mathcal{C}/\gamma(k))^{\otimes n+1}$ (i. e., the image of c in $(\mathcal{C}/\gamma(k))^{\otimes n+1}$ vanishes).

A coaugmentation of a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ is a morphism of CDG-coalgebras $(\gamma, 0)\colon (k, 0, 0) \longrightarrow (\mathcal{C}, d, h)$. A coaugmented CDG-coalgebra $(\mathcal{C}^\bullet, \gamma)$ is called conilpotent if the graded coalgebra $\mathcal{C}$ with the coaugmentation γ is conilpotent.

Under Koszul duality, lack of a chosen (co)augmentation on one side corresponds to a nonzero curvature on the other side. In particular, given a coaugmented CDG-coalgebra $(\mathcal{C}^\bullet, \gamma)$

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C}, \gamma)$ is called conilpotent if for every element $c \in \mathcal{C}$ there exists an integer $n \geqslant 0$ such that c is annihilated by the iterated comultiplication map $\mathcal{C} \longrightarrow \mathcal{C}^{\otimes n+1} \longrightarrow (\mathcal{C}/\gamma(k))^{\otimes n+1}$ (i. e., the image of c in $(\mathcal{C}/\gamma(k))^{\otimes n+1}$ vanishes).

A coaugmentation of a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ is a morphism of CDG-coalgebras $(\gamma, 0)\colon (k, 0, 0) \longrightarrow (\mathcal{C}, d, h)$. A coaugmented CDG-coalgebra $(\mathcal{C}^\bullet, \gamma)$ is called conilpotent if the graded coalgebra $\mathcal{C}$ with the coaugmentation γ is conilpotent.

Under Koszul duality, lack of a chosen (co)augmentation on one side corresponds to a nonzero curvature on the other side. In particular, given a coaugmented CDG-coalgebra $(\mathcal{C}^\bullet, \gamma)$, one can use γ in the role of the section w

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C}, \gamma)$ is called conilpotent if for every element $c \in \mathcal{C}$ there exists an integer $n \geqslant 0$ such that c is annihilated by the iterated comultiplication map $\mathcal{C} \longrightarrow \mathcal{C}^{\otimes n+1} \longrightarrow (\mathcal{C}/\gamma(k))^{\otimes n+1}$ (i. e., the image of c in $(\mathcal{C}/\gamma(k))^{\otimes n+1}$ vanishes).

A coaugmentation of a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ is a morphism of CDG-coalgebras $(\gamma, 0)\colon (k, 0, 0) \longrightarrow (\mathcal{C}, d, h)$. A coaugmented CDG-coalgebra $(\mathcal{C}^\bullet, \gamma)$ is called conilpotent if the graded coalgebra $\mathcal{C}$ with the coaugmentation γ is conilpotent.

Under Koszul duality, lack of a chosen (co)augmentation on one side corresponds to a nonzero curvature on the other side. In particular, given a coaugmented CDG-coalgebra $(\mathcal{C}^\bullet, \gamma)$, one can use γ in the role of the section w, i. e., put $w = \gamma$.

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C}, \gamma)$ is called conilpotent if for every element $c \in \mathcal{C}$ there exists an integer $n \geqslant 0$ such that c is annihilated by the iterated comultiplication map $\mathcal{C} \longrightarrow \mathcal{C}^{\otimes n+1} \longrightarrow (\mathcal{C}/\gamma(k))^{\otimes n+1}$ (i. e., the image of c in $(\mathcal{C}/\gamma(k))^{\otimes n+1}$ vanishes).

A coaugmentation of a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ is a morphism of CDG-coalgebras $(\gamma, 0)\colon (k, 0, 0) \longrightarrow (\mathcal{C}, d, h)$. A coaugmented CDG-coalgebra $(\mathcal{C}^\bullet, \gamma)$ is called conilpotent if the graded coalgebra $\mathcal{C}$ with the coaugmentation γ is conilpotent.

Under Koszul duality, lack of a chosen (co)augmentation on one side corresponds to a nonzero curvature on the other side. In particular, given a coaugmented CDG-coalgebra $(\mathcal{C}^\bullet, \gamma)$, one can use γ in the role of the section w, i. e., put $w = \gamma$. Then the cobar CDG-algebra $\mathrm{Cob}^\bullet_w(\mathcal{C}^\bullet) = (\mathrm{Cob}(\mathcal{C}), d_{\mathrm{Cob}}, h_{\mathrm{Cob}})$ is actually a DG-algebra

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C}, \gamma)$ is called conilpotent if for every element $c \in \mathcal{C}$ there exists an integer $n \geqslant 0$ such that c is annihilated by the iterated comultiplication map $\mathcal{C} \longrightarrow \mathcal{C}^{\otimes n+1} \longrightarrow (\mathcal{C}/\gamma(k))^{\otimes n+1}$ (i. e., the image of c in $(\mathcal{C}/\gamma(k))^{\otimes n+1}$ vanishes).

A coaugmentation of a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ is a morphism of CDG-coalgebras $(\gamma, 0)\colon (k, 0, 0) \longrightarrow (\mathcal{C}, d, h)$. A coaugmented CDG-coalgebra $(\mathcal{C}^\bullet, \gamma)$ is called conilpotent if the graded coalgebra $\mathcal{C}$ with the coaugmentation γ is conilpotent.

Under Koszul duality, lack of a chosen (co)augmentation on one side corresponds to a nonzero curvature on the other side. In particular, given a coaugmented CDG-coalgebra $(\mathcal{C}^\bullet, \gamma)$, one can use γ in the role of the section w, i. e., put $w = \gamma$. Then the cobar CDG-algebra $\mathrm{Cob}_w^\bullet(\mathcal{C}^\bullet) = (\mathrm{Cob}(\mathcal{C}), d_{\mathrm{Cob}}, h_{\mathrm{Cob}})$ is actually a DG-algebra, $h_{\mathrm{Cob}} = 0$.

Conilpotent CDG-coalgebras

A coaugmentation of a graded coalgebra $\mathcal{C}$ is a morphism of coalgebras $\gamma\colon k \longrightarrow \mathcal{C}$. A coaugmented coalgebra $(\mathcal{C}, \gamma)$ is called conilpotent if for every element $c \in \mathcal{C}$ there exists an integer $n \geqslant 0$ such that c is annihilated by the iterated comultiplication map $\mathcal{C} \longrightarrow \mathcal{C}^{\otimes n+1} \longrightarrow (\mathcal{C}/\gamma(k))^{\otimes n+1}$ (i. e., the image of c in $(\mathcal{C}/\gamma(k))^{\otimes n+1}$ vanishes).

A coaugmentation of a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ is a morphism of CDG-coalgebras $(\gamma, 0)\colon (k, 0, 0) \longrightarrow (\mathcal{C}, d, h)$. A coaugmented CDG-coalgebra $(\mathcal{C}^\bullet, \gamma)$ is called conilpotent if the graded coalgebra $\mathcal{C}$ with the coaugmentation γ is conilpotent.

Under Koszul duality, lack of a chosen (co)augmentation on one side corresponds to a nonzero curvature on the other side. In particular, given a coaugmented CDG-coalgebra $(\mathcal{C}^\bullet, \gamma)$, one can use γ in the role of the section w, i. e., put $w = \gamma$. Then the cobar CDG-algebra $\mathrm{Cob}^\bullet_w(\mathcal{C}^\bullet) = (\mathrm{Cob}(\mathcal{C}), d_{\mathrm{Cob}}, h_{\mathrm{Cob}})$ is actually a DG-algebra, $h_{\mathrm{Cob}} = 0$. We denote this DG-algebra by $\mathrm{Cob}^\bullet_\gamma(\mathcal{C}^\bullet)$.

Conilpotent CDG-coalgebras

An admissible filtration F on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$

An admissible filtration F on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ is an increasing filtration by homogeneous vector subspaces $F_0\mathcal{C} \subset F_1\mathcal{C} \subset F_2\mathcal{C} \subset \cdots$

Conilpotent CDG-coalgebras

An admissible filtration F on a coaugmented CDG-coalgebra $\mathcal{C}^{\bullet}$ is an increasing filtration by homogeneous vector subspaces $F_0\mathcal{C} \subset F_1\mathcal{C} \subset F_2\mathcal{C} \subset \cdots$ such that

- $\mu(F_n\mathcal{C}) \subset \sum_{p+q=n} F_p\mathcal{C} \otimes F_q\mathcal{C}$ (comultiplicativity),

Conilpotent CDG-coalgebras

An admissible filtration F on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ is an increasing filtration by homogeneous vector subspaces $F_0\mathcal{C} \subset F_1\mathcal{C} \subset F_2\mathcal{C} \subset \cdots$ such that

- $\mu(F_n\mathcal{C}) \subset \sum_{p+q=n} F_p\mathcal{C} \otimes F_q\mathcal{C}$ (comultiplicativity),
- $d(F_n\mathcal{C}) \subset F_n\mathcal{C}$ for all $n \geqslant 0$,

Conilpotent CDG-coalgebras

An admissible filtration F on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ is an increasing filtration by homogeneous vector subspaces $F_0\mathcal{C} \subset F_1\mathcal{C} \subset F_2\mathcal{C} \subset \cdots$ such that

- $\mu(F_n\mathcal{C}) \subset \sum_{p+q=n} F_p\mathcal{C} \otimes F_q\mathcal{C}$ (comultiplicativity),
- $d(F_n\mathcal{C}) \subset F_n\mathcal{C}$ for all $n \geqslant 0$,
- $\mathcal{C} = \bigcup_{n\geqslant 0} F_n\mathcal{C}$,

Conilpotent CDG-coalgebras

An admissible filtration F on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ is an increasing filtration by homogeneous vector subspaces $F_0\mathcal{C} \subset F_1\mathcal{C} \subset F_2\mathcal{C} \subset \cdots$ such that

- $\mu(F_n\mathcal{C}) \subset \sum_{p+q=n} F_p\mathcal{C} \otimes F_q\mathcal{C}$ (comultiplicativity),
- $d(F_n\mathcal{C}) \subset F_n\mathcal{C}$ for all $n \geqslant 0$,
- $\mathcal{C} = \bigcup_{n\geqslant 0} F_n\mathcal{C}$,
- $F_{-1}\mathcal{C} = 0, \;\; F_0\mathcal{C} = \gamma(k)$.

Conilpotent CDG-coalgebras

An admissible filtration F on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ is an increasing filtration by homogeneous vector subspaces $F_0\mathcal{C} \subset F_1\mathcal{C} \subset F_2\mathcal{C} \subset \cdots$ such that

- $\mu(F_n\mathcal{C}) \subset \sum_{p+q=n} F_p\mathcal{C} \otimes F_q\mathcal{C}$ (comultiplicativity),
- $d(F_n\mathcal{C}) \subset F_n\mathcal{C}$ for all $n \geqslant 0$,
- $\mathcal{C} = \bigcup_{n\geqslant 0} F_n\mathcal{C}$,
- $F_{-1}\mathcal{C} = 0$, $F_0\mathcal{C} = \gamma(k)$.

An admissible filtration exists on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ if and only if $\mathcal{C}^\bullet$ is conilpotent.

Conilpotent CDG-coalgebras

An admissible filtration F on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ is an increasing filtration by homogeneous vector subspaces $F_0\mathcal{C} \subset F_1\mathcal{C} \subset F_2\mathcal{C} \subset \cdots$ such that

- $\mu(F_n\mathcal{C}) \subset \sum_{p+q=n} F_p\mathcal{C} \otimes F_q\mathcal{C}$ (comultiplicativity),
- $d(F_n\mathcal{C}) \subset F_n\mathcal{C}$ for all $n \geqslant 0$,
- $\mathcal{C} = \bigcup_{n\geqslant 0} F_n\mathcal{C}$,
- $F_{-1}\mathcal{C} = 0$, $F_0\mathcal{C} = \gamma(k)$.

An admissible filtration exists on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ if and only if $\mathcal{C}^\bullet$ is conilpotent.

The associated graded object $\mathrm{gr}^F\mathcal{C}^\bullet = \bigoplus_{n=0}^{\infty} F_n\mathcal{C}/F_{n-1}\mathcal{C}$

Conilpotent CDG-coalgebras

An admissible filtration F on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ is an increasing filtration by homogeneous vector subspaces $F_0\mathcal{C} \subset F_1\mathcal{C} \subset F_2\mathcal{C} \subset \cdots$ such that

- $\mu(F_n\mathcal{C}) \subset \sum_{p+q=n} F_p\mathcal{C} \otimes F_q\mathcal{C}$ (comultiplicativity),
- $d(F_n\mathcal{C}) \subset F_n\mathcal{C}$ for all $n \geqslant 0$,
- $\mathcal{C} = \bigcup_{n\geqslant 0} F_n\mathcal{C}$,
- $F_{-1}\mathcal{C} = 0$, $F_0\mathcal{C} = \gamma(k)$.

An admissible filtration exists on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ if and only if $\mathcal{C}^\bullet$ is conilpotent.

The associated graded object $\mathrm{gr}^F\mathcal{C}^\bullet = \bigoplus_{n=0}^{\infty} F_n\mathcal{C}/F_{n-1}\mathcal{C}$ to a coagumented CDG-coalgebra $\mathcal{C}^\bullet$ with respect to an admissible filtration F

Conilpotent CDG-coalgebras

An admissible filtration F on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ is an increasing filtration by homogeneous vector subspaces $F_0\mathcal{C} \subset F_1\mathcal{C} \subset F_2\mathcal{C} \subset \cdots$ such that

- $\mu(F_n\mathcal{C}) \subset \sum_{p+q=n} F_p\mathcal{C} \otimes F_q\mathcal{C}$ (comultiplicativity),
- $d(F_n\mathcal{C}) \subset F_n\mathcal{C}$ for all $n \geqslant 0$,
- $\mathcal{C} = \bigcup_{n\geqslant 0} F_n\mathcal{C}$,
- $F_{-1}\mathcal{C} = 0$, $F_0\mathcal{C} = \gamma(k)$.

An admissible filtration exists on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ if and only if $\mathcal{C}^\bullet$ is conilpotent.

The associated graded object $\mathrm{gr}^F\mathcal{C}^\bullet = \bigoplus_{n=0}^{\infty} F_n\mathcal{C}/F_{n-1}\mathcal{C}$ to a coagumented CDG-coalgebra $\mathcal{C}^\bullet$ with respect to an admissible filtration F is a DG-coalgebra (in particular, a complex).

Conilpotent CDG-coalgebras

An admissible filtration F on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ is an increasing filtration by homogeneous vector subspaces $F_0\mathcal{C} \subset F_1\mathcal{C} \subset F_2\mathcal{C} \subset \cdots$ such that

- $\mu(F_n\mathcal{C}) \subset \sum_{p+q=n} F_p\mathcal{C} \otimes F_q\mathcal{C}$ (comultiplicativity),
- $d(F_n\mathcal{C}) \subset F_n\mathcal{C}$ for all $n \geqslant 0$,
- $\mathcal{C} = \bigcup_{n\geqslant 0} F_n\mathcal{C}$,
- $F_{-1}\mathcal{C} = 0, \;\; F_0\mathcal{C} = \gamma(k)$.

An admissible filtration exists on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ if and only if $\mathcal{C}^\bullet$ is conilpotent.

The associated graded object $\mathrm{gr}^F\mathcal{C}^\bullet = \bigoplus_{n=0}^{\infty} F_n\mathcal{C}/F_{n-1}\mathcal{C}$ to a coagumented CDG-coalgebra $\mathcal{C}^\bullet$ with respect to an admissible filtration F is a DG-coalgebra (in particular, a complex).

A morphism of CDG-coalgebras $(g, b)\colon \mathcal{C}^\bullet \longrightarrow \mathcal{D}^\bullet$ is called a filtered quasi-isomorphism

Conilpotent CDG-coalgebras

An admissible filtration F on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ is an increasing filtration by homogeneous vector subspaces $F_0\mathcal{C} \subset F_1\mathcal{C} \subset F_2\mathcal{C} \subset \cdots$ such that

- $\mu(F_n\mathcal{C}) \subset \sum_{p+q=n} F_p\mathcal{C} \otimes F_q\mathcal{C}$ (comultiplicativity),
- $d(F_n\mathcal{C}) \subset F_n\mathcal{C}$ for all $n \geqslant 0$,
- $\mathcal{C} = \bigcup_{n\geqslant 0} F_n\mathcal{C}$,
- $F_{-1}\mathcal{C} = 0$, $F_0\mathcal{C} = \gamma(k)$.

An admissible filtration exists on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ if and only if $\mathcal{C}^\bullet$ is conilpotent.

The associated graded object $\mathrm{gr}^F\mathcal{C}^\bullet = \bigoplus_{n=0}^{\infty} F_n\mathcal{C}/F_{n-1}\mathcal{C}$ to a coagumented CDG-coalgebra $\mathcal{C}^\bullet$ with respect to an admissible filtration F is a DG-coalgebra (in particular, a complex).

A morphism of CDG-coalgebras $(g,b)\colon \mathcal{C}^\bullet \longrightarrow \mathcal{D}^\bullet$ is called a filtered quasi-isomorphism if there exists a pair of admissible filtrations F on $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$

Conilpotent CDG-coalgebras

An admissible filtration F on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ is an increasing filtration by homogeneous vector subspaces $F_0\mathcal{C} \subset F_1\mathcal{C} \subset F_2\mathcal{C} \subset \cdots$ such that

- $\mu(F_n\mathcal{C}) \subset \sum_{p+q=n} F_p\mathcal{C} \otimes F_q\mathcal{C}$ (comultiplicativity),
- $d(F_n\mathcal{C}) \subset F_n\mathcal{C}$ for all $n \geqslant 0$,
- $\mathcal{C} = \bigcup_{n\geqslant 0} F_n\mathcal{C}$,
- $F_{-1}\mathcal{C} = 0, \;\; F_0\mathcal{C} = \gamma(k)$.

An admissible filtration exists on a coaugmented CDG-coalgebra $\mathcal{C}^\bullet$ if and only if $\mathcal{C}^\bullet$ is conilpotent.

The associated graded object $\mathrm{gr}^F\mathcal{C}^\bullet = \bigoplus_{n=0}^{\infty} F_n\mathcal{C}/F_{n-1}\mathcal{C}$ to a coagumented CDG-coalgebra $\mathcal{C}^\bullet$ with respect to an admissible filtration F is a DG-coalgebra (in particular, a complex).

A morphism of CDG-coalgebras $(g,b)\colon \mathcal{C}^\bullet \longrightarrow \mathcal{D}^\bullet$ is called a filtered quasi-isomorphism if there exists a pair of admissible filtrations F on $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$ such that $\mathrm{gr}^F g\colon \mathrm{gr}^F\mathcal{C} \longrightarrow \mathrm{gr}^F\mathcal{D}$ is a quasi-isomorphism.

Koszul duality between DG-algebras and CDG-coalgebras

Koszul duality between DG-algebras and CDG-coalgebras

Recall the notation $\mathrm{DG\text{-}alg}_k^+$ for the category of nonzero DG-algebras over a field k.

Koszul duality between DG-algebras and CDG-coalgebras

Recall the notation $\mathrm{DG\text{-}alg}_k^+$ for the category of nonzero DG-algebras over a field k. Let Quis denote the class of quasi-isomorphisms in $\mathrm{DG\text{-}alg}_k^+$

Recall the notation $\mathrm{DG\text{-}alg}_k^+$ for the category of nonzero DG-algebras over a field k. Let Quis denote the class of quasi-isomorphisms in $\mathrm{DG\text{-}alg}_k^+$ (i. e., morphisms of DG-algebras inducing isomorphisms of the cohomology algebras).

Koszul duality between DG-algebras and CDG-coalgebras

Recall the notation $\mathrm{DG\text{-}alg}_k^+$ for the category of nonzero DG-algebras over a field k. Let Quis denote the class of quasi-isomorphisms in $\mathrm{DG\text{-}alg}_k^+$ (i. e., morphisms of DG-algebras inducing isomorphisms of the cohomology algebras).

Let $\mathrm{CDG\text{-}coalg}_k^{\mathrm{conilp}}$ denote the category of conilpotent coaugmented CDG-coalgebras over k.

Koszul duality between DG-algebras and CDG-coalgebras

Recall the notation $\mathrm{DG\text{-}alg}_k^+$ for the category of nonzero DG-algebras over a field k. Let Quis denote the class of quasi-isomorphisms in $\mathrm{DG\text{-}alg}_k^+$ (i. e., morphisms of DG-algebras inducing isomorphisms of the cohomology algebras).

Let $\mathrm{CDG\text{-}coalg}_k^{\mathrm{conilp}}$ denote the category of conilpotent coaugmented CDG-coalgebras over k. Let FQuis be the class of filtered quasi-isomorphisms in $\mathrm{CDG\text{-}coalg}_k^{\mathrm{conilp}}$.

Koszul duality between DG-algebras and CDG-coalgebras

Recall the notation $\mathrm{DG\text{-}alg}_k^+$ for the category of nonzero DG-algebras over a field k. Let Quis denote the class of quasi-isomorphisms in $\mathrm{DG\text{-}alg}_k^+$ (i. e., morphisms of DG-algebras inducing isomorphisms of the cohomology algebras).

Let $\mathrm{CDG\text{-}coalg}_k^{\mathrm{conilp}}$ denote the category of conilpotent coaugmented CDG-coalgebras over k. Let FQuis be the class of filtered quasi-isomorphisms in $\mathrm{CDG\text{-}coalg}_k^{\mathrm{conilp}}$.

Theorem (Bar-Cobar duality for DG-algebras and CDG-coalgebras)

Koszul duality between DG-algebras and CDG-coalgebras

Recall the notation $\mathrm{DG\text{-}alg}_k^+$ for the category of nonzero DG-algebras over a field k. Let Quis denote the class of quasi-isomorphisms in $\mathrm{DG\text{-}alg}_k^+$ (i. e., morphisms of DG-algebras inducing isomorphisms of the cohomology algebras).

Let $\mathrm{CDG\text{-}coalg}_k^{\mathrm{conilp}}$ denote the category of conilpotent coaugmented CDG-coalgebras over k. Let FQuis be the class of filtered quasi-isomorphisms in $\mathrm{CDG\text{-}coalg}_k^{\mathrm{conilp}}$.

Theorem (Bar-Cobar duality for DG-algebras and CDG-coalgebras)

There is a natural equivalence of categories

$$\mathsf{Bar}_v \colon \mathrm{DG\text{-}alg}_k^+[\mathrm{Quis}^{-1}] \simeq \mathrm{CDG\text{-}coalg}_k^{\mathrm{conilp}}[\mathrm{FQuis}^{-1}] \; \colon \mathsf{Cob}_\gamma$$

Koszul duality between DG-algebras and CDG-coalgebras

Recall the notation $\mathrm{DG\text{-}alg}_k^+$ for the category of nonzero DG-algebras over a field k. Let Quis denote the class of quasi-isomorphisms in $\mathrm{DG\text{-}alg}_k^+$ (i. e., morphisms of DG-algebras inducing isomorphisms of the cohomology algebras).

Let $\mathrm{CDG\text{-}coalg}_k^{\mathrm{conilp}}$ denote the category of conilpotent coaugmented CDG-coalgebras over k. Let FQuis be the class of filtered quasi-isomorphisms in $\mathrm{CDG\text{-}coalg}_k^{\mathrm{conilp}}$.

Theorem (Bar-Cobar duality for DG-algebras and CDG-coalgebras)

There is a natural equivalence of categories

$$\mathsf{Bar}_v\colon \mathrm{DG\text{-}alg}_k^+[\mathrm{Quis}^{-1}] \simeq \mathrm{CDG\text{-}coalg}_k^{\mathrm{conilp}}[\mathrm{FQuis}^{-1}] \;:\!\mathsf{Cob}_\gamma$$

between the category of nonzero DG-algebras over k with quasi-isomorphisms inverted

Koszul duality between DG-algebras and CDG-coalgebras

Recall the notation $\mathrm{DG\text{-}alg}_k^+$ for the category of nonzero DG-algebras over a field k. Let Quis denote the class of quasi-isomorphisms in $\mathrm{DG\text{-}alg}_k^+$ (i. e., morphisms of DG-algebras inducing isomorphisms of the cohomology algebras).

Let $\mathrm{CDG\text{-}coalg}_k^{\mathrm{conilp}}$ denote the category of conilpotent coaugmented CDG-coalgebras over k. Let FQuis be the class of filtered quasi-isomorphisms in $\mathrm{CDG\text{-}coalg}_k^{\mathrm{conilp}}$.

Theorem (Bar-Cobar duality for DG-algebras and CDG-coalgebras)

There is a natural equivalence of categories

$$\mathsf{Bar}_v\colon \mathrm{DG\text{-}alg}_k^+[\mathrm{Quis}^{-1}] \simeq \mathrm{CDG\text{-}coalg}_k^{\mathrm{conilp}}[\mathrm{FQuis}^{-1}] \;:\! \mathsf{Cob}_\gamma$$

between the category of nonzero DG-algebras over k with quasi-isomorphisms inverted and the category of conilpotent CDG-coalgebras over k with filtered quasi-isomorphisms inverted.

Conilpotent Koszul duality

Conilpotent Koszul duality

Let $A^\bullet = (A, d_A)$ and $\mathcal{C}^\blacklozenge = (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C})$ be a DG-algebra and a conilpotent CDG-coalgebra

Conilpotent Koszul duality

Let $A^\bullet = (A, d_A)$ and $\mathcal{C}^\bullet = (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C})$ be a DG-algebra and a conilpotent CDG-coalgebra corresponding to each other under the equivalence of categories from the previous slide.

Conilpotent Koszul duality

Let $A^\bullet = (A, d_A)$ and $\mathcal{C}^\blacklozenge = (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C})$ be a DG-algebra and a conilpotent CDG-coalgebra corresponding to each other under the equivalence of categories from the previous slide.

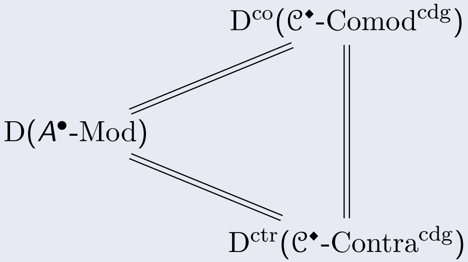
Theorem (Conilpotent Koszul triality for modules)

Conilpotent Koszul duality

Let $A^\bullet = (A, d_A)$ and $\mathcal{C}^\bullet = (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C})$ be a DG-algebra and a conilpotent CDG-coalgebra corresponding to each other under the equivalence of categories from the previous slide.

Theorem (Conilpotent Koszul triality for modules)

There is a commutative diagram of triangulated equivalences



Conilpotent Koszul duality

Let $A^\bullet = (A, d_A)$ and $\mathcal{C}^\bullet = (\mathcal{C}, d_\mathcal{C}, h_\mathcal{C})$ be a DG-algebra and a conilpotent CDG-coalgebra corresponding to each other under the equivalence of categories from the previous slide.

Theorem (Conilpotent Koszul triality for modules)

There is a commutative diagram of triangulated equivalences

$$\begin{array}{ccc} & & \mathrm{D}^{\mathrm{co}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}}) \\ & \diagup\!\!\diagup & \| \\ \mathrm{D}(A^\bullet\text{-}\mathrm{Mod}) & & \| \\ & \diagdown\!\!\diagdown & \| \\ & & \mathrm{D}^{\mathrm{ctr}}(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}}) \end{array}$$

The vertical equivalence is the derived co-contra correspondence.

Nonconilpotent Koszul duality

Nonconilpotent Koszul duality

Let $\mathcal{C}^\bullet$ be a (nonconilpotent, noncoaugmented) CDG-coalgebra

Nonconilpotent Koszul duality

Let $\mathcal{C}^\blacklozenge$ be a (nonconilpotent, noncoaugmented) CDG-coalgebra and $B^\blacklozenge = \mathrm{Cob}_w(\mathcal{C}^\blacklozenge)$ be its cobar construction (a CDG-algebra).

Let $\mathcal{C}^\bullet$ be a (nonconilpotent, noncoaugmented) CDG-coalgebra and $B^\bullet = \mathrm{Cob}_w(\mathcal{C}^\bullet)$ be its cobar construction (a CDG-algebra).

Theorem (Nonconilpotent Koszul triality for modules)

Nonconilpotent Koszul duality

Let $\mathcal{C}^\bullet$ be a (nonconilpotent, noncoaugmented) CDG-coalgebra and $B^\bullet = \mathrm{Cob}_w(\mathcal{C}^\bullet)$ be its cobar construction (a CDG-algebra).

Theorem (Nonconilpotent Koszul triality for modules)

There is a commutative diagram of triangulated equivalences

$$
\begin{array}{ccc}
 & & \mathsf{D}^{\mathrm{co}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}}) \\
 & \diagup\!\!\diagup & \| \\
\mathsf{D}^{\mathrm{co}=\mathrm{abs}=\mathrm{ctr}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}) & & \| \\
 & \diagdown\!\!\diagdown & \| \\
 & & \mathsf{D}^{\mathrm{ctr}}(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}})
\end{array}
$$

Nonconilpotent Koszul duality

Let $\mathcal{C}^\bullet$ be a (nonconilpotent, noncoaugmented) CDG-coalgebra and $B^\bullet = \mathrm{Cob}_w(\mathcal{C}^\bullet)$ be its cobar construction (a CDG-algebra).

Theorem (Nonconilpotent Koszul triality for modules)

There is a commutative diagram of triangulated equivalences

$$\mathsf{D}^{\mathrm{co}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}})$$

$$\mathsf{D}^{\mathrm{co=abs=ctr}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}})$$

$$\mathsf{D}^{\mathrm{ctr}}(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}})$$

The classes of absolutely acyclic, coacyclic, and contraacyclic CDG-modules over $B^\bullet$ coincide

Nonconilpotent Koszul duality

Let $\mathcal{C}^\bullet$ be a (nonconilpotent, noncoaugmented) CDG-coalgebra and $B^\bullet = \mathrm{Cob}_w(\mathcal{C}^\bullet)$ be its cobar construction (a CDG-algebra).

Theorem (Nonconilpotent Koszul triality for modules)

There is a commutative diagram of triangulated equivalences

$$\begin{array}{ccc} & & \mathsf{D}^{\mathrm{co}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}}) \\ & \nearrow\!\!\!\!= & \| \\ \mathsf{D}^{\mathrm{co}=\mathrm{abs}=\mathrm{ctr}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}) & & \| \\ & \searrow\!\!\!\!= & \| \\ & & \mathsf{D}^{\mathrm{ctr}}(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}}) \end{array}$$

The classes of absolutely acyclic, coacyclic, and contraacyclic CDG-modules over $B^\bullet$ coincide, because the graded algebra B has finite global dimension

Nonconilpotent Koszul duality

Let $\mathcal{C}^\bullet$ be a (nonconilpotent, noncoaugmented) CDG-coalgebra and $B^\bullet = \mathrm{Cob}_w(\mathcal{C}^\bullet)$ be its cobar construction (a CDG-algebra).

Theorem (Nonconilpotent Koszul triality for modules)

There is a commutative diagram of triangulated equivalences

$$\begin{array}{ccc} & & \mathsf{D}^{\mathrm{co}}(\mathcal{C}^\bullet\text{-}\mathrm{Comod}^{\mathrm{cdg}}) \\ & \diagup\!\!\diagup & \| \\ \mathsf{D}^{\mathrm{co}=\mathrm{abs}=\mathrm{ctr}}(B^\bullet\text{-}\mathrm{Mod}^{\mathrm{cdg}}) & & \| \\ & \diagdown\!\!\diagdown & \| \\ & & \mathsf{D}^{\mathrm{ctr}}(\mathcal{C}^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}}) \end{array}$$

The classes of absolutely acyclic, coacyclic, and contraacyclic CDG-modules over $B^\bullet$ coincide, because the graded algebra B has finite global dimension (in fact, global dimension 1 in this case).

Koszul duality functors

Koszul duality functors

The functors providing the “diagonal” category equivalences on the previous two slides are constructed as follows.

Koszul duality functors

The functors providing the "diagonal" category equivalences on the previous two slides are constructed as follows.

Given a CDG-module $M^\blacklozenge$ over $B^\blacklozenge$

Koszul duality functors

The functors providing the "diagonal" category equivalences on the previous two slides are constructed as follows.

Given a CDG-module $M^\bullet$ over $B^\bullet$, the related CDG-comodule over $\mathcal{C}^\bullet$ is $\mathcal{C}^\bullet \otimes_k^\tau M^\bullet$

Koszul duality functors

The functors providing the "diagonal" category equivalences on the previous two slides are constructed as follows.

Given a CDG-module $M^\bullet$ over $B^\bullet$, the related CDG-comodule over $\mathcal{C}^\bullet$ is $\mathcal{C}^\bullet \otimes_k^\tau M^\bullet$, while the related CDG-contramodule is $\mathrm{Hom}_k^\tau(\mathcal{C}^\bullet, M^\bullet)$.

Koszul duality functors

The functors providing the "diagonal" category equivalences on the previous two slides are constructed as follows.

Given a CDG-module $M^\bullet$ over $B^\bullet$, the related CDG-comodule over $\mathcal{C}^\bullet$ is $\mathcal{C}^\bullet \otimes_k^\tau M^\bullet$, while the related CDG-contramodule is $\mathrm{Hom}_k^\tau(\mathcal{C}^\bullet, M^\bullet)$. The tensor product and Hom are taken over k.

Koszul duality functors

The functors providing the "diagonal" category equivalences on the previous two slides are constructed as follows.

Given a CDG-module $M^\bullet$ over $B^\bullet$, the related CDG-comodule over $\mathcal{C}^\bullet$ is $\mathcal{C}^\bullet \otimes_k^\tau M^\bullet$, while the related CDG-contramodule is $\mathrm{Hom}_k^\tau(\mathcal{C}^\bullet, M^\bullet)$. The tensor product and Hom are taken over k. The co/contraaction of $\mathcal{C}$ on this tensor product/Hom space is induced by the coaction of $\mathcal{C}$ on itself.

Koszul duality functors

The functors providing the "diagonal" category equivalences on the previous two slides are constructed as follows.

Given a CDG-module $M^\bullet$ over $B^\bullet$, the related CDG-comodule over $\mathcal{C}^\bullet$ is $\mathcal{C}^\bullet \otimes_k^\tau M^\bullet$, while the related CDG-contramodule is $\mathrm{Hom}_k^\tau(\mathcal{C}^\bullet, M^\bullet)$. The tensor product and Hom are taken over k. The co/contraaction of $\mathcal{C}$ on this tensor product/Hom space is induced by the coaction of $\mathcal{C}$ on itself. The symbol τ means a twisted differential

Koszul duality functors

The functors providing the “diagonal” category equivalences on the previous two slides are constructed as follows.

Given a CDG-module $M^\bullet$ over $B^\bullet$, the related CDG-comodule over $\mathcal{C}^\bullet$ is $\mathcal{C}^\bullet \otimes_k^\tau M^\bullet$, while the related CDG-contramodule is $\mathrm{Hom}_k^\tau(\mathcal{C}^\bullet, M^\bullet)$. The tensor product and Hom are taken over k. The co/contraaction of $\mathcal{C}$ on this tensor product/Hom space is induced by the coaction of $\mathcal{C}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on M).

Koszul duality functors

The functors providing the "diagonal" category equivalences on the previous two slides are constructed as follows.

Given a CDG-module $M^\bullet$ over $B^\bullet$, the related CDG-comodule over $\mathcal{C}^\bullet$ is $\mathcal{C}^\bullet \otimes_k^\tau M^\bullet$, while the related CDG-contramodule is $\mathrm{Hom}_k^\tau(\mathcal{C}^\bullet, M^\bullet)$. The tensor product and Hom are taken over k. The co/contraaction of $\mathcal{C}$ on this tensor product/Hom space is induced by the coaction of $\mathcal{C}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on M).

Givan a CDG-comodule $\mathcal{N}^\bullet$ over $\mathcal{C}^\bullet$, the related CDG-module over $B^\bullet$ is $B^\bullet \otimes_k^\tau \mathcal{N}^\bullet$.

The functors providing the "diagonal" category equivalences on the previous two slides are constructed as follows.

Given a CDG-module $M^\bullet$ over $B^\bullet$, the related CDG-comodule over $\mathcal{C}^\bullet$ is $\mathcal{C}^\bullet \otimes_k^\tau M^\bullet$, while the related CDG-contramodule is $\mathrm{Hom}_k^\tau(\mathcal{C}^\bullet, M^\bullet)$. The tensor product and Hom are taken over k. The co/contraaction of $\mathcal{C}$ on this tensor product/Hom space is induced by the coaction of $\mathcal{C}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on M).

Givan a CDG-comodule $\mathcal{N}^\bullet$ over $\mathcal{C}^\bullet$, the related CDG-module over $B^\bullet$ is $B^\bullet \otimes_k^\tau \mathcal{N}^\bullet$. Given a CDG-contramodule $\mathfrak{Q}^\bullet$ over $\mathcal{C}^\bullet$, the related CDG-module over $B^\bullet$ is $\mathrm{Hom}_k^\tau(B^\bullet, \mathfrak{Q}^\bullet)$.

Koszul duality functors

The functors providing the “diagonal” category equivalences on the previous two slides are constructed as follows.

Given a CDG-module $M^\bullet$ over $B^\bullet$, the related CDG-comodule over $\mathcal{C}^\bullet$ is $\mathcal{C}^\bullet \otimes_k^\tau M^\bullet$, while the related CDG-contramodule is $\mathrm{Hom}_k^\tau(\mathcal{C}^\bullet, M^\bullet)$. The tensor product and Hom are taken over k. The co/contraaction of $\mathcal{C}$ on this tensor product/Hom space is induced by the coaction of $\mathcal{C}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on M).

Givan a CDG-comodule $\mathcal{N}^\bullet$ over $\mathcal{C}^\bullet$, the related CDG-module over $B^\bullet$ is $B^\bullet \otimes_k^\tau \mathcal{N}^\bullet$. Given a CDG-contramodule $\mathfrak{Q}^\bullet$ over $\mathcal{C}^\bullet$, the related CDG-module over $B^\bullet$ is $\mathrm{Hom}_k^\tau(B^\bullet, \mathfrak{Q}^\bullet)$. The tensor product and Hom are taken over k.

Koszul duality functors

The functors providing the “diagonal” category equivalences on the previous two slides are constructed as follows.

Given a CDG-module $M^\bullet$ over $B^\bullet$, the related CDG-comodule over $\mathcal{C}^\bullet$ is $\mathcal{C}^\bullet \otimes_k^\tau M^\bullet$, while the related CDG-contramodule is $\mathrm{Hom}_k^\tau(\mathcal{C}^\bullet, M^\bullet)$. The tensor product and Hom are taken over k. The co/contraaction of $\mathcal{C}$ on this tensor product/Hom space is induced by the coaction of $\mathcal{C}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on M).

Givan a CDG-comodule $\mathcal{N}^\bullet$ over $\mathcal{C}^\bullet$, the related CDG-module over $B^\bullet$ is $B^\bullet \otimes_k^\tau \mathcal{N}^\bullet$. Given a CDG-contramodule $\mathfrak{Q}^\bullet$ over $\mathcal{C}^\bullet$, the related CDG-module over $B^\bullet$ is $\mathrm{Hom}_k^\tau(B^\bullet, \mathfrak{Q}^\bullet)$. The tensor product and Hom are taken over k. The action of B on this tensor product/Hom space is induced by the action of B on itself.

Koszul duality functors

The functors providing the "diagonal" category equivalences on the previous two slides are constructed as follows.

Given a CDG-module $M^\bullet$ over $B^\bullet$, the related CDG-comodule over $\mathcal{C}^\bullet$ is $\mathcal{C}^\bullet \otimes_k^\tau M^\bullet$, while the related CDG-contramodule is $\mathrm{Hom}_k^\tau(\mathcal{C}^\bullet, M^\bullet)$. The tensor product and Hom are taken over k. The co/contraaction of $\mathcal{C}$ on this tensor product/Hom space is induced by the coaction of $\mathcal{C}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on M).

Givan a CDG-comodule $\mathcal{N}^\bullet$ over $\mathcal{C}^\bullet$, the related CDG-module over $B^\bullet$ is $B^\bullet \otimes_k^\tau \mathcal{N}^\bullet$. Given a CDG-contramodule $\mathfrak{Q}^\bullet$ over $\mathcal{C}^\bullet$, the related CDG-module over $B^\bullet$ is $\mathrm{Hom}_k^\tau(B^\bullet, \mathfrak{Q}^\bullet)$. The tensor product and Hom are taken over k. The action of B on this tensor product/Hom space is induced by the action of B on itself. The symbol τ means a twisted differential

Koszul duality functors

The functors providing the "diagonal" category equivalences on the previous two slides are constructed as follows.

Given a CDG-module $M^\bullet$ over $B^\bullet$, the related CDG-comodule over $\mathcal{C}^\bullet$ is $\mathcal{C}^\bullet \otimes_k^\tau M^\bullet$, while the related CDG-contramodule is $\mathrm{Hom}_k^\tau(\mathcal{C}^\bullet, M^\bullet)$. The tensor product and Hom are taken over k. The co/contraaction of $\mathcal{C}$ on this tensor product/Hom space is induced by the coaction of $\mathcal{C}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on M).

Givan a CDG-comodule $\mathcal{N}^\bullet$ over $\mathcal{C}^\bullet$, the related CDG-module over $B^\bullet$ is $B^\bullet \otimes_k^\tau \mathcal{N}^\bullet$. Given a CDG-contramodule $\mathfrak{Q}^\bullet$ over $\mathcal{C}^\bullet$, the related CDG-module over $B^\bullet$ is $\mathrm{Hom}_k^\tau(B^\bullet, \mathfrak{Q}^\bullet)$. The tensor product and Hom are taken over k. The action of B on this tensor product/Hom space is induced by the action of B on itself. The symbol τ means a twisted differential (depending on the $\mathcal{C}$-comodule structure on $\mathcal{N}$ or the $\mathcal{C}$-contramodule structure on $\mathfrak{Q}$).

Incompatible invariance properties

Incompatible invariance properties

It is well-known that quasi-isomorphisms of DG-rings induce equivalences of the conventional derived categories of DG-modules.

Incompatible invariance properties

It is well-known that quasi-isomorphisms of DG-rings induce equivalences of the conventional derived categories of DG-modules.

On the other hand, change-of-connection isomorphisms of CDG-rings

Incompatible invariance properties

It is well-known that quasi-isomorphisms of DG-rings induce equivalences of the conventional derived categories of DG-modules.

On the other hand, change-of-connection isomorphisms of CDG-rings (in particular, such isomorphisms between DG-rings)

Incompatible invariance properties

It is well-known that quasi-isomorphisms of DG-rings induce equivalences of the conventional derived categories of DG-modules.

On the other hand, change-of-connection isomorphisms of CDG-rings (in particular, such isomorphisms between DG-rings) induce equivalences (in fact, isomorphisms) of the absolute derived, coderived, and contraderived categories of (C)DG-modules.

Incompatible invariance properties

It is well-known that quasi-isomorphisms of DG-rings induce equivalences of the conventional derived categories of DG-modules.

On the other hand, change-of-connection isomorphisms of CDG-rings (in particular, such isomorphisms between DG-rings) induce equivalences (in fact, isomorphisms) of the absolute derived, coderived, and contraderived categories of (C)DG-modules.

Let us explain that these two kinds of invariance properties are incompatible for DG-algebras over k.

Incompatible invariance properties

It is well-known that quasi-isomorphisms of DG-rings induce equivalences of the conventional derived categories of DG-modules.

On the other hand, change-of-connection isomorphisms of CDG-rings (in particular, such isomorphisms between DG-rings) induce equivalences (in fact, isomorphisms) of the absolute derived, coderived, and contraderived categories of (C)DG-modules.

Let us explain that these two kinds of invariance properties are incompatible for DG-algebras over k. In fact, any two DG-algebras can be connected by a chain of transformations

Incompatible invariance properties

It is well-known that quasi-isomorphisms of DG-rings induce equivalences of the conventional derived categories of DG-modules.

On the other hand, change-of-connection isomorphisms of CDG-rings (in particular, such isomorphisms between DG-rings) induce equivalences (in fact, isomorphisms) of the absolute derived, coderived, and contraderived categories of (C)DG-modules.

Let us explain that these two kinds of invariance properties are incompatible for DG-algebras over k. In fact, any two DG-algebras can be connected by a chain of transformations consisting of quasi-isomorphisms and change-of-connection isomorphisms.

Incompatible invariance properties

Let $A^\bullet$ and $B^\bullet$ be any two nonzero DG-algebras over k.

Incompatible invariance properties

Let $A^\bullet$ and $B^\bullet$ be any two nonzero DG-algebras over k. Put $\mathcal{C}^\blacklozenge = \mathrm{Bar}_{v'}(A^\bullet)$ and $\mathcal{D}^\blacklozenge = \mathrm{Bar}_{v''}(B^\bullet)$.

Incompatible invariance properties

Let $A^\bullet$ and $B^\bullet$ be any two nonzero DG-algebras over k. Put $\mathcal{C}^\blacklozenge = \mathrm{Bar}_{v'}(A^\bullet)$ and $\mathcal{D}^\blacklozenge = \mathrm{Bar}_{v''}(B^\bullet)$. So $\mathcal{C}^\blacklozenge$ and $\mathcal{D}^\blacklozenge$ are conilpotent CDG-coalgebras.

Incompatible invariance properties

Let $A^\bullet$ and $B^\bullet$ be any two nonzero DG-algebras over k. Put $\mathcal{C}^\bullet = \mathrm{Bar}_{v'}(A^\bullet)$ and $\mathcal{D}^\bullet = \mathrm{Bar}_{v''}(B^\bullet)$. So $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$ are conilpotent CDG-coalgebras. The direct sum $\mathcal{E}^\bullet = \mathcal{C}^\bullet \oplus \mathcal{D}^\bullet$ is a nonconilpotent CDG-coalgebra

Incompatible invariance properties

Let $A^\bullet$ and $B^\bullet$ be any two nonzero DG-algebras over k. Put $\mathcal{C}^\blacklozenge = \mathrm{Bar}_{v'}(A^\bullet)$ and $\mathcal{D}^\blacklozenge = \mathrm{Bar}_{v''}(B^\bullet)$. So $\mathcal{C}^\blacklozenge$ and $\mathcal{D}^\blacklozenge$ are conilpotent CDG-coalgebras. The direct sum $\mathcal{E}^\blacklozenge = \mathcal{C}^\blacklozenge \oplus \mathcal{D}^\blacklozenge$ is a nonconilpotent CDG-coalgebra with two natural coaugmentations γ and δ coming from the coaugmentations of $\mathcal{C}^\blacklozenge$ and $\mathcal{D}^\blacklozenge$.

Incompatible invariance properties

Let $A^\bullet$ and $B^\bullet$ be any two nonzero DG-algebras over k. Put $\mathcal{C}^\bullet = \mathrm{Bar}_{v'}(A^\bullet)$ and $\mathcal{D}^\bullet = \mathrm{Bar}_{v''}(B^\bullet)$. So $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$ are conilpotent CDG-coalgebras. The direct sum $\mathcal{E}^\bullet = \mathcal{C}^\bullet \oplus \mathcal{D}^\bullet$ is a nonconilpotent CDG-coalgebra with two natural coaugmentations γ and δ coming from the coaugmentations of $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$.

Specifically, γ and δ are the compositions $k \longrightarrow \mathcal{C}^\bullet \longrightarrow \mathcal{E}^\bullet$ and $k \longrightarrow \mathcal{D}^\bullet \longrightarrow \mathcal{E}^\bullet$

Incompatible invariance properties

Let $A^\bullet$ and $B^\bullet$ be any two nonzero DG-algebras over k. Put $\mathcal{C}^\blacklozenge = \mathrm{Bar}_{v'}(A^\bullet)$ and $\mathcal{D}^\blacklozenge = \mathrm{Bar}_{v''}(B^\bullet)$. So $\mathcal{C}^\blacklozenge$ and $\mathcal{D}^\blacklozenge$ are conilpotent CDG-coalgebras. The direct sum $\mathcal{E}^\blacklozenge = \mathcal{C}^\blacklozenge \oplus \mathcal{D}^\blacklozenge$ is a nonconilpotent CDG-coalgebra with two natural coaugmentations γ and δ coming from the coaugmentations of $\mathcal{C}^\blacklozenge$ and $\mathcal{D}^\blacklozenge$.

Specifically, γ and δ are the compositions $k \longrightarrow \mathcal{C}^\blacklozenge \longrightarrow \mathcal{E}^\blacklozenge$ and $k \longrightarrow \mathcal{D}^\blacklozenge \longrightarrow \mathcal{E}^\blacklozenge$, where $k \longrightarrow \mathcal{C}^\blacklozenge$ and $k \longrightarrow \mathcal{D}^\blacklozenge$ are the coaugmentations of $\mathcal{C}^\blacklozenge$ and $\mathcal{D}^\blacklozenge$.

Incompatible invariance properties

Let $A^\bullet$ and $B^\bullet$ be any two nonzero DG-algebras over k. Put $\mathcal{C}^\bullet = \mathrm{Bar}_{v'}(A^\bullet)$ and $\mathcal{D}^\bullet = \mathrm{Bar}_{v''}(B^\bullet)$. So $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$ are conilpotent CDG-coalgebras. The direct sum $\mathcal{E}^\bullet = \mathcal{C}^\bullet \oplus \mathcal{D}^\bullet$ is a nonconilpotent CDG-coalgebra with two natural coaugmentations γ and δ coming from the coaugmentations of $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$.

Specifically, γ and δ are the compositions $k \longrightarrow \mathcal{C}^\bullet \longrightarrow \mathcal{E}^\bullet$ and $k \longrightarrow \mathcal{D}^\bullet \longrightarrow \mathcal{E}^\bullet$, where $k \longrightarrow \mathcal{C}^\bullet$ and $k \longrightarrow \mathcal{D}^\bullet$ are the coaugmentations of $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$.

Now there are natural quasi-isomorphisms of DG-algebras $\mathrm{Cob}_\gamma(\mathcal{E}^\bullet) \longrightarrow A^\bullet$ and $\mathrm{Cob}_\delta(\mathcal{E}^\bullet) \longrightarrow B^\bullet$.

Incompatible invariance properties

Let $A^\bullet$ and $B^\bullet$ be any two nonzero DG-algebras over k. Put $\mathcal{C}^\bullet = \mathrm{Bar}_{v'}(A^\bullet)$ and $\mathcal{D}^\bullet = \mathrm{Bar}_{v''}(B^\bullet)$. So $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$ are conilpotent CDG-coalgebras. The direct sum $\mathcal{E}^\bullet = \mathcal{C}^\bullet \oplus \mathcal{D}^\bullet$ is a nonconilpotent CDG-coalgebra with two natural coaugmentations γ and δ coming from the coaugmentations of $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$.

Specifically, γ and δ are the compositions $k \longrightarrow \mathcal{C}^\bullet \longrightarrow \mathcal{E}^\bullet$ and $k \longrightarrow \mathcal{D}^\bullet \longrightarrow \mathcal{E}^\bullet$, where $k \longrightarrow \mathcal{C}^\bullet$ and $k \longrightarrow \mathcal{D}^\bullet$ are the coaugmentations of $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$.

Now there are natural quasi-isomorphisms of DG-algebras $\mathrm{Cob}_\gamma(\mathcal{E}^\bullet) \longrightarrow A^\bullet$ and $\mathrm{Cob}_\delta(\mathcal{E}^\bullet) \longrightarrow B^\bullet$.

At the same time, the two DG-algebras $\mathrm{Cob}_\gamma(\mathcal{E}^\bullet)$ and $\mathrm{Cob}_\delta(\mathcal{E}^\bullet)$ are change-of-connection isomorphic (as CDG-algebras).

Incompatible invariance properties

Let $A^\bullet$ and $B^\bullet$ be any two nonzero DG-algebras over k. Put $\mathcal{C}^\bullet = \mathrm{Bar}_{v'}(A^\bullet)$ and $\mathcal{D}^\bullet = \mathrm{Bar}_{v''}(B^\bullet)$. So $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$ are conilpotent CDG-coalgebras. The direct sum $\mathcal{E}^\bullet = \mathcal{C}^\bullet \oplus \mathcal{D}^\bullet$ is a nonconilpotent CDG-coalgebra with two natural coaugmentations γ and δ coming from the coaugmentations of $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$.

Specifically, γ and δ are the compositions $k \longrightarrow \mathcal{C}^\bullet \longrightarrow \mathcal{E}^\bullet$ and $k \longrightarrow \mathcal{D}^\bullet \longrightarrow \mathcal{E}^\bullet$, where $k \longrightarrow \mathcal{C}^\bullet$ and $k \longrightarrow \mathcal{D}^\bullet$ are the coaugmentations of $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$.

Now there are natural quasi-isomorphisms of DG-algebras $\mathrm{Cob}_\gamma(\mathcal{E}^\bullet) \longrightarrow A^\bullet$ and $\mathrm{Cob}_\delta(\mathcal{E}^\bullet) \longrightarrow B^\bullet$.

At the same time, the two DG-algebras $\mathrm{Cob}_\gamma(\mathcal{E}^\bullet)$ and $\mathrm{Cob}_\delta(\mathcal{E}^\bullet)$ are change-of-connection isomorphic (as CDG-algebras). Indeed, they are the cobar constructions of one and the same CDG-coalgebra $\mathcal{E}^\bullet$

Incompatible invariance properties

Let $A^\bullet$ and $B^\bullet$ be any two nonzero DG-algebras over k. Put $\mathcal{C}^\bullet = \mathrm{Bar}_{v'}(A^\bullet)$ and $\mathcal{D}^\bullet = \mathrm{Bar}_{v''}(B^\bullet)$. So $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$ are conilpotent CDG-coalgebras. The direct sum $\mathcal{E}^\bullet = \mathcal{C}^\bullet \oplus \mathcal{D}^\bullet$ is a nonconilpotent CDG-coalgebra with two natural coaugmentations γ and δ coming from the coaugmentations of $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$.

Specifically, γ and δ are the compositions $k \longrightarrow \mathcal{C}^\bullet \longrightarrow \mathcal{E}^\bullet$ and $k \longrightarrow \mathcal{D}^\bullet \longrightarrow \mathcal{E}^\bullet$, where $k \longrightarrow \mathcal{C}^\bullet$ and $k \longrightarrow \mathcal{D}^\bullet$ are the coaugmentations of $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$.

Now there are natural quasi-isomorphisms of DG-algebras $\mathrm{Cob}_\gamma(\mathcal{E}^\bullet) \longrightarrow A^\bullet$ and $\mathrm{Cob}_\delta(\mathcal{E}^\bullet) \longrightarrow B^\bullet$.

At the same time, the two DG-algebras $\mathrm{Cob}_\gamma(\mathcal{E}^\bullet)$ and $\mathrm{Cob}_\delta(\mathcal{E}^\bullet)$ are change-of-connection isomorphic (as CDG-algebras). Indeed, they are the cobar constructions of one and the same CDG-coalgebra $\mathcal{E}^\bullet$ with respect to two different sections γ and δ of the counit map $\epsilon\colon \mathcal{E} \longrightarrow k$.

Koszul filtrations

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k \cdot 1)$, which is pretty big.

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k\cdot 1)$, which is pretty big. Under suitable Koszulity assumptions, one can construct a smaller CDG-coalgebra $\mathcal{C}^\bullet$

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k\cdot 1)$, which is pretty big. Under suitable Koszulity assumptions, one can construct a smaller CDG-coalgebra $\mathcal{C}^\blacklozenge$ connected with $\mathrm{Bar}_v^\blacklozenge(A^\bullet)$ by a chain of filtered quasi-isomorphisms.

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k\cdot 1)$, which is pretty big. Under suitable Koszulity assumptions, one can construct a smaller CDG-coalgebra $\mathcal{C}^\blacklozenge$ connected with $\mathrm{Bar}_v^\blacklozenge(A^\bullet)$ by a chain of filtered quasi-isomorphisms. Let me elaborate on this approach in a context both more and less general than the above one.

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k \cdot 1)$, which is pretty big. Under suitable Koszulity assumptions, one can construct a smaller CDG-coalgebra $\mathcal{C}^\bullet$ connected with $\mathrm{Bar}_v^\bullet(A^\bullet)$ by a chain of filtered quasi-isomorphisms. Let me elaborate on this approach in a context both more and less general than the above one.

Let $A = k \oplus A_1 \oplus A_2 \oplus A_3 \oplus \cdots$ be a positively graded associative algebra.

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k\cdot 1)$, which is pretty big. Under suitable Koszulity assumptions, one can construct a smaller CDG-coalgebra $\mathcal{C}^\bullet$ connected with $\mathrm{Bar}_v^\bullet(A^\bullet)$ by a chain of filtered quasi-isomorphisms. Let me elaborate on this approach in a context both more and less general than the above one.

Let $A = k \oplus A_1 \oplus A_2 \oplus A_3 \oplus \cdots$ be a positively graded associative algebra. ("Positively graded" means nonnegatively graded with $A_0 = k$.)

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k \cdot 1)$, which is pretty big. Under suitable Koszulity assumptions, one can construct a smaller CDG-coalgebra $\mathcal{C}^\bullet$ connected with $\mathrm{Bar}_v^\bullet(A^\bullet)$ by a chain of filtered quasi-isomorphisms. Let me elaborate on this approach in a context both more and less general than the above one.

Let $A = k \oplus A_1 \oplus A_2 \oplus A_3 \oplus \cdots$ be a positively graded associative algebra. ("Positively graded" means nonnegatively graded with $A_0 = k$.) The algebra A is called Koszul

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k \cdot 1)$, which is pretty big. Under suitable Koszulity assumptions, one can construct a smaller CDG-coalgebra $\mathcal{C}^\blacklozenge$ connected with $\mathrm{Bar}_v^\blacklozenge(A^\bullet)$ by a chain of filtered quasi-isomorphisms. Let me elaborate on this approach in a context both more and less general than the above one.

Let $A = k \oplus A_1 \oplus A_2 \oplus A_3 \oplus \cdots$ be a positively graded associative algebra. ("Positively graded" means nonnegatively graded with $A_0 = k$.) The algebra A is called Koszul if $\mathrm{Tor}^A_{ij}(k,k) = 0$ for all $i \neq j$.

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k\cdot 1)$, which is pretty big. Under suitable Koszulity assumptions, one can construct a smaller CDG-coalgebra $\mathcal{C}^\bullet$ connected with $\mathrm{Bar}^\bullet_v(A^\bullet)$ by a chain of filtered quasi-isomorphisms. Let me elaborate on this approach in a context both more and less general than the above one.

Let $A = k\oplus A_1\oplus A_2\oplus A_3\oplus\cdots$ be a positively graded associative algebra. ("Positively graded" means nonnegatively graded with $A_0 = k$.) The algebra A is called Koszul if $\mathrm{Tor}^A_{ij}(k,k) = 0$ for all $i\neq j$. Here the first grading i is the usual homological grading on the Tor

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k \cdot 1)$, which is pretty big. Under suitable Koszulity assumptions, one can construct a smaller CDG-coalgebra $\mathcal{C}^\blacklozenge$ connected with $\mathrm{Bar}_v^\blacklozenge(A^\bullet)$ by a chain of filtered quasi-isomorphisms. Let me elaborate on this approach in a context both more and less general than the above one.

Let $A = k \oplus A_1 \oplus A_2 \oplus A_3 \oplus \cdots$ be a positively graded associative algebra. ("Positively graded" means nonnegatively graded with $A_0 = k$.) The algebra A is called Koszul if $\mathrm{Tor}^A_{ij}(k,k) = 0$ for all $i \neq j$. Here the first grading i is the usual homological grading on the Tor, while the second grading j is the internal grading induced by the grading of A.

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k \cdot 1)$, which is pretty big. Under suitable Koszulity assumptions, one can construct a smaller CDG-coalgebra $\mathcal{C}^\bullet$ connected with $\mathrm{Bar}_v^\bullet(A^\bullet)$ by a chain of filtered quasi-isomorphisms. Let me elaborate on this approach in a context both more and less general than the above one.

Let $A = k \oplus A_1 \oplus A_2 \oplus A_3 \oplus \cdots$ be a positively graded associative algebra. ("Positively graded" means nonnegatively graded with $A_0 = k$.) The algebra A is called Koszul if $\mathrm{Tor}_{ij}^A(k,k) = 0$ for all $i \neq j$. Here the first grading i is the usual homological grading on the Tor, while the second grading j is the internal grading induced by the grading of A.

In particular, all Koszul algebras are quadratic, i. e., generated by A_1 with relations in degree 2.

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k\cdot 1)$, which is pretty big. Under suitable Koszulity assumptions, one can construct a smaller CDG-coalgebra $\mathcal{C}^\bullet$ connected with $\mathrm{Bar}_v^\bullet(A^\bullet)$ by a chain of filtered quasi-isomorphisms. Let me elaborate on this approach in a context both more and less general than the above one.

Let $A = k \oplus A_1 \oplus A_2 \oplus A_3 \oplus \cdots$ be a positively graded associative algebra. ("Positively graded" means nonnegatively graded with $A_0 = k$.) The algebra A is called Koszul if $\mathrm{Tor}^A_{ij}(k,k) = 0$ for all $i \neq j$. Here the first grading i is the usual homological grading on the Tor, while the second grading j is the internal grading induced by the grading of A.

In particular, all Koszul algebras are quadratic, i. e., generated by A_1 with relations in degree 2. A quadratic algebra has the form $A = T(V)/(I)$

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k\cdot 1)$, which is pretty big. Under suitable Koszulity assumptions, one can construct a smaller CDG-coalgebra $\mathcal{C}^\bullet$ connected with $\mathrm{Bar}_v^\bullet(A^\bullet)$ by a chain of filtered quasi-isomorphisms. Let me elaborate on this approach in a context both more and less general than the above one.

Let $A = k \oplus A_1 \oplus A_2 \oplus A_3 \oplus \cdots$ be a positively graded associative algebra. ("Positively graded" means nonnegatively graded with $A_0 = k$.) The algebra A is called Koszul if $\mathrm{Tor}^A_{ij}(k,k) = 0$ for all $i \neq j$. Here the first grading i is the usual homological grading on the Tor, while the second grading j is the internal grading induced by the grading of A.

In particular, all Koszul algebras are quadratic, i. e., generated by A_1 with relations in degree 2. A quadratic algebra has the form $A = T(V)/(I)$, where V is a vector space, $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ is the tensor algebra of V

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k\cdot 1)$, which is pretty big. Under suitable Koszulity assumptions, one can construct a smaller CDG-coalgebra $\mathcal{C}^\blacklozenge$ connected with $\mathrm{Bar}_v^\blacklozenge(A^\bullet)$ by a chain of filtered quasi-isomorphisms. Let me elaborate on this approach in a context both more and less general than the above one.

Let $A = k \oplus A_1 \oplus A_2 \oplus A_3 \oplus \cdots$ be a positively graded associative algebra. ("Positively graded" means nonnegatively graded with $A_0 = k$.) The algebra A is called Koszul if $\mathrm{Tor}^A_{ij}(k,k) = 0$ for all $i \neq j$. Here the first grading i is the usual homological grading on the Tor, while the second grading j is the internal grading induced by the grading of A.

In particular, all Koszul algebras are quadratic, i. e., generated by A_1 with relations in degree 2. A quadratic algebra has the form $A = T(V)/(I)$, where V is a vector space, $T(V) = \bigoplus_{n=0}^\infty V^{\otimes n}$ is the tensor algebra of V, while $I \subset V \otimes V$ is the space of relations of degree 2

Koszul filtrations

The bar construction of a DG-algebra $A^\bullet$ is the tensor coalgebra of $A^\bullet/(k \cdot 1)$, which is pretty big. Under suitable Koszulity assumptions, one can construct a smaller CDG-coalgebra $\mathcal{C}^\bullet$ connected with $\mathrm{Bar}_v^\bullet(A^\bullet)$ by a chain of filtered quasi-isomorphisms. Let me elaborate on this approach in a context both more and less general than the above one.

Let $A = k \oplus A_1 \oplus A_2 \oplus A_3 \oplus \cdots$ be a positively graded associative algebra. ("Positively graded" means nonnegatively graded with $A_0 = k$.) The algebra A is called Koszul if $\mathrm{Tor}^A_{ij}(k, k) = 0$ for all $i \neq j$. Here the first grading i is the usual homological grading on the Tor, while the second grading j is the internal grading induced by the grading of A.

In particular, all Koszul algebras are quadratic, i. e., generated by A_1 with relations in degree 2. A quadratic algebra has the form $A = T(V)/(I)$, where V is a vector space, $T(V) = \bigoplus_{n=0}^\infty V^{\otimes n}$ is the tensor algebra of V, while $I \subset V \otimes V$ is the space of relations of degree 2 and (I) is the ideal generated by I in $T(V)$.

Quadratic dual coalgebra

Quadratic dual coalgebra

Given a quadratic algebra A, one constructs the quadratic dual graded coalgebra $\mathcal{C} = A^!$ as follows.

Quadratic dual coalgebra

Given a quadratic algebra A, one constructs the quadratic dual graded coalgebra $\mathcal{C} = A^!$ as follows. Suppose $A = T(V)/(I)$, so

- $A_0 = k, \; A_1 = V, \; A_2 = (V \otimes V)/I,$

Quadratic dual coalgebra

Given a quadratic algebra A, one constructs the quadratic dual graded coalgebra $\mathcal{C} = A^!$ as follows. Suppose $A = T(V)/(I)$, so

- $A_0 = k, \ \ A_1 = V, \ \ A_2 = (V \otimes V)/I,$
- $A_3 = (V \otimes V \otimes V)/(I \otimes V + V \otimes I),$

Quadratic dual coalgebra

Given a quadratic algebra A, one constructs the quadratic dual graded coalgebra $\mathcal{C} = A^!$ as follows. Suppose $A = T(V)/(I)$, so

- $A_0 = k, \ \ A_1 = V, \ \ A_2 = (V \otimes V)/I,$
- $A_3 = (V \otimes V \otimes V)/(I \otimes V + V \otimes I),$
- $A_n = V^{\otimes n}/\sum_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1}, \ \ n \geqslant 2.$

Quadratic dual coalgebra

Given a quadratic algebra A, one constructs the quadratic dual graded coalgebra $\mathcal{C} = A^!$ as follows. Suppose $A = T(V)/(I)$, so

- $A_0 = k, \; A_1 = V, \; A_2 = (V \otimes V)/I,$
- $A_3 = (V \otimes V \otimes V)/(I \otimes V + V \otimes I),$
- $A_n = V^{\otimes n} / \sum_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1}, \; n \geqslant 2.$

Put

- $\mathcal{C}_0 = k, \; \mathcal{C}_1 = V, \; \mathcal{C}_2 = I,$

Quadratic dual coalgebra

Given a quadratic algebra A, one constructs the quadratic dual graded coalgebra $\mathcal{C} = A^!$ as follows. Suppose $A = T(V)/(I)$, so

- $A_0 = k$, $A_1 = V$, $A_2 = (V \otimes V)/I$,
- $A_3 = (V \otimes V \otimes V)/(I \otimes V + V \otimes I)$,
- $A_n = V^{\otimes n} / \sum_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1}$, $n \geqslant 2$.

Put

- $\mathcal{C}_0 = k$, $\mathcal{C}_1 = V$, $\mathcal{C}_2 = I$,
- $\mathcal{C}_3 = (I \otimes V) \cap (V \otimes I) \subset V^{\otimes 3}$,

Quadratic dual coalgebra

Given a quadratic algebra A, one constructs the quadratic dual graded coalgebra $\mathcal{C} = A^!$ as follows. Suppose $A = T(V)/(I)$, so

- $A_0 = k,\ \ A_1 = V,\ \ A_2 = (V \otimes V)/I,$
- $A_3 = (V \otimes V \otimes V)/(I \otimes V + V \otimes I),$
- $A_n = V^{\otimes n}/\sum_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1},\ \ n \geqslant 2.$

Put

- $\mathcal{C}_0 = k,\ \ \mathcal{C}_1 = V,\ \ \mathcal{C}_2 = I,$
- $\mathcal{C}_3 = (I \otimes V) \cap (V \otimes I) \subset V^{\otimes 3},$
- $\mathcal{C}_n = \bigcap_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1} \subset V^{\otimes n},\ \ n \geqslant 2.$

Quadratic dual coalgebra

Given a quadratic algebra A, one constructs the quadratic dual graded coalgebra $\mathcal{C} = A^!$ as follows. Suppose $A = T(V)/(I)$, so

- $A_0 = k,\ \ A_1 = V,\ \ A_2 = (V \otimes V)/I,$
- $A_3 = (V \otimes V \otimes V)/(I \otimes V + V \otimes I),$
- $A_n = V^{\otimes n} / \sum_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1},\ \ n \geqslant 2.$

Put

- $\mathcal{C}_0 = k,\ \ \mathcal{C}_1 = V,\ \ \mathcal{C}_2 = I,$
- $\mathcal{C}_3 = (I \otimes V) \cap (V \otimes I) \subset V^{\otimes 3},$
- $\mathcal{C}_n = \bigcap_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1} \subset V^{\otimes n},\ \ n \geqslant 2.$

So $\mathcal{C}$ is a subcoalgebra of the tensor coalgebra $\bigoplus_{n=0}^{\infty} V^{\otimes n}$.

Quadratic dual coalgebra

Given a quadratic algebra A, one constructs the quadratic dual graded coalgebra $\mathcal{C} = A^!$ as follows. Suppose $A = T(V)/(I)$, so

- $A_0 = k, \;\; A_1 = V, \;\; A_2 = (V \otimes V)/I,$
- $A_3 = (V \otimes V \otimes V)/(I \otimes V + V \otimes I),$
- $A_n = V^{\otimes n} / \sum_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1}, \;\; n \geqslant 2.$

Put

- $\mathcal{C}_0 = k, \;\; \mathcal{C}_1 = V, \;\; \mathcal{C}_2 = I,$
- $\mathcal{C}_3 = (I \otimes V) \cap (V \otimes I) \subset V^{\otimes 3},$
- $\mathcal{C}_n = \bigcap_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1} \subset V^{\otimes n}, \;\; n \geqslant 2.$

So $\mathcal{C}$ is a subcoalgebra of the tensor coalgebra $\bigoplus_{n=0}^{\infty} V^{\otimes n}$.

A positively graded coalgebra $\mathcal{C} = k \oplus \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \cdots$ is called Koszul if $\mathrm{Ext}_{\mathcal{C}}^{ij}(k, k) = 0$ for $i \neq j$

Quadratic dual coalgebra

Given a quadratic algebra A, one constructs the quadratic dual graded coalgebra $\mathcal{C} = A^!$ as follows. Suppose $A = T(V)/(I)$, so

- $A_0 = k$, $A_1 = V$, $A_2 = (V \otimes V)/I$,
- $A_3 = (V \otimes V \otimes V)/(I \otimes V + V \otimes I)$,
- $A_n = V^{\otimes n}/\sum_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1}$, $n \geqslant 2$.

Put

- $\mathcal{C}_0 = k$, $\mathcal{C}_1 = V$, $\mathcal{C}_2 = I$,
- $\mathcal{C}_3 = (I \otimes V) \cap (V \otimes I) \subset V^{\otimes 3}$,
- $\mathcal{C}_n = \bigcap_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1} \subset V^{\otimes n}$, $n \geqslant 2$.

So $\mathcal{C}$ is a subcoalgebra of the tensor coalgebra $\bigoplus_{n=0}^{\infty} V^{\otimes n}$.

A positively graded coalgebra $\mathcal{C} = k \oplus \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \cdots$ is called Koszul if $\operatorname{Ext}_{\mathcal{C}}^{ij}(k, k) = 0$ for $i \neq j$ (Ext in the category of left $\mathcal{C}$-comodules, or equivalently, right $\mathcal{C}$-comodules).

Quadratic dual coalgebra

Given a quadratic algebra A, one constructs the quadratic dual graded coalgebra $\mathcal{C} = A^!$ as follows. Suppose $A = T(V)/(I)$, so

- $A_0 = k, \ \ A_1 = V, \ \ A_2 = (V \otimes V)/I$,
- $A_3 = (V \otimes V \otimes V)/(I \otimes V + V \otimes I)$,
- $A_n = V^{\otimes n}/\sum_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1}, \ \ n \geqslant 2$.

Put

- $\mathcal{C}_0 = k, \ \ \mathcal{C}_1 = V, \ \ \mathcal{C}_2 = I$,
- $\mathcal{C}_3 = (I \otimes V) \cap (V \otimes I) \subset V^{\otimes 3}$,
- $\mathcal{C}_n = \bigcap_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1} \subset V^{\otimes n}, \ \ n \geqslant 2$.

So $\mathcal{C}$ is a subcoalgebra of the tensor coalgebra $\bigoplus_{n=0}^{\infty} V^{\otimes n}$.

A positively graded coalgebra $\mathcal{C} = k \oplus \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \cdots$ is called Koszul if $\operatorname{Ext}_{\mathcal{C}}^{ij}(k,k) = 0$ for $i \neq j$ (Ext in the category of left $\mathcal{C}$-comodules, or equivalently, right $\mathcal{C}$-comodules). A quadratic algebra A is Koszul if and only if its quadratic dual coalgebra $\mathcal{C}$ is Koszul.

Quadratic dual coalgebra

Given a quadratic algebra A, one constructs the quadratic dual graded coalgebra $\mathcal{C} = A^!$ as follows. Suppose $A = T(V)/(I)$, so

- $A_0 = k, \ A_1 = V, \ A_2 = (V \otimes V)/I,$
- $A_3 = (V \otimes V \otimes V)/(I \otimes V + V \otimes I),$
- $A_n = V^{\otimes n}/\sum_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1}, \ n \geqslant 2.$

Put

- $\mathcal{C}_0 = k, \ \mathcal{C}_1 = V, \ \mathcal{C}_2 = I,$
- $\mathcal{C}_3 = (I \otimes V) \cap (V \otimes I) \subset V^{\otimes 3},$
- $\mathcal{C}_n = \bigcap_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1} \subset V^{\otimes n}, \ n \geqslant 2.$

So $\mathcal{C}$ is a subcoalgebra of the tensor coalgebra $\bigoplus_{n=0}^{\infty} V^{\otimes n}$.

A positively graded coalgebra $\mathcal{C} = k \oplus \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \cdots$ is called Koszul if $\mathrm{Ext}_{\mathcal{C}}^{ij}(k,k) = 0$ for $i \neq j$ (Ext in the category of left $\mathcal{C}$-comodules, or equivalently, right $\mathcal{C}$-comodules). A quadratic algebra A is Koszul if and only if its quadratic dual coalgebra $\mathcal{C}$ is Koszul.

Furthermore, put $\mathcal{C}^i = \mathcal{C}_{-i}$.

Quadratic dual coalgebra

Given a quadratic algebra A, one constructs the quadratic dual graded coalgebra $\mathcal{C} = A^!$ as follows. Suppose $A = T(V)/(I)$, so

- $A_0 = k, \ \ A_1 = V, \ \ A_2 = (V \otimes V)/I$,
- $A_3 = (V \otimes V \otimes V)/(I \otimes V + V \otimes I)$,
- $A_n = V^{\otimes n}/\sum_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1}, \ \ n \geqslant 2$.

Put

- $\mathcal{C}_0 = k, \ \ \mathcal{C}_1 = V, \ \ \mathcal{C}_2 = I$,
- $\mathcal{C}_3 = (I \otimes V) \cap (V \otimes I) \subset V^{\otimes 3}$,
- $\mathcal{C}_n = \bigcap_{k=1}^{n-1} V^{\otimes k-1} \otimes I \otimes V^{\otimes n-k-1} \subset V^{\otimes n}, \ \ n \geqslant 2$.

So $\mathcal{C}$ is a subcoalgebra of the tensor coalgebra $\bigoplus_{n=0}^{\infty} V^{\otimes n}$.

A positively graded coalgebra $\mathcal{C} = k \oplus \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \cdots$ is called Koszul if $\operatorname{Ext}_{\mathcal{C}}^{ij}(k, k) = 0$ for $i \neq j$ (Ext in the category of left $\mathcal{C}$-comodules, or equivalently, right $\mathcal{C}$-comodules). A quadratic algebra A is Koszul if and only if its quadratic dual coalgebra $\mathcal{C}$ is Koszul.

Furthermore, put $\mathcal{C}^i = \mathcal{C}_{-i}$. So $\mathcal{C} = \bigoplus_{n \geqslant 0} \mathcal{C}_n = \bigoplus_{i \leqslant 0} \mathcal{C}^i$.

Koszul filtrations

Koszul filtrations

A nonhomogeneous Koszul algebra $\widetilde{A}$ over a field k

Koszul filtrations

A nonhomogeneous Koszul algebra $\widetilde{A}$ over a field k is a filtered k-algebra $F_0\widetilde{A} \subset F_1\widetilde{A} \subset F_2\widetilde{A} \subset \cdots \subset \widetilde{A}$

Koszul filtrations

A nonhomogeneous Koszul algebra $\widetilde{A}$ over a field k is a filtered k-algebra $F_0\widetilde{A} \subset F_1\widetilde{A} \subset F_2\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all i, $j \geqslant 0$;

Koszul filtrations

A nonhomogeneous Koszul algebra $\widetilde{A}$ over a field k is a filtered k-algebra $F_0\widetilde{A} \subset F_1\widetilde{A} \subset F_2\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $F_0\widetilde{A} = k \cdot 1 = k$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;

Koszul filtrations

A nonhomogeneous Koszul algebra $\widetilde{A}$ over a field k is a filtered k-algebra $F_0\widetilde{A} \subset F_1\widetilde{A} \subset F_2\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $F_0\widetilde{A} = k \cdot 1 = k$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;
- the associated graded algebra $A = \operatorname{gr}^F\widetilde{A} = \bigoplus_{n \geqslant 0} F_n\widetilde{A}/F_{n-1}\widetilde{A}$ is Koszul.

Koszul filtrations

A nonhomogeneous Koszul algebra $\widetilde{A}$ over a field k is a filtered k-algebra $F_0\widetilde{A} \subset F_1\widetilde{A} \subset F_2\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $F_0\widetilde{A} = k \cdot 1 = k$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;
- the associated graded algebra $A = \mathrm{gr}^F\widetilde{A} = \bigoplus_{n\geqslant 0} F_n\widetilde{A}/F_{n-1}\widetilde{A}$ is Koszul.

Theorem (Poincaré–Birkhoff–Witt theorem)

Koszul filtrations

A nonhomogeneous Koszul algebra $\widetilde{A}$ over a field k is a filtered k-algebra $F_0\widetilde{A} \subset F_1\widetilde{A} \subset F_2\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $F_0\widetilde{A} = k \cdot 1 = k$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;
- the associated graded algebra $A = \mathrm{gr}^F \widetilde{A} = \bigoplus_{n\geqslant 0} F_n\widetilde{A}/F_{n-1}\widetilde{A}$ is Koszul.

Theorem (Poincaré–Birkhoff–Witt theorem)

There is a natural equivalence of categories between the category of nonhomogeneous Koszul algebras $(\widetilde{A}, F)$ over k

Koszul filtrations

A nonhomogeneous Koszul algebra $\widetilde{A}$ over a field k is a filtered k-algebra $F_0\widetilde{A} \subset F_1\widetilde{A} \subset F_2\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $F_0\widetilde{A} = k \cdot 1 = k$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;
- the associated graded algebra $A = \mathrm{gr}^F\widetilde{A} = \bigoplus_{n\geqslant 0} F_n\widetilde{A}/F_{n-1}\widetilde{A}$ is Koszul.

Theorem (Poincaré–Birkhoff–Witt theorem)

There is a natural equivalence of categories between the category of nonhomogeneous Koszul algebras $(\widetilde{A}, F)$ over k and the category of CDG-coalgebras $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C} = \bigoplus_{i\leqslant 0} \mathcal{C}^i$.

Koszul filtrations

A nonhomogeneous Koszul algebra $\widetilde{A}$ over a field k is a filtered k-algebra $F_0\widetilde{A} \subset F_1\widetilde{A} \subset F_2\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $F_0\widetilde{A} = k \cdot 1 = k$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;
- the associated graded algebra $A = \mathrm{gr}^F\widetilde{A} = \bigoplus_{n\geqslant 0} F_n\widetilde{A}/F_{n-1}\widetilde{A}$ is Koszul.

Theorem (Poincaré–Birkhoff–Witt theorem)

There is a natural equivalence of categories between the category of nonhomogeneous Koszul algebras $(\widetilde{A}, F)$ over k and the category of CDG-coalgebras $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C} = \bigoplus_{i\leqslant 0} \mathcal{C}^i$. Nonhomogeneous Koszul algebras $(\widetilde{A}, F)$ with $\mathrm{gr}^F\widetilde{A} = A$

Koszul filtrations

A nonhomogeneous Koszul algebra $\widetilde{A}$ over a field k is a filtered k-algebra $F_0\widetilde{A} \subset F_1\widetilde{A} \subset F_2\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $F_0\widetilde{A} = k \cdot 1 = k$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;
- the associated graded algebra $A = \operatorname{gr}^F\widetilde{A} = \bigoplus_{n\geqslant 0} F_n\widetilde{A}/F_{n-1}\widetilde{A}$ is Koszul.

Theorem (Poincaré–Birkhoff–Witt theorem)

There is a natural equivalence of categories between the category of nonhomogeneous Koszul algebras $(\widetilde{A}, F)$ over k and the category of CDG-coalgebras $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C} = \bigoplus_{i\leqslant 0} \mathcal{C}^i$. Nonhomogeneous Koszul algebras $(\widetilde{A}, F)$ with $\operatorname{gr}^F\widetilde{A} = A$ correspond to CDG-coalgebra structures on $\mathcal{C} = A^!$ under this equivalence.

Koszul filtrations

A nonhomogeneous Koszul algebra $\widetilde{A}$ over a field k is a filtered k-algebra $F_0\widetilde{A} \subset F_1\widetilde{A} \subset F_2\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $F_0\widetilde{A} = k \cdot 1 = k$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;
- the associated graded algebra $A = \mathrm{gr}^F\widetilde{A} = \bigoplus_{n\geqslant 0} F_n\widetilde{A}/F_{n-1}\widetilde{A}$ is Koszul.

Theorem (Poincaré–Birkhoff–Witt theorem)

There is a natural equivalence of categories between the category of nonhomogeneous Koszul algebras $(\widetilde{A}, F)$ over k and the category of CDG-coalgebras $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C} = \bigoplus_{i\leqslant 0} \mathcal{C}^i$. Nonhomogeneous Koszul algebras $(\widetilde{A}, F)$ with $\mathrm{gr}^F\widetilde{A} = A$ correspond to CDG-coalgebra structures on $\mathcal{C} = A^!$ under this equivalence. There is a natural quasi-isomorphism of DG-algebras $\mathrm{Cob}_\gamma(\mathcal{C}^\bullet) \longrightarrow \widetilde{A}$.

Koszul filtrations

A nonhomogeneous Koszul algebra $\widetilde{A}$ over a field k is a filtered k-algebra $F_0\widetilde{A} \subset F_1\widetilde{A} \subset F_2\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $F_0\widetilde{A} = k \cdot 1 = k$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;
- the associated graded algebra $A = \mathrm{gr}^F\widetilde{A} = \bigoplus_{n \geqslant 0} F_n\widetilde{A}/F_{n-1}\widetilde{A}$ is Koszul.

Theorem (Poincaré–Birkhoff–Witt theorem)

There is a natural equivalence of categories between the category of nonhomogeneous Koszul algebras $(\widetilde{A}, F)$ over k and the category of CDG-coalgebras $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C} = \bigoplus_{i \leqslant 0} \mathcal{C}^i$. Nonhomogeneous Koszul algebras $(\widetilde{A}, F)$ with $\mathrm{gr}^F\widetilde{A} = A$ correspond to CDG-coalgebra structures on $\mathcal{C} = A^!$ under this equivalence. There is a natural quasi-isomorphism of DG-algebras $\mathrm{Cob}_\gamma(\mathcal{C}^\bullet) \longrightarrow \widetilde{A}$.

So one can use $\mathcal{C}^\bullet$ as the Koszul dual CDG-coalgebra to $\widetilde{A}$ in the conilpotent Koszul triality theorem above.

Koszul filtrations

Koszul filtrations

The Poincaré–Birkhoff–Witt theorem stated above is called this way

Koszul filtrations

The Poincaré–Birkhoff–Witt theorem stated above is called this way because it specializes to the classical Poincaré–Birkhoff–Witt theorem for Lie algebras

Koszul filtrations

The Poincaré–Birkhoff–Witt theorem stated above is called this way because it specializes to the classical Poincaré–Birkhoff–Witt theorem for Lie algebras when $A = \mathrm{Sym}(V)$ is the symmetric algebra and $\mathcal{C} = \bigwedge(V)$ is the exterior coalgebra of a vector space V.

Koszul filtrations

The Poincaré–Birkhoff–Witt theorem stated above is called this way because it specializes to the classical Poincaré–Birkhoff–Witt theorem for Lie algebras when $A = \mathrm{Sym}(V)$ is the symmetric algebra and $\mathcal{C} = \bigwedge(V)$ is the exterior coalgebra of a vector space V.

Given a Lie algebra structure on V

Koszul filtrations

The Poincaré–Birkhoff–Witt theorem stated above is called this way because it specializes to the classical Poincaré–Birkhoff–Witt theorem for Lie algebras when $A = \mathrm{Sym}(V)$ is the symmetric algebra and $\mathcal{C} = \bigwedge(V)$ is the exterior coalgebra of a vector space V.

Given a Lie algebra structure on V, it is easy to construct the related DG-coalgebra structure on $\bigwedge(V)$

Koszul filtrations

The Poincaré–Birkhoff–Witt theorem stated above is called this way because it specializes to the classical Poincaré–Birkhoff–Witt theorem for Lie algebras when $A = \mathrm{Sym}(V)$ is the symmetric algebra and $\mathcal{C} = \bigwedge(V)$ is the exterior coalgebra of a vector space V.

Given a Lie algebra structure on V, it is easy to construct the related DG-coalgebra structure on $\bigwedge(V)$ (called the standard homological complex of the Lie algebra V).

The Poincaré–Birkhoff–Witt theorem stated above is called this way because it specializes to the classical Poincaré–Birkhoff–Witt theorem for Lie algebras when $A = \mathrm{Sym}(V)$ is the symmetric algebra and $\mathcal{C} = \bigwedge(V)$ is the exterior coalgebra of a vector space V.

Given a Lie algebra structure on V, it is easy to construct the related DG-coalgebra structure on $\bigwedge(V)$ (called the standard homological complex of the Lie algebra V). Then the theorem claims that the corresponding filtered algebra $\widetilde{A}$ (the enveloping algebra of V) is of the correct size.

The Poincaré–Birkhoff–Witt theorem stated above is called this way because it specializes to the classical Poincaré–Birkhoff–Witt theorem for Lie algebras when $A = \mathrm{Sym}(V)$ is the symmetric algebra and $\mathcal{C} = \bigwedge(V)$ is the exterior coalgebra of a vector space V.

Given a Lie algebra structure on V, it is easy to construct the related DG-coalgebra structure on $\bigwedge(V)$ (called the standard homological complex of the Lie algebra V). Then the theorem claims that the corresponding filtered algebra $\widetilde{A}$ (the enveloping algebra of V) is of the correct size.

The standard approach to constructing the equivalence of categories in the theorem

Koszul filtrations

The Poincaré–Birkhoff–Witt theorem stated above is called this way because it specializes to the classical Poincaré–Birkhoff–Witt theorem for Lie algebras when $A = \mathrm{Sym}(V)$ is the symmetric algebra and $\mathcal{C} = \bigwedge(V)$ is the exterior coalgebra of a vector space V.

Given a Lie algebra structure on V, it is easy to construct the related DG-coalgebra structure on $\bigwedge(V)$ (called the standard homological complex of the Lie algebra V). Then the theorem claims that the corresponding filtered algebra $\widetilde{A}$ (the enveloping algebra of V) is of the correct size.

The standard approach to constructing the equivalence of categories in the theorem is to write down explicit formulas connecting the linear and scalar components of the nonhomogeous quadratic relations defining $\widetilde{A}$

Koszul filtrations

The Poincaré–Birkhoff–Witt theorem stated above is called this way because it specializes to the classical Poincaré–Birkhoff–Witt theorem for Lie algebras when $A = \mathrm{Sym}(V)$ is the symmetric algebra and $\mathcal{C} = \bigwedge(V)$ is the exterior coalgebra of a vector space V.

Given a Lie algebra structure on V, it is easy to construct the related DG-coalgebra structure on $\bigwedge(V)$ (called the standard homological complex of the Lie algebra V). Then the theorem claims that the corresponding filtered algebra $\widetilde{A}$ (the enveloping algebra of V) is of the correct size.

The standard approach to constructing the equivalence of categories in the theorem is to write down explicit formulas connecting the linear and scalar components of the nonhomogeneous quadratic relations defining $\widetilde{A}$ with the differential d and the curvature linear function h on $\mathcal{C}$.

Koszul filtrations

The Poincaré–Birkhoff–Witt theorem stated above is called this way because it specializes to the classical Poincaré–Birkhoff–Witt theorem for Lie algebras when $A = \mathrm{Sym}(V)$ is the symmetric algebra and $\mathcal{C} = \bigwedge(V)$ is the exterior coalgebra of a vector space V.

Given a Lie algebra structure on V, it is easy to construct the related DG-coalgebra structure on $\bigwedge(V)$ (called the standard homological complex of the Lie algebra V). Then the theorem claims that the corresponding filtered algebra $\widetilde{A}$ (the enveloping algebra of V) is of the correct size.

The standard approach to constructing the equivalence of categories in the theorem is to write down explicit formulas connecting the linear and scalar components of the nonhomogeous quadratic relations defining $\widetilde{A}$ with the differential d and the curvature linear function h on $\mathcal{C}$. A more high-tech approach uses the hat construction for coalgebras.

Central element theorem

Central element theorem

Theorem (Central element theorem)

Let L be a positively graded algebra over k and $t \in L_1$ be a central element.

Central element theorem

Theorem (Central element theorem)

Let L be a positively graded algebra over k and $t \in L_1$ be a central element. Assume that t is a non-zero-divisor in L, and let $(t) \subset L$ be the principal ideal spanned by t.

Central element theorem

Theorem (Central element theorem)

Let L be a positively graded algebra over k and $t \in L_1$ be a central element. Assume that t is a non-zero-divisor in L, and let $(t) \subset L$ be the principal ideal spanned by t. Then the graded algebra L is Koszul if and only if the graded algebra $L/(t)$ is Koszul.

Central element theorem

Theorem (Central element theorem)

Let L be a positively graded algebra over k and $t \in L_1$ be a central element. Assume that t is a non-zero-divisor in L, and let $(t) \subset L$ be the principal ideal spanned by t. Then the graded algebra L is Koszul if and only if the graded algebra $L/(t)$ is Koszul.

Lemma (Central element and acyclic coderivation lemma)

Central element theorem

Theorem (Central element theorem)

Let L be a positively graded algebra over k and $t \in L_1$ be a central element. Assume that t is a non-zero-divisor in L, and let $(t) \subset L$ be the principal ideal spanned by t. Then the graded algebra L is Koszul if and only if the graded algebra $L/(t)$ is Koszul.

Lemma (Central element and acyclic coderivation lemma)

Let L be a quadratic algebra and $\mathcal{D} = L^!$ be the quadratic dual coalgebra.

Central element theorem

Theorem (Central element theorem)

Let L be a positively graded algebra over k and $t \in L_1$ be a central element. Assume that t is a non-zero-divisor in L, and let $(t) \subset L$ be the principal ideal spanned by t. Then the graded algebra L is Koszul if and only if the graded algebra $L/(t)$ is Koszul.

Lemma (Central element and acyclic coderivation lemma)

Let L be a quadratic algebra and $\mathcal{D} = L^!$ be the quadratic dual coalgebra. Let $t \in L_1 \simeq \mathcal{D}_1$ be an element.

Central element theorem

Theorem (Central element theorem)

Let L be a positively graded algebra over k and $t \in L_1$ be a central element. Assume that t is a non-zero-divisor in L, and let $(t) \subset L$ be the principal ideal spanned by t. Then the graded algebra L is Koszul if and only if the graded algebra $L/(t)$ is Koszul.

Lemma (Central element and acyclic coderivation lemma)

Let L be a quadratic algebra and $\mathcal{D} = L^!$ be the quadratic dual coalgebra. Let $t \in L_1 \simeq \mathcal{D}_1$ be an element. Then the element t is central in L if and only if

Central element theorem

Theorem (Central element theorem)

Let L be a positively graded algebra over k and $t \in L_1$ be a central element. Assume that t is a non-zero-divisor in L, and let $(t) \subset L$ be the principal ideal spanned by t. Then the graded algebra L is Koszul if and only if the graded algebra $L/(t)$ is Koszul.

Lemma (Central element and acyclic coderivation lemma)

Let L be a quadratic algebra and $\mathcal{D} = L^!$ be the quadratic dual coalgebra. Let $t \in L_1 \simeq \mathcal{D}_1$ be an element. Then the element t is central in L if and only if there exists an (always unique) odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$, $n \geqslant 0$ on $\mathcal{D}$

Central element theorem

Theorem (Central element theorem)

Let L be a positively graded algebra over k and $t \in L_1$ be a central element. Assume that t is a non-zero-divisor in L, and let $(t) \subset L$ be the principal ideal spanned by t. Then the graded algebra L is Koszul if and only if the graded algebra $L/(t)$ is Koszul.

Lemma (Central element and acyclic coderivation lemma)

Let L be a quadratic algebra and $\mathcal{D} = L^!$ be the quadratic dual coalgebra. Let $t \in L_1 \simeq \mathcal{D}_1$ be an element. Then the element t is central in L if and only if there exists an (always unique) odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$, $n \geqslant 0$ on $\mathcal{D}$ taking $1 \in k = \mathcal{D}_0$ to $t \in \mathcal{D}_1$.

Central element theorem

Theorem (Central element theorem)

Let L be a positively graded algebra over k and $t \in L_1$ be a central element. Assume that t is a non-zero-divisor in L, and let $(t) \subset L$ be the principal ideal spanned by t. Then the graded algebra L is Koszul if and only if the graded algebra $L/(t)$ is Koszul.

Lemma (Central element and acyclic coderivation lemma)

Let L be a quadratic algebra and $\mathcal{D} = L^!$ be the quadratic dual coalgebra. Let $t \in L_1 \simeq \mathcal{D}_1$ be an element. Then the element t is central in L if and only if there exists an (always unique) odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$, $n \geqslant 0$ on $\mathcal{D}$ taking $1 \in k = \mathcal{D}_0$ to $t \in \mathcal{D}_1$. If this is the case, then one has $\partial^2 = 0$; so $(\mathcal{D}, \partial)$ is a DG-coalgebra.

Central element theorem

Theorem (Central element theorem)

Let L be a positively graded algebra over k and $t \in L_1$ be a central element. Assume that t is a non-zero-divisor in L, and let $(t) \subset L$ be the principal ideal spanned by t. Then the graded algebra L is Koszul if and only if the graded algebra $L/(t)$ is Koszul.

Lemma (Central element and acyclic coderivation lemma)

Let L be a quadratic algebra and $\mathcal{D} = L^!$ be the quadratic dual coalgebra. Let $t \in L_1 \simeq \mathcal{D}_1$ be an element. Then the element t is central in L if and only if there exists an (always unique) odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$, $n \geqslant 0$ on $\mathcal{D}$ taking $1 \in k = \mathcal{D}_0$ to $t \in \mathcal{D}_1$. If this is the case, then one has $\partial^2 = 0$; so $(\mathcal{D}, \partial)$ is a DG-coalgebra. The differential ∂ on $\mathcal{D}$ is acyclic (i. e., $H_*^\partial(\mathcal{D}) = 0$)

Central element theorem

Theorem (Central element theorem)

Let L be a positively graded algebra over k and $t \in L_1$ be a central element. Assume that t is a non-zero-divisor in L, and let $(t) \subset L$ be the principal ideal spanned by t. Then the graded algebra L is Koszul if and only if the graded algebra $L/(t)$ is Koszul.

Lemma (Central element and acyclic coderivation lemma)

Let L be a quadratic algebra and $\mathcal{D} = L^!$ be the quadratic dual coalgebra. Let $t \in L_1 \simeq \mathcal{D}_1$ be an element. Then the element t is central in L if and only if there exists an (always unique) odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$, $n \geqslant 0$ on $\mathcal{D}$ taking $1 \in k = \mathcal{D}_0$ to $t \in \mathcal{D}_1$. If this is the case, then one has $\partial^2 = 0$; so $(\mathcal{D}, \partial)$ is a DG-coalgebra. The differential ∂ on $\mathcal{D}$ is acyclic (i. e., $H^{\partial}_*(\mathcal{D}) = 0$) if and only if $t \neq 0$.

Construction of the category equivalence

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$.

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Denote by t element $1 \in F_0\widetilde{A} \subset F_1\widetilde{A}$

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Denote by t element $1 \in F_0\widetilde{A} \subset F_1\widetilde{A}$, viewed as an element of L_1.

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Denote by t element $1 \in F_0\widetilde{A} \subset F_1\widetilde{A}$, viewed as an element of L_1. Then t is a central non-zero-divisor in L

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Denote by t element $1 \in F_0\widetilde{A} \subset F_1\widetilde{A}$, viewed as an element of L_1. Then t is a central non-zero-divisor in L, and $L/(t) \simeq A = \operatorname{gr}^F \widetilde{A}$.

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Denote by t element $1 \in F_0\widetilde{A} \subset F_1\widetilde{A}$, viewed as an element of L_1. Then t is a central non-zero-divisor in L, and $L/(t) \simeq A = \operatorname{gr}^F\widetilde{A}$. This is a Koszul algebra by assumption.

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Denote by t element $1 \in F_0\widetilde{A} \subset F_1\widetilde{A}$, viewed as an element of L_1. Then t is a central non-zero-divisor in L, and $L/(t) \simeq A = \mathrm{gr}^F \widetilde{A}$. This is a Koszul algebra by assumption. By the central element theorem, the graded algebra L is Koszul, too.

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Denote by t element $1 \in F_0\widetilde{A} \subset F_1\widetilde{A}$, viewed as an element of L_1. Then t is a central non-zero-divisor in L, and $L/(t) \simeq A = \mathrm{gr}^F\widetilde{A}$. This is a Koszul algebra by assumption. By the central element theorem, the graded algebra L is Koszul, too.

Put $\mathcal{D} = L^!$.

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Denote by t element $1 \in F_0\widetilde{A} \subset F_1\widetilde{A}$, viewed as an element of L_1. Then t is a central non-zero-divisor in L, and $L/(t) \simeq A = \operatorname{gr}^F \widetilde{A}$. This is a Koszul algebra by assumption. By the central element theorem, the graded algebra L is Koszul, too.

Put $\mathcal{D} = L^!$. According to the lemma, we have a coderivation ∂ on $\mathcal{D}$ corresponding to the element $t \in \mathcal{D}_1$.

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Denote by t element $1 \in F_0\widetilde{A} \subset F_1\widetilde{A}$, viewed as an element of L_1. Then t is a central non-zero-divisor in L, and $L/(t) \simeq A = \mathrm{gr}^F\widetilde{A}$. This is a Koszul algebra by assumption. By the central element theorem, the graded algebra L is Koszul, too.

Put $\mathcal{D} = L^!$. According to the lemma, we have a coderivation ∂ on $\mathcal{D}$ corresponding to the element $t \in \mathcal{D}_1$. Furthermore, $(\mathcal{D}, \partial)$ is an acyclic DG-coalgebra (as $t \neq 0$).

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Denote by t element $1 \in F_0\widetilde{A} \subset F_1\widetilde{A}$, viewed as an element of L_1. Then t is a central non-zero-divisor in L, and $L/(t) \simeq A = \mathrm{gr}^F\widetilde{A}$. This is a Koszul algebra by assumption. By the central element theorem, the graded algebra L is Koszul, too.

Put $\mathcal{D} = L^!$. According to the lemma, we have a coderivation ∂ on $\mathcal{D}$ corresponding to the element $t \in \mathcal{D}_1$. Furthermore, $(\mathcal{D}, \partial)$ is an acyclic DG-coalgebra (as $t \neq 0$).

Let $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ be the CDG-coalgebra corresponding to the acyclic DG-coalgebra $\mathcal{D}^\bullet = (\mathcal{D}, \partial)$.

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Denote by t element $1 \in F_0\widetilde{A} \subset F_1\widetilde{A}$, viewed as an element of L_1. Then t is a central non-zero-divisor in L, and $L/(t) \simeq A = \mathrm{gr}^F\widetilde{A}$. This is a Koszul algebra by assumption. By the central element theorem, the graded algebra L is Koszul, too.

Put $\mathcal{D} = L^!$. According to the lemma, we have a coderivation ∂ on $\mathcal{D}$ corresponding to the element $t \in \mathcal{D}_1$. Furthermore, $(\mathcal{D}, \partial)$ is an acyclic DG-coalgebra (as $t \neq 0$).

Let $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ be the CDG-coalgebra corresponding to the acyclic DG-coalgebra $\mathcal{D}^\bullet = (\mathcal{D}, \partial)$. So $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$ and $\mathcal{C} = \mathrm{coker}(\partial\colon \mathcal{D} \to \mathcal{D})$.

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Denote by t element $1 \in F_0\widetilde{A} \subset F_1\widetilde{A}$, viewed as an element of L_1. Then t is a central non-zero-divisor in L, and $L/(t) \simeq A = \mathrm{gr}^F \widetilde{A}$. This is a Koszul algebra by assumption. By the central element theorem, the graded algebra L is Koszul, too.

Put $\mathcal{D} = L^!$. According to the lemma, we have a coderivation ∂ on $\mathcal{D}$ corresponding to the element $t \in \mathcal{D}_1$. Furthermore, $(\mathcal{D}, \partial)$ is an acyclic DG-coalgebra (as $t \neq 0$).

Let $\mathcal{C}^{\bullet} = (\mathcal{C}, d, h)$ be the CDG-coalgebra corresponding to the acyclic DG-coalgebra $\mathcal{D}^\bullet = (\mathcal{D}, \partial)$. So $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$ and $\mathcal{C} = \mathrm{coker}(\partial\colon \mathcal{D} \to \mathcal{D})$. One can show that $\mathcal{C} \simeq A^!$.

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Denote by t element $1 \in F_0\widetilde{A} \subset F_1\widetilde{A}$, viewed as an element of L_1. Then t is a central non-zero-divisor in L, and $L/(t) \simeq A = \operatorname{gr}^F \widetilde{A}$. This is a Koszul algebra by assumption. By the central element theorem, the graded algebra L is Koszul, too.

Put $\mathcal{D} = L^!$. According to the lemma, we have a coderivation ∂ on $\mathcal{D}$ corresponding to the element $t \in \mathcal{D}_1$. Furthermore, $(\mathcal{D}, \partial)$ is an acyclic DG-coalgebra (as $t \neq 0$).

Let $\mathcal{C}^{\blacklozenge} = (\mathcal{C}, d, h)$ be the CDG-coalgebra corresponding to the acyclic DG-coalgebra $\mathcal{D}^\bullet = (\mathcal{D}, \partial)$. So $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$ and $\mathcal{C} = \operatorname{coker}(\partial\colon \mathcal{D} \to \mathcal{D})$. One can show that $\mathcal{C} \simeq A^!$.

The category equivalence from the theorem

Construction of the category equivalence

Suppose given a nonhomogeneous Koszul algebra $(\widetilde{A}, F)$. Consider the Rees algebra $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Denote by t element $1 \in F_0\widetilde{A} \subset F_1\widetilde{A}$, viewed as an element of L_1. Then t is a central non-zero-divisor in L, and $L/(t) \simeq A = \mathrm{gr}^F \widetilde{A}$. This is a Koszul algebra by assumption. By the central element theorem, the graded algebra L is Koszul, too.

Put $\mathcal{D} = L^!$. According to the lemma, we have a coderivation ∂ on $\mathcal{D}$ corresponding to the element $t \in \mathcal{D}_1$. Furthermore, $(\mathcal{D}, \partial)$ is an acyclic DG-coalgebra (as $t \neq 0$).

Let $\mathcal{C}^\blacklozenge = (\mathcal{C}, d, h)$ be the CDG-coalgebra corresponding to the acyclic DG-coalgebra $\mathcal{D}^\bullet = (\mathcal{D}, \partial)$. So $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$ and $\mathcal{C} = \mathrm{coker}(\partial\colon \mathcal{D} \to \mathcal{D})$. One can show that $\mathcal{C} \simeq A^!$.

The category equivalence from the theorem assigns the CDG-coalgebra $\mathcal{C}^\blacklozenge$ to the filtered algebra $(\widetilde{A}, F)$.

Construction of the category equivalence

Conversely, starting from a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C}$

Construction of the category equivalence

Conversely, starting from a CDG-coalgebra $\mathcal{C}^\blacklozenge = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C}$, one applies the hat construction to obtain an acyclic DG-coalgebra $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$.

Construction of the category equivalence

Conversely, starting from a CDG-coalgebra $\mathcal{C}^\blacklozenge = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C}$, one applies the hat construction to obtain an acyclic DG-coalgebra $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$.

Then one recovers the Rees algebra L with the central non-zero-divisor $t \in L_1$

Construction of the category equivalence

Conversely, starting from a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C}$, one applies the hat construction to obtain an acyclic DG-coalgebra $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$.

Then one recovers the Rees algebra L with the central non-zero-divisor $t \in L_1$ as the quadratic dual algebra to $\mathcal{D}$.

Construction of the category equivalence

Conversely, starting from a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C}$, one applies the hat construction to obtain an acyclic DG-coalgebra $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$.

Then one recovers the Rees algebra L with the central non-zero-divisor $t \in L_1$ as the quadratic dual algebra to $\mathcal{D}$.
The difficult part of the Poincaré–Birkhoff–Witt theorem in this approach

Construction of the category equivalence

Conversely, starting from a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C}$, one applies the hat construction to obtain an acyclic DG-coalgebra $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$.

Then one recovers the Rees algebra L with the central non-zero-divisor $t \in L_1$ as the quadratic dual algebra to $\mathcal{D}$. The difficult part of the Poincaré–Birkhoff–Witt theorem in this approach is to prove that t is indeed a non-zero-divisor in L.

Construction of the category equivalence

Conversely, starting from a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C}$, one applies the hat construction to obtain an acyclic DG-coalgebra $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$.

Then one recovers the Rees algebra L with the central non-zero-divisor $t \in L_1$ as the quadratic dual algebra to $\mathcal{D}$. The difficult part of the Poincaré–Birkhoff–Witt theorem in this approach is to prove that t is indeed a non-zero-divisor in L.

Then it remains to consider the ideal $(t-1)$ generated by the nonhomogeneous element $t - 1 \in L$

Construction of the category equivalence

Conversely, starting from a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C}$, one applies the hat construction to obtain an acyclic DG-coalgebra $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$.

Then one recovers the Rees algebra L with the central non-zero-divisor $t \in L_1$ as the quadratic dual algebra to $\mathcal{D}$. The difficult part of the Poincaré–Birkhoff–Witt theorem in this approach is to prove that t is indeed a non-zero-divisor in L.

Then it remains to consider the ideal $(t-1)$ generated by the nonhomogeneous element $t-1 \in L$, and put $\widetilde{A} = L/(t-1)$.

Construction of the category equivalence

Conversely, starting from a CDG-coalgebra $\mathcal{C}^\blacklozenge = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C}$, one applies the hat construction to obtain an acyclic DG-coalgebra $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$.

Then one recovers the Rees algebra L with the central non-zero-divisor $t \in L_1$ as the quadratic dual algebra to $\mathcal{D}$. The difficult part of the Poincaré–Birkhoff–Witt theorem in this approach is to prove that t is indeed a non-zero-divisor in L.

Then it remains to consider the ideal $(t-1)$ generated by the nonhomogeneous element $t-1 \in L$, and put $\widetilde{A} = L/(t-1)$.

For example, any nonzero k-algebra $\widetilde{A}$ can be endowed with the trivial filtration

Conversely, starting from a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C}$, one applies the hat construction to obtain an acyclic DG-coalgebra $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$.

Then one recovers the Rees algebra L with the central non-zero-divisor $t \in L_1$ as the quadratic dual algebra to $\mathcal{D}$. The difficult part of the Poincaré–Birkhoff–Witt theorem in this approach is to prove that t is indeed a non-zero-divisor in L.

Then it remains to consider the ideal $(t-1)$ generated by the nonhomogeneous element $t-1 \in L$, and put $\widetilde{A} = L/(t-1)$.

For example, any nonzero k-algebra $\widetilde{A}$ can be endowed with the trivial filtration defined by the rules

- $F_{-1}\widetilde{A} = 0, \;\; F_0\widetilde{A} = k \cdot 1,$

Construction of the category equivalence

Conversely, starting from a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C}$, one applies the hat construction to obtain an acyclic DG-coalgebra $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$.

Then one recovers the Rees algebra L with the central non-zero-divisor $t \in L_1$ as the quadratic dual algebra to $\mathcal{D}$. The difficult part of the Poincaré–Birkhoff–Witt theorem in this approach is to prove that t is indeed a non-zero-divisor in L.

Then it remains to consider the ideal $(t-1)$ generated by the nonhomogeneous element $t-1 \in L$, and put $\widetilde{A} = L/(t-1)$.

For example, any nonzero k-algebra $\widetilde{A}$ can be endowed with the trivial filtration defined by the rules

- $F_{-1}\widetilde{A} = 0$, $F_0\widetilde{A} = k \cdot 1$,
- $F_n\widetilde{A} = \widetilde{A}$ for $n \geqslant 1$.

Construction of the category equivalence

Conversely, starting from a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C}$, one applies the hat construction to obtain an acyclic DG-coalgebra $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$.

Then one recovers the Rees algebra L with the central non-zero-divisor $t \in L_1$ as the quadratic dual algebra to $\mathcal{D}$. The difficult part of the Poincaré–Birkhoff–Witt theorem in this approach is to prove that t is indeed a non-zero-divisor in L.

Then it remains to consider the ideal $(t-1)$ generated by the nonhomogeneous element $t-1 \in L$, and put $\widetilde{A} = L/(t-1)$.

For example, any nonzero k-algebra $\widetilde{A}$ can be endowed with the trivial filtration defined by the rules

- $F_{-1}\widetilde{A} = 0, \quad F_0\widetilde{A} = k \cdot 1,$
- $F_n\widetilde{A} = \widetilde{A}$ for $n \geqslant 1$.

Then $(\widetilde{A}, F)$ is a nonhomogeneous Koszul algebra.

Conversely, starting from a CDG-coalgebra $\mathcal{C}^\blacklozenge = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C}$, one applies the hat construction to obtain an acyclic DG-coalgebra $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$.

Then one recovers the Rees algebra L with the central non-zero-divisor $t \in L_1$ as the quadratic dual algebra to $\mathcal{D}$. The difficult part of the Poincaré–Birkhoff–Witt theorem in this approach is to prove that t is indeed a non-zero-divisor in L.

Then it remains to consider the ideal $(t-1)$ generated by the nonhomogeneous element $t - 1 \in L$, and put $\widetilde{A} = L/(t-1)$.

For example, any nonzero k-algebra $\widetilde{A}$ can be endowed with the trivial filtration defined by the rules

- $F_{-1}\widetilde{A} = 0, \quad F_0\widetilde{A} = k \cdot 1,$
- $F_n\widetilde{A} = \widetilde{A}$ for $n \geqslant 1$.

Then $(\widetilde{A}, F)$ is a nonhomogeneous Koszul algebra.

The related CDG-coalgebra $\mathcal{C}^\blacklozenge = (\mathcal{C}, d, h)$

Construction of the category equivalence

Conversely, starting from a CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ with a Koszul underlying graded coalgebra $\mathcal{C}$, one applies the hat construction to obtain an acyclic DG-coalgebra $\widehat{\mathcal{C}}^\bullet = (\mathcal{D}, \partial)$.

Then one recovers the Rees algebra L with the central non-zero-divisor $t \in L_1$ as the quadratic dual algebra to $\mathcal{D}$. The difficult part of the Poincaré–Birkhoff–Witt theorem in this approach is to prove that t is indeed a non-zero-divisor in L.

Then it remains to consider the ideal $(t-1)$ generated by the nonhomogeneous element $t - 1 \in L$, and put $\widetilde{A} = L/(t-1)$.

For example, any nonzero k-algebra $\widetilde{A}$ can be endowed with the trivial filtration defined by the rules

- $F_{-1}\widetilde{A} = 0, \quad F_0\widetilde{A} = k \cdot 1,$
- $F_n\widetilde{A} = \widetilde{A}$ for $n \geqslant 1$.

Then $(\widetilde{A}, F)$ is a nonhomogeneous Koszul algebra.

The related CDG-coalgebra $\mathcal{C}^\bullet = (\mathcal{C}, d, h)$ is the bar construction of $\widetilde{A}$, i. e., $\mathcal{C}^\bullet = \mathrm{Bar}_v(\widetilde{A})$.

Koszul graded rings

Koszul graded rings

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$.

Koszul graded rings

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$. We will say that A is left flat Koszul

Koszul graded rings

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$. We will say that A is left flat Koszul if A is a flat left R-module

Koszul graded rings

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$. We will say that A is left flat Koszul if A is a flat left R-module and $\operatorname{Tor}^A_{ij}(R, R) = 0$ for all $i \neq j$.

Koszul graded rings

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$. We will say that A is left flat Koszul if A is a flat left R-module and $\operatorname{Tor}_{ij}^A(R, R) = 0$ for all $i \neq j$. Similarly, A is called left projective Koszul

Koszul graded rings

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$. We will say that A is left flat Koszul if A is a flat left R-module and $\mathrm{Tor}^A_{ij}(R, R) = 0$ for all $i \neq j$. Similarly, A is called left projective Koszul if A is a projective left R-module and $\mathrm{Tor}^A_{ij}(R, R) = 0$ for all $i \neq j$.

Koszul graded rings

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$. We will say that A is left flat Koszul if A is a flat left R-module and $\mathrm{Tor}^A_{ij}(R, R) = 0$ for all $i \neq j$. Similarly, A is called left projective Koszul if A is a projective left R-module and $\mathrm{Tor}^A_{ij}(R, R) = 0$ for all $i \neq j$. Finally, A is called left finitely projective Koszul if A_n is a finitely generated projective left R-module for every $n \geqslant 0$

Koszul graded rings

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$. We will say that A is left flat Koszul if A is a flat left R-module and $\operatorname{Tor}_{ij}^A(R, R) = 0$ for all $i \neq j$. Similarly, A is called left projective Koszul if A is a projective left R-module and $\operatorname{Tor}_{ij}^A(R, R) = 0$ for all $i \neq j$. Finally, A is called left finitely projective Koszul if A_n is a finitely generated projective left R-module for every $n \geqslant 0$ and $\operatorname{Tor}_{ij}^A(R, R) = 0$ for all $i \neq j$.

Koszul graded rings

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$. We will say that A is left flat Koszul if A is a flat left R-module and $\mathrm{Tor}_{ij}^A(R,R) = 0$ for all $i \neq j$. Similarly, A is called left projective Koszul if A is a projective left R-module and $\mathrm{Tor}_{ij}^A(R,R) = 0$ for all $i \neq j$. Finally, A is called left finitely projective Koszul if A_n is a finitely generated projective left R-module for every $n \geqslant 0$ and $\mathrm{Tor}_{ij}^A(R,R) = 0$ for all $i \neq j$. All left flat Koszul graded rings are quadratic (in an obvious sense).

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$. We will say that A is left flat Koszul if A is a flat left R-module and $\mathrm{Tor}_{ij}^A(R, R) = 0$ for all $i \neq j$. Similarly, A is called left projective Koszul if A is a projective left R-module and $\mathrm{Tor}_{ij}^A(R, R) = 0$ for all $i \neq j$. Finally, A is called left finitely projective Koszul if A_n is a finitely generated projective left R-module for every $n \geqslant 0$ and $\mathrm{Tor}_{ij}^A(R, R) = 0$ for all $i \neq j$. All left flat Koszul graded rings are quadratic (in an obvious sense).

Theorem (Homogeneous quadratic duality over a base ring)

Koszul graded rings

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$. We will say that A is left flat Koszul if A is a flat left R-module and $\mathrm{Tor}_{ij}^A(R, R) = 0$ for all $i \neq j$. Similarly, A is called left projective Koszul if A is a projective left R-module and $\mathrm{Tor}_{ij}^A(R, R) = 0$ for all $i \neq j$. Finally, A is called left finitely projective Koszul if A_n is a finitely generated projective left R-module for every $n \geqslant 0$ and $\mathrm{Tor}_{ij}^A(R, R) = 0$ for all $i \neq j$. All left flat Koszul graded rings are quadratic (in an obvious sense).

Theorem (Homogeneous quadratic duality over a base ring)

There is a natural anti-equivalence between the categories of left finitely projective Koszul graded rings A

Koszul graded rings

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$. We will say that A is left flat Koszul if A is a flat left R-module and $\mathrm{Tor}_{ij}^{A}(R, R) = 0$ for all $i \neq j$. Similarly, A is called left projective Koszul if A is a projective left R-module and $\mathrm{Tor}_{ij}^{A}(R, R) = 0$ for all $i \neq j$. Finally, A is called left finitely projective Koszul if A_n is a finitely generated projective left R-module for every $n \geqslant 0$ and $\mathrm{Tor}_{ij}^{A}(R, R) = 0$ for all $i \neq j$. All left flat Koszul graded rings are quadratic (in an obvious sense).

Theorem (Homogeneous quadratic duality over a base ring)

There is a natural anti-equivalence between the categories of left finitely projective Koszul graded rings A and right finitely projective Koszul graded rings B over a fixed base ring $A_0 = R = B_0$.

Koszul graded rings

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$. We will say that A is left flat Koszul if A is a flat left R-module and $\mathrm{Tor}_{ij}^A(R,R) = 0$ for all $i \neq j$. Similarly, A is called left projective Koszul if A is a projective left R-module and $\mathrm{Tor}_{ij}^A(R,R) = 0$ for all $i \neq j$. Finally, A is called left finitely projective Koszul if A_n is a finitely generated projective left R-module for every $n \geqslant 0$ and $\mathrm{Tor}_{ij}^A(R,R) = 0$ for all $i \neq j$. All left flat Koszul graded rings are quadratic (in an obvious sense).

Theorem (Homogeneous quadratic duality over a base ring)

There is a natural anti-equivalence between the categories of left finitely projective Koszul graded rings A and right finitely projective Koszul graded rings B over a fixed base ring $A_0 = R = B_0$. The anti-equivalence is given by the rules

- $B_1 = \mathrm{Hom}_R(A_1, R)$, $A_1 = \mathrm{Hom}_{R^{\mathrm{op}}}(B_1, R)$;

Koszul graded rings

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$. We will say that A is left flat Koszul if A is a flat left R-module and $\mathrm{Tor}_{ij}^A(R, R) = 0$ for all $i \neq j$. Similarly, A is called left projective Koszul if A is a projective left R-module and $\mathrm{Tor}_{ij}^A(R, R) = 0$ for all $i \neq j$. Finally, A is called left finitely projective Koszul if A_n is a finitely generated projective left R-module for every $n \geqslant 0$ and $\mathrm{Tor}_{ij}^A(R, R) = 0$ for all $i \neq j$. All left flat Koszul graded rings are quadratic (in an obvious sense).

Theorem (Homogeneous quadratic duality over a base ring)

There is a natural anti-equivalence between the categories of left finitely projective Koszul graded rings A and right finitely projective Koszul graded rings B over a fixed base ring $A_0 = R = B_0$. The anti-equivalence is given by the rules

- $B_1 = \mathrm{Hom}_R(A_1, R)$, $A_1 = \mathrm{Hom}_{R^{\mathrm{op}}}(B_1, R)$;
- $I_A = \ker(A_1 \otimes_R A_1 \to A_2)$, $I_B = \ker(B_1 \otimes_R B_1 \to B_2)$;

Koszul graded rings

Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a nonnegatively graded ring with the degree 0 component $R = A_0$. We will say that A is left flat Koszul if A is a flat left R-module and $\mathrm{Tor}^A_{ij}(R, R) = 0$ for all $i \neq j$. Similarly, A is called left projective Koszul if A is a projective left R-module and $\mathrm{Tor}^A_{ij}(R, R) = 0$ for all $i \neq j$. Finally, A is called left finitely projective Koszul if A_n is a finitely generated projective left R-module for every $n \geqslant 0$ and $\mathrm{Tor}^A_{ij}(R, R) = 0$ for all $i \neq j$. All left flat Koszul graded rings are quadratic (in an obvious sense).

Theorem (Homogeneous quadratic duality over a base ring)

There is a natural anti-equivalence between the categories of left finitely projective Koszul graded rings A and right finitely projective Koszul graded rings B over a fixed base ring $A_0 = R = B_0$. The anti-equivalence is given by the rules

- $B_1 = \mathrm{Hom}_R(A_1, R)$, $A_1 = \mathrm{Hom}_{R^{op}}(B_1, R)$;
- $I_A = \ker(A_1 \otimes_R A_1 \to A_2)$, $I_B = \ker(B_1 \otimes_R B_1 \to B_2)$;
- $B_2 = \mathrm{Hom}_R(I_A, R)$, $A_2 = \mathrm{Hom}_{R^{op}}(I_B, R)$.

Nonhomogeneous Koszul graded rings

Nonhomogeneous Koszul graded rings

A left flat/left projective/left finitely projective nonhomogeneous Koszul ring $\widetilde{A}$

Nonhomogeneous Koszul graded rings

A left flat/left projective/left finitely projective nonhomogeneous Koszul ring $\widetilde{A}$ is a filtered ring $F_0\widetilde{A} \subset F_1\widetilde{A} \subset \cdots \subset \widetilde{A}$

Nonhomogeneous Koszul graded rings

A left flat/left projective/left finitely projective nonhomogeneous Koszul ring $\widetilde{A}$ is a filtered ring $F_0\widetilde{A} \subset F_1\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;

Nonhomogeneous Koszul graded rings

A left flat/left projective/left finitely projective nonhomogeneous Koszul ring $\widetilde{A}$ is a filtered ring $F_0\widetilde{A} \subset F_1\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;

Nonhomogeneous Koszul graded rings

A left flat/left projective/left finitely projective nonhomogeneous Koszul ring $\widetilde{A}$ is a filtered ring $F_0\widetilde{A} \subset F_1\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;
- the associated graded ring $A = \mathrm{gr}^F \widetilde{A} = \bigoplus_{n \geqslant 0} F_n\widetilde{A}/F_{n-1}\widetilde{A}$ is left flat/left projective/left finitely projective Koszul.

Nonhomogeneous Koszul graded rings

A left flat/left projective/left finitely projective nonhomogeneous Koszul ring $\widetilde{A}$ is a filtered ring $F_0\widetilde{A} \subset F_1\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;
- the associated graded ring $A = \mathrm{gr}^F \widetilde{A} = \bigoplus_{n \geqslant 0} F_n\widetilde{A}/F_{n-1}\widetilde{A}$ is left flat/left projective/left finitely projective Koszul.

Theorem (Poincaré–Birkhoff–Witt theorem over a base ring)

Nonhomogeneous Koszul graded rings

A left flat/left projective/left finitely projective nonhomogeneous Koszul ring $\widetilde{A}$ is a filtered ring $F_0\widetilde{A} \subset F_1\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;
- the associated graded ring $A = \mathrm{gr}^F \widetilde{A} = \bigoplus_{n\geqslant 0} F_n\widetilde{A}/F_{n-1}\widetilde{A}$ is left flat/left projective/left finitely projective Koszul.

Theorem (Poincaré–Birkhoff–Witt theorem over a base ring)

There is a natural anti-equivalence between the category of left finitely projective nonhomogeneous Koszul rings $(\widetilde{A}, F)$ over a fixed base ring $R = F_0\widetilde{A}$

Nonhomogeneous Koszul graded rings

A left flat/left projective/left finitely projective nonhomogeneous Koszul ring $\widetilde{A}$ is a filtered ring $F_0\widetilde{A} \subset F_1\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;
- the associated graded ring $A = \mathrm{gr}^F\widetilde{A} = \bigoplus_{n\geqslant 0} F_n\widetilde{A}/F_{n-1}\widetilde{A}$ is left flat/left projective/left finitely projective Koszul.

Theorem (Poincaré–Birkhoff–Witt theorem over a base ring)

There is a natural anti-equivalence between the category of left finitely projective nonhomogeneous Koszul rings $(\widetilde{A}, F)$ over a fixed base ring $R = F_0\widetilde{A}$ and the category of CDG-rings $B^\bullet = (B, d, h)$ with a right finitely projective Koszul underlying graded ring $B = \bigoplus_{n=0}^{\infty} B^n$ over the same fixed base ring $R = B^0$.

Nonhomogeneous Koszul graded rings

A left flat/left projective/left finitely projective nonhomogeneous Koszul ring $\widetilde{A}$ is a filtered ring $F_0\widetilde{A} \subset F_1\widetilde{A} \subset \cdots \subset \widetilde{A}$ such that

- $F_i\widetilde{A} \cdot F_j\widetilde{A} \subset F_{i+j}\widetilde{A}$ for all $i, j \geqslant 0$;
- $F_{-1}\widetilde{A} = 0$; $\widetilde{A} = \bigcup_{n=0}^{\infty} F_n\widetilde{A}$;
- the associated graded ring $A = \mathrm{gr}^F\widetilde{A} = \bigoplus_{n\geqslant 0} F_n\widetilde{A}/F_{n-1}\widetilde{A}$ is left flat/left projective/left finitely projective Koszul.

Theorem (Poincaré–Birkhoff–Witt theorem over a base ring)

There is a natural anti-equivalence between the category of left finitely projective nonhomogeneous Koszul rings $(\widetilde{A}, F)$ over a fixed base ring $R = F_0\widetilde{A}$ and the category of CDG-rings $B^\bullet = (B, d, h)$ with a right finitely projective Koszul underlying graded ring $B = \bigoplus_{n=0}^{\infty} B^n$ over the same fixed base ring $R = B^0$. Under this anti-equivalence, the Koszul graded rings $A = \mathrm{gr}^F\widetilde{A}$ and B are quadratic dual to each other, as per the previous slide.

Example: D-Ω duality

Example: D-Ω duality

Let M be a compact smooth real manifold or a smooth affine algebraic variety over a field

Example: D-Ω duality

Let M be a compact smooth real manifold or a smooth affine algebraic variety over a field, $\mathcal{E}$ be a vector bundle on M

Example: D-Ω duality

Let M be a compact smooth real manifold or a smooth affine algebraic variety over a field, $\mathcal{E}$ be a vector bundle on M, and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$.

Example: D-Ω duality

Let M be a compact smooth real manifold or a smooth affine algebraic variety over a field, $\mathcal{E}$ be a vector bundle on M, and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$. Consider the ring $\mathrm{Diff}(M, \mathcal{E})$ of differential operators acting on the sections of $\mathcal{E}$.

Example: D-Ω duality

Let M be a compact smooth real manifold or a smooth affine algebraic variety over a field, $\mathcal{E}$ be a vector bundle on M, and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$. Consider the ring $\mathrm{Diff}(M,\mathcal{E})$ of differential operators acting on the sections of $\mathcal{E}$.

The ring $\mathrm{Diff}(M,\mathcal{E})$ has a natural increasing filtration F by the orders of differential operators.

Example: D-Ω duality

Let M be a compact smooth real manifold or a smooth affine algebraic variety over a field, $\mathcal{E}$ be a vector bundle on M, and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$. Consider the ring $\mathrm{Diff}(M, \mathcal{E})$ of differential operators acting on the sections of $\mathcal{E}$.

The ring $\mathrm{Diff}(M, \mathcal{E})$ has a natural increasing filtration F by the orders of differential operators. Endowed with this filtration, $\mathrm{Diff}(M, \mathcal{E})$ becomes a left and right finitely projective nonhomogeneous Koszul ring.

Example: D-Ω duality

Let M be a compact smooth real manifold or a smooth affine algebraic variety over a field, $\mathcal{E}$ be a vector bundle on M, and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$. Consider the ring $\mathrm{Diff}(M, \mathcal{E})$ of differential operators acting on the sections of $\mathcal{E}$.

The ring $\mathrm{Diff}(M, \mathcal{E})$ has a natural increasing filtration F by the orders of differential operators. Endowed with this filtration, $\mathrm{Diff}(M, \mathcal{E})$ becomes a left and right finitely projective nonhomogeneous Koszul ring.

The CDG-ring of differential forms $(\Omega(M, \mathcal{E}nd(\mathcal{E})), d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}, h_{\nabla_{\mathcal{E}}})$ mentioned in the beginning of this talk

Example: D-Ω duality

Let M be a compact smooth real manifold or a smooth affine algebraic variety over a field, $\mathcal{E}$ be a vector bundle on M, and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$. Consider the ring $\mathrm{Diff}(M, \mathcal{E})$ of differential operators acting on the sections of $\mathcal{E}$.

The ring $\mathrm{Diff}(M, \mathcal{E})$ has a natural increasing filtration F by the orders of differential operators. Endowed with this filtration, $\mathrm{Diff}(M, \mathcal{E})$ becomes a left and right finitely projective nonhomogeneous Koszul ring.

The CDG-ring of differential forms $(\Omega(M, \mathcal{E}nd(\mathcal{E})), d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}, h_{\nabla_{\mathcal{E}}})$ mentioned in the beginning of this talk corresponds to the nonhomogeneous Koszul ring $(\mathrm{Diff}(M, \mathcal{E}), F)$ under the equivalence of categories from the previous slide.

Example: D-Ω duality

Let M be a compact smooth real manifold or a smooth affine algebraic variety over a field, $\mathcal{E}$ be a vector bundle on M, and $\nabla_{\mathcal{E}}$ be a connection in $\mathcal{E}$. Consider the ring $\mathrm{Diff}(M, \mathcal{E})$ of differential operators acting on the sections of $\mathcal{E}$.

The ring $\mathrm{Diff}(M, \mathcal{E})$ has a natural increasing filtration F by the orders of differential operators. Endowed with this filtration, $\mathrm{Diff}(M, \mathcal{E})$ becomes a left and right finitely projective nonhomogeneous Koszul ring.

The CDG-ring of differential forms $(\Omega(M, \mathcal{E}nd(\mathcal{E})), d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}, h_{\nabla_{\mathcal{E}}})$ mentioned in the beginning of this talk corresponds to the nonhomogeneous Koszul ring $(\mathrm{Diff}(M, \mathcal{E}), F)$ under the equivalence of categories from the previous slide.

Reference: L. Positselski, "Relative Nonhomogeneous Koszul duality", Frontiers in Math., Birkhäuser, 2021.

Comodules and contramodules over graded rings

Comodules and contramodules over graded rings

Let $B = \bigoplus_{n=0}^{\infty} B^n$ be a nonnegatively graded ring.

Comodules and contramodules over graded rings

Let $B = \bigoplus_{n=0}^{\infty} B^n$ be a nonnegatively graded ring. A graded right B-module N is said to be a B-comodule

Comodules and contramodules over graded rings

Let $B = \bigoplus_{n=0}^{\infty} B^n$ be a nonnegatively graded ring. A graded right B-module N is said to be a B-comodule if for every (homogeneous) element $x \in N$ there exists an integer $m \geqslant 0$

Comodules and contramodules over graded rings

Let $B = \bigoplus_{n=0}^{\infty} B^n$ be a nonnegatively graded ring. A graded right B-module N is said to be a B-comodule if for every (homogeneous) element $x \in N$ there exists an integer $m \geqslant 0$ such that $xB^{>m} = 0$.

Comodules and contramodules over graded rings

Let $B = \bigoplus_{n=0}^{\infty} B^n$ be a nonnegatively graded ring. A graded right B-module N is said to be a B-comodule if for every (homogeneous) element $x \in N$ there exists an integer $m \geqslant 0$ such that $xB^{>m} = 0$. The category of graded right B-comodules $\mathrm{Comod}\text{-}B$ is abelian.

Comodules and contramodules over graded rings

Let $B = \bigoplus_{n=0}^{\infty} B^n$ be a nonnegatively graded ring. A graded right B-module N is said to be a *B-comodule* if for every (homogeneous) element $x \in N$ there exists an integer $m \geqslant 0$ such that $xB^{>m} = 0$. The category of graded right B-comodules $\mathrm{Comod}\text{-}B$ is abelian.

Let K be an associative ring and $K \longrightarrow B^0$ be a ring homomorphism.

Comodules and contramodules over graded rings

Let $B = \bigoplus_{n=0}^{\infty} B^n$ be a nonnegatively graded ring. A graded right B-module N is said to be a B-comodule if for every (homogeneous) element $x \in N$ there exists an integer $m \geqslant 0$ such that $xB^{>m} = 0$. The category of graded right B-comodules $\mathrm{Comod}\text{-}B$ is abelian.

Let K be an associative ring and $K \longrightarrow B^0$ be a ring homomorphism. A graded left B-contramodule P is a graded K-module

Comodules and contramodules over graded rings

Let $B = \bigoplus_{n=0}^{\infty} B^n$ be a nonnegatively graded ring. A graded right B-module N is said to be a B-comodule if for every (homogeneous) element $x \in N$ there exists an integer $m \geqslant 0$ such that $xB^{>m} = 0$. The category of graded right B-comodules $\mathrm{Comod}\text{-}B$ is abelian.

Let K be an associative ring and $K \longrightarrow B^0$ be a ring homomorphism. A graded left B-contramodule P is a graded K-module endowed with a morphism of graded K-modules

$$\prod\nolimits_{m \in \mathbb{Z}} (B^m \otimes_K P^{n-m}) \longrightarrow P^n$$

Comodules and contramodules over graded rings

Let $B = \bigoplus_{n=0}^{\infty} B^n$ be a nonnegatively graded ring. A graded right B-module N is said to be a B-comodule if for every (homogeneous) element $x \in N$ there exists an integer $m \geqslant 0$ such that $xB^{>m} = 0$. The category of graded right B-comodules $\mathrm{Comod}\text{-}B$ is abelian.

Let K be an associative ring and $K \longrightarrow B^0$ be a ring homomorphism. A graded left B-contramodule P is a graded K-module endowed with a morphism of graded K-modules

$$\prod\nolimits_{m\in\mathbb{Z}}(B^m \otimes_K P^{n-m}) \longrightarrow P^n$$

for every $n \in \mathbb{Z}$

Comodules and contramodules over graded rings

Let $B = \bigoplus_{n=0}^{\infty} B^n$ be a nonnegatively graded ring. A graded right B-module N is said to be a B-comodule if for every (homogeneous) element $x \in N$ there exists an integer $m \geqslant 0$ such that $xB^{>m} = 0$. The category of graded right B-comodules $\mathrm{Comod}\text{-}B$ is abelian.

Let K be an associative ring and $K \longrightarrow B^0$ be a ring homomorphism. A graded left B-contramodule P is a graded K-module endowed with a morphism of graded K-modules

$$\prod\nolimits_{m\in\mathbb{Z}} (B^m \otimes_K P^{n-m}) \longrightarrow P^n$$

for every $n \in \mathbb{Z}$, satisfying natural (contra)associativity and (contra)unitality equations.

Comodules and contramodules over graded rings

Let $B = \bigoplus_{n=0}^{\infty} B^n$ be a nonnegatively graded ring. A graded right B-module N is said to be a B-comodule if for every (homogeneous) element $x \in N$ there exists an integer $m \geqslant 0$ such that $xB^{>m} = 0$. The category of graded right B-comodules $\mathrm{Comod}\text{-}B$ is abelian.

Let K be an associative ring and $K \longrightarrow B^0$ be a ring homomorphism. A graded left B-contramodule P is a graded K-module endowed with a morphism of graded K-modules

$$\prod\nolimits_{m\in\mathbb{Z}} (B^m \otimes_K P^{n-m}) \longrightarrow P^n$$

for every $n \in \mathbb{Z}$, satisfying natural (contra)associativity and (contra)unitality equations. So left B-contramodules are left B-modules with infinite summation operations.

Comodules and contramodules over graded rings

Let $B = \bigoplus_{n=0}^{\infty} B^n$ be a nonnegatively graded ring. A graded right B-module N is said to be a B-comodule if for every (homogeneous) element $x \in N$ there exists an integer $m \geqslant 0$ such that $xB^{>m} = 0$. The category of graded right B-comodules $\mathrm{Comod}\text{-}B$ is abelian.

Let K be an associative ring and $K \longrightarrow B^0$ be a ring homomorphism. A graded left B-contramodule P is a graded K-module endowed with a morphism of graded K-modules

$$\prod\nolimits_{m\in\mathbb{Z}}(B^m \otimes_K P^{n-m}) \longrightarrow P^n$$

for every $n \in \mathbb{Z}$, satisfying natural (contra)associativity and (contra)unitality equations. So left B-contramodules are left B-modules with infinite summation operations. The category of graded left B-contramodules $B\text{-}\mathrm{Contra}$ is abelian

Comodules and contramodules over graded rings

Let $B = \bigoplus_{n=0}^{\infty} B^n$ be a nonnegatively graded ring. A graded right B-module N is said to be a B-comodule if for every (homogeneous) element $x \in N$ there exists an integer $m \geqslant 0$ such that $xB^{>m} = 0$. The category of graded right B-comodules $\mathrm{Comod}\text{-}B$ is abelian.

Let K be an associative ring and $K \longrightarrow B^0$ be a ring homomorphism. A graded left B-contramodule P is a graded K-module endowed with a morphism of graded K-modules

$$\prod\nolimits_{m\in\mathbb{Z}}(B^m \otimes_K P^{n-m}) \longrightarrow P^n$$

for every $n \in \mathbb{Z}$, satisfying natural (contra)associativity and (contra)unitality equations. So left B-contramodules are left B-modules with infinite summation operations. The category of graded left B-contramodules $B\text{-}\mathrm{Contra}$ is abelian and does not depend on the choice of a ring K.

CDG-comodules and CDG-contramodules over CDG-rings

CDG-comodules and CDG-contramodules over CDG-rings

Let $B^\blacklozenge = (B, d, h)$ be a CDG-ring whose underlying graded ring B is nonnegatively graded.

CDG-comodules and CDG-contramodules over CDG-rings

Let $B^\bullet = (B, d, h)$ be a CDG-ring whose underlying graded ring B is nonnegatively graded. A right CDG-module $N^\bullet = (N, d_N)$ over $B^\bullet$ is said to be CDG-comodule over $B^\bullet$

CDG-comodules and CDG-contramodules over CDG-rings

Let $B^{\bullet} = (B, d, h)$ be a CDG-ring whose underlying graded ring B is nonnegatively graded. A right CDG-module $N^{\bullet} = (N, d_N)$ over $B^{\bullet}$ is said to be CDG-comodule over $B^{\bullet}$ if the graded B-module N is a B-comodule.

CDG-comodules and CDG-contramodules over CDG-rings

Let $B^{\bullet} = (B, d, h)$ be a CDG-ring whose underlying graded ring B is nonnegatively graded. A right CDG-module $N^{\bullet} = (N, d_N)$ over $B^{\bullet}$ is said to be CDG-comodule over $B^{\bullet}$ if the graded B-module N is a B-comodule. Right CDG-comodules over $B^{\bullet}$ form an abelian DG-category $\mathrm{Comod}^{\mathrm{cdg}}\text{-}B^{\bullet}$.

CDG-comodules and CDG-contramodules over CDG-rings

Let $B^\bullet = (B, d, h)$ be a CDG-ring whose underlying graded ring B is nonnegatively graded. A right CDG-module $N^\bullet = (N, d_N)$ over $B^\bullet$ is said to be CDG-comodule over $B^\bullet$ if the graded B-module N is a B-comodule. Right CDG-comodules over $B^\bullet$ form an abelian DG-category $\mathrm{Comod}^{\mathrm{cdg}}\text{-}B^\bullet$.

Given a graded left B-contramodule P, one can define what it means for a differential $d_P\colon P^i \longrightarrow P^{i+1}$ to be a d_B-contraderivation

CDG-comodules and CDG-contramodules over CDG-rings

Let $B^\bullet = (B, d, h)$ be a CDG-ring whose underlying graded ring B is nonnegatively graded. A right CDG-module $N^\bullet = (N, d_N)$ over $B^\bullet$ is said to be CDG-comodule over $B^\bullet$ if the graded B-module N is a B-comodule. Right CDG-comodules over $B^\bullet$ form an abelian DG-category $\mathrm{Comod}^{\mathrm{cdg}}\text{-}B^\bullet$.

Given a graded left B-contramodule P, one can define what it means for a differential $d_P\colon P^i \longrightarrow P^{i+1}$ to be a d_B-contraderivation (i. e., an odd derivation of the contramodule P compatible with the derivation d on B).

CDG-comodules and CDG-contramodules over CDG-rings

Let $B^\bullet = (B, d, h)$ be a CDG-ring whose underlying graded ring B is nonnegatively graded. A right CDG-module $N^\bullet = (N, d_N)$ over $B^\bullet$ is said to be CDG-comodule over $B^\bullet$ if the graded B-module N is a B-comodule. Right CDG-comodules over $B^\bullet$ form an abelian DG-category $\mathrm{Comod}^{\mathrm{cdg}}\text{-}B^\bullet$.

Given a graded left B-contramodule P, one can define what it means for a differential $d_P\colon P^i \longrightarrow P^{i+1}$ to be a d_B-contraderivation (i. e., an odd derivation of the contramodule P compatible with the derivation d on B). Imposing the equation $d_P^2(p) = hp$ for all $p \in P$

CDG-comodules and CDG-contramodules over CDG-rings

Let $B^\bullet = (B, d, h)$ be a CDG-ring whose underlying graded ring B is nonnegatively graded. A right CDG-module $N^\bullet = (N, d_N)$ over $B^\bullet$ is said to be CDG-comodule over $B^\bullet$ if the graded B-module N is a B-comodule. Right CDG-comodules over $B^\bullet$ form an abelian DG-category $\mathrm{Comod}^{\mathrm{cdg}}\text{-}B^\bullet$.

Given a graded left B-contramodule P, one can define what it means for a differential $d_P\colon P^i \longrightarrow P^{i+1}$ to be a d_B-contraderivation (i. e., an odd derivation of the contramodule P compatible with the derivation d on B). Imposing the equation $d_P^2(p) = hp$ for all $p \in P$, one obtains the definition of a left CDG-contramodule $P^\bullet$ over $B^\bullet$.

CDG-comodules and CDG-contramodules over CDG-rings

Let $B^\bullet = (B, d, h)$ be a CDG-ring whose underlying graded ring B is nonnegatively graded. A right CDG-module $N^\bullet = (N, d_N)$ over $B^\bullet$ is said to be CDG-comodule over $B^\bullet$ if the graded B-module N is a B-comodule. Right CDG-comodules over $B^\bullet$ form an abelian DG-category $\mathrm{Comod}^{\mathrm{cdg}}\text{-}B^\bullet$.

Given a graded left B-contramodule P, one can define what it means for a differential $d_P\colon P^i \longrightarrow P^{i+1}$ to be a d_B-contraderivation (i. e., an odd derivation of the contramodule P compatible with the derivation d on B). Imposing the equation $d_P^2(p) = hp$ for all $p \in P$, one obtains the definition of a left CDG-contramodule $P^\bullet$ over $B^\bullet$. Left CDG-contramodules over $B^\bullet$ form an abelian DG-category $B^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}}$.

Semiderived categories

Semiderived categories

Let $R \longrightarrow A$ be a homomorphism of associative rings.

Semiderived categories

Let $R \longrightarrow A$ be a homomorphism of associative rings.
The semicoderived category $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A)$ of right A-modules relative to R

Let $R \longrightarrow A$ be a homomorphism of associative rings. The semicoderived category $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A)$ of right A-modules relative to R is defined as the triangulated Verdier quotient category of the homotopy category of complexes of right A-modules

Let $R \longrightarrow A$ be a homomorphism of associative rings.
The semicoderived category $\mathrm{D}_R^{\mathrm{sico}}(\mathrm{Mod}\text{-}A)$ of right A-modules relative to R is defined as the triangulated Verdier quotient category of the homotopy category of complexes of right A-modules by the thick subcategory of complexes of A-modules that are coacyclic as complexes of R-modules.

Let $R \longrightarrow A$ be a homomorphism of associative rings. The semicoderived category $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A)$ of right A-modules relative to R is defined as the triangulated Verdier quotient category of the homotopy category of complexes of right A-modules by the thick subcategory of complexes of A-modules that are coacyclic as complexes of R-modules.

For example, if $R = A$, then $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A) = \mathrm{D}^{\mathrm{co}}(\mathrm{Mod}\text{-}A)$.

Semiderived categories

Let $R \longrightarrow A$ be a homomorphism of associative rings. The semicoderived category $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A)$ of right A-modules relative to R is defined as the triangulated Verdier quotient category of the homotopy category of complexes of right A-modules by the thick subcategory of complexes of A-modules that are coacyclic as complexes of R-modules.

For example, if $R = A$, then $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A) = \mathrm{D}^{\mathrm{co}}(\mathrm{Mod}\text{-}A)$.
If $R = \mathbb{Z}$, then $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A) = \mathrm{D}(\mathrm{Mod}\text{-}A)$.

Semiderived categories

Let $R \longrightarrow A$ be a homomorphism of associative rings. The semicoderived category $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A)$ of right A-modules relative to R is defined as the triangulated Verdier quotient category of the homotopy category of complexes of right A-modules by the thick subcategory of complexes of A-modules that are coacyclic as complexes of R-modules.

For example, if $R = A$, then $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A) = \mathrm{D}^{\mathrm{co}}(\mathrm{Mod}\text{-}A)$.
If $R = \mathbb{Z}$, then $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A) = \mathrm{D}(\mathrm{Mod}\text{-}A)$.

The semicontraderived category $\mathrm{D}^{\mathrm{sictr}}_R(A\text{-}\mathrm{Mod})$ of left A-modules relative to R

Let $R \longrightarrow A$ be a homomorphism of associative rings. The semicoderived category $\mathrm{D}_R^{\mathrm{sico}}(\mathrm{Mod}\text{-}A)$ of right A-modules relative to R is defined as the triangulated Verdier quotient category of the homotopy category of complexes of right A-modules by the thick subcategory of complexes of A-modules that are coacyclic as complexes of R-modules.

For example, if $R = A$, then $\mathrm{D}_R^{\mathrm{sico}}(\mathrm{Mod}\text{-}A) = \mathrm{D}^{\mathrm{co}}(\mathrm{Mod}\text{-}A)$.
If $R = \mathbb{Z}$, then $\mathrm{D}_R^{\mathrm{sico}}(\mathrm{Mod}\text{-}A) = \mathrm{D}(\mathrm{Mod}\text{-}A)$.

The semicontraderived category $\mathrm{D}_R^{\mathrm{sictr}}(A\text{-}\mathrm{Mod})$ of left A-modules relative to R is defined as the triangulated Verdier quotient category of the homotopy category of complexes of left A-modules

Semiderived categories

Let $R \longrightarrow A$ be a homomorphism of associative rings.
The semicoderived category $\mathrm{D}_R^{\mathrm{sico}}(\mathrm{Mod}\text{-}A)$ of right A-modules relative to R is defined as the triangulated Verdier quotient category of the homotopy category of complexes of right A-modules by the thick subcategory of complexes of A-modules that are coacyclic as complexes of R-modules.

For example, if $R = A$, then $\mathrm{D}_R^{\mathrm{sico}}(\mathrm{Mod}\text{-}A) = \mathrm{D}^{\mathrm{co}}(\mathrm{Mod}\text{-}A)$.
If $R = \mathbb{Z}$, then $\mathrm{D}_R^{\mathrm{sico}}(\mathrm{Mod}\text{-}A) = \mathrm{D}(\mathrm{Mod}\text{-}A)$.

The semicontraderived category $\mathrm{D}_R^{\mathrm{sictr}}(A\text{-}\mathrm{Mod})$ of left A-modules relative to R is defined as the triangulated Verdier quotient category of the homotopy category of complexes of left A-modules by the thick subcategory of complexes of A-modules that are contraacyclic as complexes of R-modules.

Semiderived categories

Let $R \longrightarrow A$ be a homomorphism of associative rings. The semicoderived category $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A)$ of right A-modules relative to R is defined as the triangulated Verdier quotient category of the homotopy category of complexes of right A-modules by the thick subcategory of complexes of A-modules that are coacyclic as complexes of R-modules.

For example, if $R = A$, then $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A) = \mathrm{D}^{\mathrm{co}}(\mathrm{Mod}\text{-}A)$. If $R = \mathbb{Z}$, then $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A) = \mathrm{D}(\mathrm{Mod}\text{-}A)$.

The semicontraderived category $\mathrm{D}^{\mathrm{sictr}}_R(A\text{-}\mathrm{Mod})$ of left A-modules relative to R is defined as the triangulated Verdier quotient category of the homotopy category of complexes of left A-modules by the thick subcategory of complexes of A-modules that are contraacyclic as complexes of R-modules.

For example, if $R = A$, then $\mathrm{D}^{\mathrm{sictr}}_R(A\text{-}\mathrm{Mod}) = \mathrm{D}^{\mathrm{ctr}}(A\text{-}\mathrm{Mod})$.

Semiderived categories

Let $R \longrightarrow A$ be a homomorphism of associative rings.
The semicoderived category $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A)$ of right A-modules relative to R is defined as the triangulated Verdier quotient category of the homotopy category of complexes of right A-modules by the thick subcategory of complexes of A-modules that are coacyclic as complexes of R-modules.

For example, if $R = A$, then $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A) = \mathrm{D}^{\mathrm{co}}(\mathrm{Mod}\text{-}A)$.
If $R = \mathbb{Z}$, then $\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}A) = \mathrm{D}(\mathrm{Mod}\text{-}A)$.

The semicontraderived category $\mathrm{D}^{\mathrm{sictr}}_R(A\text{-}\mathrm{Mod})$ of left A-modules relative to R is defined as the triangulated Verdier quotient category of the homotopy category of complexes of left A-modules by the thick subcategory of complexes of A-modules that are contraacyclic as complexes of R-modules.

For example, if $R = A$, then $\mathrm{D}^{\mathrm{sictr}}_R(A\text{-}\mathrm{Mod}) = \mathrm{D}^{\mathrm{ctr}}(A\text{-}\mathrm{Mod})$.
If $R = \mathbb{Z}$, then $\mathrm{D}^{\mathrm{sictr}}_R(A\text{-}\mathrm{Mod}) = \mathrm{D}(A\text{-}\mathrm{Mod})$.

Relative nonhomogeneous derived Koszul duality

Relative nonhomogeneous derived Koszul duality

Theorem (Derived Koszul duality over a base ring)

Relative nonhomogeneous derived Koszul duality

Theorem (Derived Koszul duality over a base ring)

Let $(\widetilde{A}, F)$ be a left finitely projective nonhomogeneous Koszul ring

Relative nonhomogeneous derived Koszul duality

Theorem (Derived Koszul duality over a base ring)

Let $(\widetilde{A}, F)$ be a left finitely projective nonhomogeneous Koszul ring and $B^{\blacklozenge} = (B, d, h)$ be the corresponding CDG-ring with a right finitely projective Koszul underlying graded ring B.

Relative nonhomogeneous derived Koszul duality

Theorem (Derived Koszul duality over a base ring)

Let $(\widetilde{A}, F)$ be a left finitely projective nonhomogeneous Koszul ring and $B^\bullet = (B, d, h)$ be the corresponding CDG-ring with a right finitely projective Koszul underlying graded ring B. Put $R = F_0\widetilde{A} = B^0$. In this context:

Relative nonhomogeneous derived Koszul duality

Theorem (Derived Koszul duality over a base ring)

Let $(\widetilde{A}, F)$ *be a left finitely projective nonhomogeneous Koszul ring and* $B^\bullet = (B, d, h)$ *be the corresponding CDG-ring with a right finitely projective Koszul underlying graded ring* B*. Put* $R = F_0\widetilde{A} = B^0$*. In this context:*

(a) *There is a natural equivalence of triangulated categories*

$$\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}\widetilde{A}) \simeq \mathrm{D}^{\mathrm{co}}(\mathrm{Comod}^{\mathrm{cdg}}\text{-}B^\bullet).$$

Relative nonhomogeneous derived Koszul duality

Theorem (Derived Koszul duality over a base ring)

Let $(\widetilde{A}, F)$ be a left finitely projective nonhomogeneous Koszul ring and $B^\bullet = (B, d, h)$ be the corresponding CDG-ring with a right finitely projective Koszul underlying graded ring B. Put $R = F_0\widetilde{A} = B^0$. In this context:

(a) *There is a natural equivalence of triangulated categories*

$$\mathrm{D}^{\mathrm{sico}}_R(\mathrm{Mod}\text{-}\widetilde{A}) \simeq \mathrm{D}^{\mathrm{co}}(\mathrm{Comod}^{\mathrm{cdg}}\text{-}B^\bullet).$$

(b) *There is a natural equivalence of triangulated categories*

$$\mathrm{D}^{\mathrm{sictr}}_R(\widetilde{A}\text{-}\mathrm{Mod}) \simeq \mathrm{D}^{\mathrm{ctr}}(B^\bullet\text{-}\mathrm{Contra}^{\mathrm{cdg}}).$$

Koszul duality functors

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Given a right CDG-comodule $N^\blacklozenge$ over $B^\blacklozenge$

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Given a right CDG-comodule $N^\bullet$ over $B^\bullet$, the related complex of right $\widetilde{A}$-modules is $N^\bullet \otimes_R^\tau \widetilde{A}$.

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Given a right CDG-comodule $N^\bullet$ over $B^\bullet$, the related complex of right $\widetilde{A}$-modules is $N^\bullet \otimes_R^\tau \widetilde{A}$. Given a left CDG-contramodule $Q^\bullet$ over $B^\bullet$

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Given a right CDG-comodule $N^\bullet$ over $B^\bullet$, the related complex of right $\widetilde{A}$-modules is $N^\bullet \otimes_R^\tau \widetilde{A}$. Given a left CDG-contramodule $Q^\bullet$ over $B^\bullet$, the related complex of left $\widetilde{A}$-modules is $\mathrm{Hom}_R^\tau(\widetilde{A}, Q^\bullet)$.

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Given a right CDG-comodule $N^\bullet$ over $B^\bullet$, the related complex of right $\widetilde{A}$-modules is $N^\bullet \otimes_R^\tau \widetilde{A}$. Given a left CDG-contramodule $Q^\bullet$ over $B^\bullet$, the related complex of left $\widetilde{A}$-modules is $\mathrm{Hom}_R^\tau(\widetilde{A}, Q^\bullet)$. The action of $\widetilde{A}$ on this tensor product/Hom space is induced by the action of $\widetilde{A}$ on itself.

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Given a right CDG-comodule $N^\bullet$ over $B^\bullet$, the related complex of right $\widetilde{A}$-modules is $N^\bullet \otimes_R^\tau \widetilde{A}$. Given a left CDG-contramodule $Q^\bullet$ over $B^\bullet$, the related complex of left $\widetilde{A}$-modules is $\mathrm{Hom}_R^\tau(\widetilde{A}, Q^\bullet)$. The action of $\widetilde{A}$ on this tensor product/Hom space is induced by the action of $\widetilde{A}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on N or Q).

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Given a right CDG-comodule $N^\blacklozenge$ over $B^\blacklozenge$, the related complex of right $\widetilde{A}$-modules is $N^\blacklozenge \otimes_R^\tau \widetilde{A}$. Given a left CDG-contramodule $Q^\blacklozenge$ over $B^\blacklozenge$, the related complex of left $\widetilde{A}$-modules is $\mathrm{Hom}_R^\tau(\widetilde{A}, Q^\blacklozenge)$. The action of $\widetilde{A}$ on this tensor product/Hom space is induced by the action of $\widetilde{A}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on N or Q).

Given a complex of right $\widetilde{A}$-modules $M^\bullet$

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Given a right CDG-comodule $N^\blacklozenge$ over $B^\blacklozenge$, the related complex of right $\widetilde{A}$-modules is $N^\blacklozenge \otimes_R^\tau \widetilde{A}$. Given a left CDG-contramodule $Q^\blacklozenge$ over $B^\blacklozenge$, the related complex of left $\widetilde{A}$-modules is $\mathrm{Hom}_R^\tau(\widetilde{A}, Q^\blacklozenge)$. The action of $\widetilde{A}$ on this tensor product/Hom space is induced by the action of $\widetilde{A}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on N or Q).

Given a complex of right $\widetilde{A}$-modules $M^\bullet$, the related right CDG-module over $B^\blacklozenge$ is $\mathrm{Hom}_R^{\Sigma,\tau}(B^\blacklozenge, M^\bullet)$

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Given a right CDG-comodule $N^\blacklozenge$ over $B^\blacklozenge$, the related complex of right $\widetilde{A}$-modules is $N^\blacklozenge \otimes_R^\tau \widetilde{A}$. Given a left CDG-contramodule $Q^\blacklozenge$ over $B^\blacklozenge$, the related complex of left $\widetilde{A}$-modules is $\mathrm{Hom}_R^\tau(\widetilde{A}, Q^\blacklozenge)$. The action of $\widetilde{A}$ on this tensor product/Hom space is induced by the action of $\widetilde{A}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on N or Q).

Given a complex of right $\widetilde{A}$-modules $M^\bullet$, the related right CDG-module over $B^\blacklozenge$ is $\mathrm{Hom}_R^{\Sigma,\tau}(B^\blacklozenge, M^\bullet) = \mathrm{Hom}_{R^{\mathrm{op}}}(B^\blacklozenge, R) \otimes_R^\tau M^\bullet$.

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Given a right CDG-comodule $N^\blacklozenge$ over $B^\blacklozenge$, the related complex of right $\widetilde{A}$-modules is $N^\blacklozenge \otimes_R^\tau \widetilde{A}$. Given a left CDG-contramodule $Q^\blacklozenge$ over $B^\blacklozenge$, the related complex of left $\widetilde{A}$-modules is $\mathrm{Hom}_R^\tau(\widetilde{A}, Q^\blacklozenge)$. The action of $\widetilde{A}$ on this tensor product/Hom space is induced by the action of $\widetilde{A}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on N or Q).

Given a complex of right $\widetilde{A}$-modules $M^\bullet$, the related right CDG-module over $B^\blacklozenge$ is $\mathrm{Hom}_R^{\Sigma,\tau}(B^\blacklozenge, M^\bullet) =$ $\mathrm{Hom}_{R^{\mathrm{op}}}(B^\blacklozenge, R) \otimes_R^\tau M^\bullet$. The symbol Σ means the direct sum totalization of the bigraded $\mathrm{Hom}_{R^{\mathrm{op}}}$ module.

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Given a right CDG-comodule $N^\blacklozenge$ over $B^\blacklozenge$, the related complex of right $\widetilde{A}$-modules is $N^\blacklozenge \otimes_R^\tau \widetilde{A}$. Given a left CDG-contramodule $Q^\blacklozenge$ over $B^\blacklozenge$, the related complex of left $\widetilde{A}$-modules is $\mathrm{Hom}_R^\tau(\widetilde{A}, Q^\blacklozenge)$. The action of $\widetilde{A}$ on this tensor product/Hom space is induced by the action of $\widetilde{A}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on N or Q).

Given a complex of right $\widetilde{A}$-modules $M^\bullet$, the related right CDG-module over $B^\blacklozenge$ is $\mathrm{Hom}_R^{\Sigma,\tau}(B^\blacklozenge, M^\bullet) =$
$\mathrm{Hom}_{R^{\mathrm{op}}}(B^\blacklozenge, R) \otimes_R^\tau M^\bullet$. The symbol Σ means the direct sum totalization of the bigraded $\mathrm{Hom}_{R^{\mathrm{op}}}$ module.

Given a complex of left $\widetilde{A}$-modules $P^\bullet$

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Given a right CDG-comodule $N^{\blacklozenge}$ over $B^{\blacklozenge}$, the related complex of right $\widetilde{A}$-modules is $N^{\blacklozenge} \otimes_R^{\tau} \widetilde{A}$. Given a left CDG-contramodule $Q^{\blacklozenge}$ over $B^{\blacklozenge}$, the related complex of left $\widetilde{A}$-modules is $\mathrm{Hom}_R^{\tau}(\widetilde{A}, Q^{\blacklozenge})$. The action of $\widetilde{A}$ on this tensor product/Hom space is induced by the action of $\widetilde{A}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on N or Q).

Given a complex of right $\widetilde{A}$-modules $M^{\bullet}$, the related right CDG-module over $B^{\blacklozenge}$ is $\mathrm{Hom}_R^{\Sigma,\tau}(B^{\blacklozenge}, M^{\bullet}) = \mathrm{Hom}_{R^{\mathrm{op}}}(B^{\blacklozenge}, R) \otimes_R^{\tau} M^{\bullet}$. The symbol Σ means the direct sum totalization of the bigraded $\mathrm{Hom}_{R^{\mathrm{op}}}$ module.

Given a complex of left $\widetilde{A}$-modules $P^{\bullet}$, the related left CDG-module over $B^{\blacklozenge}$ is $B^{\blacklozenge} \otimes_R^{\Pi} P^{\bullet}$

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Given a right CDG-comodule $N^\bullet$ over $B^\bullet$, the related complex of right $\widetilde{A}$-modules is $N^\bullet \otimes_R^\tau \widetilde{A}$. Given a left CDG-contramodule $Q^\bullet$ over $B^\bullet$, the related complex of left $\widetilde{A}$-modules is $\mathrm{Hom}_R^\tau(\widetilde{A}, Q^\bullet)$. The action of $\widetilde{A}$ on this tensor product/Hom space is induced by the action of $\widetilde{A}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on N or Q).

Given a complex of right $\widetilde{A}$-modules $M^\bullet$, the related right CDG-module over $B^\bullet$ is $\mathrm{Hom}_R^{\Sigma,\tau}(B^\bullet, M^\bullet) = \mathrm{Hom}_{R^{\mathrm{op}}}(B^\bullet, R) \otimes_R^\tau M^\bullet$. The symbol Σ means the direct sum totalization of the bigraded $\mathrm{Hom}_{R^{\mathrm{op}}}$ module.

Given a complex of left $\widetilde{A}$-modules $P^\bullet$, the related left CDG-module over $B^\bullet$ is $B^\bullet \otimes_R^\Pi P^\bullet = \mathrm{Hom}_R^\tau(\mathrm{Hom}_{R^{\mathrm{op}}}(B^\bullet, R),\ M^\bullet)$.

Koszul duality functors

The functors providing the category equivalences on the previous slide are constructed as follows.

Given a right CDG-comodule $N^\bullet$ over $B^\bullet$, the related complex of right $\widetilde{A}$-modules is $N^\bullet \otimes_R^\tau \widetilde{A}$. Given a left CDG-contramodule $Q^\bullet$ over $B^\bullet$, the related complex of left $\widetilde{A}$-modules is $\mathrm{Hom}_R^\tau(\widetilde{A}, Q^\bullet)$. The action of $\widetilde{A}$ on this tensor product/Hom space is induced by the action of $\widetilde{A}$ on itself. The symbol τ means a twisted differential (depending on the B-module structure on N or Q).

Given a complex of right $\widetilde{A}$-modules $M^\bullet$, the related right CDG-module over $B^\bullet$ is $\mathrm{Hom}_R^{\Sigma,\tau}(B^\bullet, M^\bullet) = \mathrm{Hom}_{R^{\mathrm{op}}}(B^\bullet, R) \otimes_R^\tau M^\bullet$. The symbol Σ means the direct sum totalization of the bigraded $\mathrm{Hom}_{R^{\mathrm{op}}}$ module.

Given a complex of left $\widetilde{A}$-modules $P^\bullet$, the related left CDG-module over $B^\bullet$ is $B^\bullet \otimes_R^\Pi P^\bullet = \mathrm{Hom}_R^\tau(\mathrm{Hom}_{R^{\mathrm{op}}}(B^\bullet, R), M^\bullet)$. The symbol Π means the direct product totalization of the bigraded module of tensor products over R.

Questions and explanations

Questions and explanations

Why the unusual direct sum totalization of Hom

Questions and explanations

Why the unusual direct sum totalization of Hom and the unusual direct product totalization of the tensor product?

Questions and explanations

Why the unusual direct sum totalization of Hom and the unusual direct product totalization of the tensor product?

One could also make one step back and ask a related question:

Questions and explanations

Why the unusual direct sum totalization of Hom and the unusual direct product totalization of the tensor product?

One could also make one step back and ask a related question: Why the strange concepts of comodules and contramodules over a graded ring?

Questions and explanations

Why the unusual direct sum totalization of Hom and the unusual direct product totalization of the tensor product?

One could also make one step back and ask a related question: Why the strange concepts of comodules and contramodules over a graded ring? Why not just graded modules?

Questions and explanations

Why the unusual direct sum totalization of Hom and the unusual direct product totalization of the tensor product?

One could also make one step back and ask a related question: Why the strange concepts of comodules and contramodules over a graded ring? Why not just graded modules?

One could make another step back and ask how the anti-equivalence between the categories

Questions and explanations

Why the unusual direct sum totalization of Hom and the unusual direct product totalization of the tensor product?

One could also make one step back and ask a related question: Why the strange concepts of comodules and contramodules over a graded ring? Why not just graded modules?

One could make another step back and ask how the anti-equivalence between the categories of nonhomogeneous Koszul rings and Koszul CDG-rings over a base ring R is constructed.

Questions and explanations

Why the unusual direct sum totalization of Hom and the unusual direct product totalization of the tensor product?

One could also make one step back and ask a related question: Why the strange concepts of comodules and contramodules over a graded ring? Why not just graded modules?

One could make another step back and ask how the anti-equivalence between the categories of nonhomogeneous Koszul rings and Koszul CDG-rings over a base ring R is constructed.

Concerning the latter question

Questions and explanations

Why the unusual direct sum totalization of Hom and the unusual direct product totalization of the tensor product?

One could also make one step back and ask a related question: Why the strange concepts of comodules and contramodules over a graded ring? Why not just graded modules?

One could make another step back and ask how the anti-equivalence between the categories of nonhomogeneous Koszul rings and Koszul CDG-rings over a base ring R is constructed.

Concerning the latter question, one answer is to write down explicit formulas expressing the differential and curvature element of $B^{\bullet}$

Questions and explanations

Why the unusual direct sum totalization of Hom and the unusual direct product totalization of the tensor product?

One could also make one step back and ask a related question: Why the strange concepts of comodules and contramodules over a graded ring? Why not just graded modules?

One could make another step back and ask how the anti-equivalence between the categories of nonhomogeneous Koszul rings and Koszul CDG-rings over a base ring R is constructed.

Concerning the latter question, one answer is to write down explicit formulas expressing the differential and curvature element of $B^{\bullet}$ in terms of the linear and scalar components of the nonhomogeneous quadratic relations in $\widetilde{A}$.

Questions and explanations

Why the unusual direct sum totalization of Hom and the unusual direct product totalization of the tensor product?

One could also make one step back and ask a related question: Why the strange concepts of comodules and contramodules over a graded ring? Why not just graded modules?

One could make another step back and ask how the anti-equivalence between the categories of nonhomogeneous Koszul rings and Koszul CDG-rings over a base ring R is constructed.

Concerning the latter question, one answer is to write down explicit formulas expressing the differential and curvature element of $B^{\bullet}$ in terms of the linear and scalar components of the nonhomogeneous quadratic relations in $\widetilde{A}$. The formulas are complicated and beautiful.

Questions and explanations

Why the unusual direct sum totalization of Hom and the unusual direct product totalization of the tensor product?

One could also make one step back and ask a related question: Why the strange concepts of comodules and contramodules over a graded ring? Why not just graded modules?

One could make another step back and ask how the anti-equivalence between the categories of nonhomogeneous Koszul rings and Koszul CDG-rings over a base ring R is constructed.

Concerning the latter question, one answer is to write down explicit formulas expressing the differential and curvature element of $B^\bullet$ in terms of the linear and scalar components of the nonhomogeneous quadratic relations in $\widetilde{A}$. The formulas are complicated and beautiful. (They are more complicated than in the absolute situation over a field k

Questions and explanations

Why the unusual direct sum totalization of Hom and the unusual direct product totalization of the tensor product?

One could also make one step back and ask a related question: Why the strange concepts of comodules and contramodules over a graded ring? Why not just graded modules?

One could make another step back and ask how the anti-equivalence between the categories of nonhomogeneous Koszul rings and Koszul CDG-rings over a base ring R is constructed.

Concerning the latter question, one answer is to write down explicit formulas expressing the differential and curvature element of $B^{\bullet}$ in terms of the linear and scalar components of the nonhomogeneous quadratic relations in $\widetilde{A}$. The formulas are complicated and beautiful. (They are more complicated than in the absolute situation over a field k, because R is not a central subring in $\widetilde{A}$

Questions and explanations

Why the unusual direct sum totalization of Hom and the unusual direct product totalization of the tensor product?

One could also make one step back and ask a related question: Why the strange concepts of comodules and contramodules over a graded ring? Why not just graded modules?

One could make another step back and ask how the anti-equivalence between the categories of nonhomogeneous Koszul rings and Koszul CDG-rings over a base ring R is constructed.

Concerning the latter question, one answer is to write down explicit formulas expressing the differential and curvature element of $B^\bullet$ in terms of the linear and scalar components of the nonhomogeneous quadratic relations in $\widetilde{A}$. The formulas are complicated and beautiful. (They are more complicated than in the absolute situation over a field k, because R is not a central subring in $\widetilde{A}$ and the differential d on B is not R-linear.)

Questions and explanations

Why the unusual direct sum totalization of Hom and the unusual direct product totalization of the tensor product?

One could also make one step back and ask a related question: Why the strange concepts of comodules and contramodules over a graded ring? Why not just graded modules?

One could make another step back and ask how the anti-equivalence between the categories of nonhomogeneous Koszul rings and Koszul CDG-rings over a base ring R is constructed.

Concerning the latter question, one answer is to write down explicit formulas expressing the differential and curvature element of $B^\bullet$ in terms of the linear and scalar components of the nonhomogeneous quadratic relations in $\widetilde{A}$. The formulas are complicated and beautiful. (They are more complicated than in the absolute situation over a field k, because R is not a central subring in $\widetilde{A}$ and the differential d on B is not R-linear.) That is how I originally arrived at this category anti-equivalence back in early '90s.

Questions and explanations

But one can also answer all the three questions with one observation

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^\blacklozenge$ "really wants to be" a coalgebra

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^{\blacklozenge}$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^{\blacklozenge}$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Viewed in these terms, the relative nonhomogeneous quadratic duality (the Poincaré–Birkhoff–Witt theorem)

But one can also answer all the three questions with one observation that the CDG-ring $B^\blacklozenge$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Viewed in these terms, the relative nonhomogeneous quadratic duality (the Poincaré–Birkhoff–Witt theorem) assigns left flat Koszul quasi-differential corings (= acyclic DG-corings)

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^\blacklozenge$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Viewed in these terms, the relative nonhomogeneous quadratic duality (the Poincaré–Birkhoff–Witt theorem) assigns left flat Koszul quasi-differential corings (= acyclic DG-corings) to left flat nonhomogeneous Koszul rings and back.

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^\blacklozenge$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Viewed in these terms, the relative nonhomogeneous quadratic duality (the Poincaré–Birkhoff–Witt theorem) assigns left flat Koszul quasi-differential corings (= acyclic DG-corings) to left flat nonhomogeneous Koszul rings and back.

However, the inverse hat construction only works for rings or coalgebras over a field

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^\blacklozenge$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Viewed in these terms, the relative nonhomogeneous quadratic duality (the Poincaré–Birkhoff–Witt theorem) assigns left flat Koszul quasi-differential corings (= acyclic DG-corings) to left flat nonhomogeneous Koszul rings and back.

However, the inverse hat construction only works for rings or coalgebras over a field, but not for corings.

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^\blacklozenge$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Viewed in these terms, the relative nonhomogeneous quadratic duality (the Poincaré–Birkhoff–Witt theorem) assigns left flat Koszul quasi-differential corings (= acyclic DG-corings) to left flat nonhomogeneous Koszul rings and back.

However, the inverse hat construction only works for rings or coalgebras over a field, but not for corings. One cannot extract a CDG-coring from an acyclic DG-coring.

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^{\blacklozenge}$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Viewed in these terms, the relative nonhomogeneous quadratic duality (the Poincaré–Birkhoff–Witt theorem) assigns left flat Koszul quasi-differential corings (= acyclic DG-corings) to left flat nonhomogeneous Koszul rings and back.

However, the inverse hat construction only works for rings or coalgebras over a field, but not for corings. One cannot extract a CDG-coring from an acyclic DG-coring. One can only assume finite projectivity

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^{\blacklozenge}$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Viewed in these terms, the relative nonhomogeneous quadratic duality (the Poincaré–Birkhoff–Witt theorem) assigns left flat Koszul quasi-differential corings (= acyclic DG-corings) to left flat nonhomogeneous Koszul rings and back.

However, the inverse hat construction only works for rings or coalgebras over a field, but not for corings. One cannot extract a CDG-coring from an acyclic DG-coring. One can only assume finite projectivity, apply $\mathrm{Hom}_R(-, R)$ functor to the acyclic DG-coring

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^\bullet$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Viewed in these terms, the relative nonhomogeneous quadratic duality (the Poincaré–Birkhoff–Witt theorem) assigns left flat Koszul quasi-differential corings (= acyclic DG-corings) to left flat nonhomogeneous Koszul rings and back.

However, the inverse hat construction only works for rings or coalgebras over a field, but not for corings. One cannot extract a CDG-coring from an acyclic DG-coring. One can only assume finite projectivity, apply $\mathrm{Hom}_R(-, R)$ functor to the acyclic DG-coring, obtain an acyclic DG-ring

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^{\blacklozenge}$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Viewed in these terms, the relative nonhomogeneous quadratic duality (the Poincaré–Birkhoff–Witt theorem) assigns left flat Koszul quasi-differential corings (= acyclic DG-corings) to left flat nonhomogeneous Koszul rings and back.

However, the inverse hat construction only works for rings or coalgebras over a field, but not for corings. One cannot extract a CDG-coring from an acyclic DG-coring. One can only assume finite projectivity, apply $\mathrm{Hom}_R(-, R)$ functor to the acyclic DG-coring, obtain an acyclic DG-ring, and extract a CDG-ring from it.

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^\blacklozenge$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Viewed in these terms, the relative nonhomogeneous quadratic duality (the Poincaré–Birkhoff–Witt theorem) assigns left flat Koszul quasi-differential corings (= acyclic DG-corings) to left flat nonhomogeneous Koszul rings and back.

However, the inverse hat construction only works for rings or coalgebras over a field, but not for corings. One cannot extract a CDG-coring from an acyclic DG-coring. One can only assume finite projectivity, apply $\mathrm{Hom}_R(-, R)$ functor to the acyclic DG-coring, obtain an acyclic DG-ring, and extract a CDG-ring from it.

For example, if $\widetilde{A} = \mathrm{Diff}(M)$ is a ring of differential operators on a variety M

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^\blacklozenge$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Viewed in these terms, the relative nonhomogeneous quadratic duality (the Poincaré–Birkhoff–Witt theorem) assigns left flat Koszul quasi-differential corings (= acyclic DG-corings) to left flat nonhomogeneous Koszul rings and back.

However, the inverse hat construction only works for rings or coalgebras over a field, but not for corings. One cannot extract a CDG-coring from an acyclic DG-coring. One can only assume finite projectivity, apply $\mathrm{Hom}_R(-, R)$ functor to the acyclic DG-coring, obtain an acyclic DG-ring, and extract a CDG-ring from it.

For example, if $\widetilde{A} = \mathrm{Diff}(M)$ is a ring of differential operators on a variety M, then the related (C)DG-ring is the de Rham DG-algebra $(\Omega(M), d_{dR})$.

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^\blacklozenge$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Viewed in these terms, the relative nonhomogeneous quadratic duality (the Poincaré–Birkhoff–Witt theorem) assigns left flat Koszul quasi-differential corings (= acyclic DG-corings) to left flat nonhomogeneous Koszul rings and back.

However, the inverse hat construction only works for rings or coalgebras over a field, but not for corings. One cannot extract a CDG-coring from an acyclic DG-coring. One can only assume finite projectivity, apply $\mathrm{Hom}_R(-, R)$ functor to the acyclic DG-coring, obtain an acyclic DG-ring, and extract a CDG-ring from it.

For example, if $\widetilde{A} = \mathrm{Diff}(M)$ is a ring of differential operators on a variety M, then the related (C)DG-ring is the de Rham DG-algebra $(\Omega(M), d_{dR})$. A related CDG-coring would involve a "de Rham differential" on the coring of polyvector fields on M.

Questions and explanations

But one can also answer all the three questions with one observation that the CDG-ring $B^\bullet$ "really wants to be" a coalgebra, or rather a coring over the ring R.

Viewed in these terms, the relative nonhomogeneous quadratic duality (the Poincaré–Birkhoff–Witt theorem) assigns left flat Koszul quasi-differential corings (= acyclic DG-corings) to left flat nonhomogeneous Koszul rings and back.

However, the inverse hat construction only works for rings or coalgebras over a field, but not for corings. One cannot extract a CDG-coring from an acyclic DG-coring. One can only assume finite projectivity, apply $\mathrm{Hom}_R(-,R)$ functor to the acyclic DG-coring, obtain an acyclic DG-ring, and extract a CDG-ring from it.

For example, if $\widetilde{A} = \mathrm{Diff}(M)$ is a ring of differential operators on a variety M, then the related (C)DG-ring is the de Rham DG-algebra $(\Omega(M), d_{dR})$. A related CDG-coring would involve a "de Rham differential" on the coring of polyvector fields on M. There is no such differential on the polyvector fields.

Quadratic dual coring

Quadratic dual coring

A coring over an associative ring R is comonoid object in the monoidal category of R-R-bimodules.

Quadratic dual coring

A coring over an associative ring R is comonoid object in the monoidal category of R-R-bimodules. In other words, an R-coring $\mathcal{C}$ is an R-R-bimodule

Quadratic dual coring

A coring over an associative ring R is comonoid object in the monoidal category of R-R-bimodules. In other words, an R-coring $\mathcal{C}$ is an R-R-bimodule endowed with R-R-bimodule maps of comultiplication $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_R \mathcal{C}$ and counit $\epsilon\colon \mathcal{C} \longrightarrow R$.

Quadratic dual coring

A coring over an associative ring R is comonoid object in the monoidal category of R-R-bimodules. In other words, an R-coring $\mathcal{C}$ is an R-R-bimodule endowed with R-R-bimodule maps of comultiplication $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_R \mathcal{C}$ and counit $\epsilon\colon \mathcal{C} \longrightarrow R$.
The usual coassociativity and counitality axioms are imposed.

Quadratic dual coring

A coring over an associative ring R is comonoid object in the monoidal category of R-R-bimodules. In other words, an R-coring $\mathcal{C}$ is an R-R-bimodule endowed with R-R-bimodule maps of comultiplication $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_R \mathcal{C}$ and counit $\epsilon\colon \mathcal{C} \longrightarrow R$. The usual coassociativity and counitality axioms are imposed.

Let A be a left flat Koszul graded ring over R.

Quadratic dual coring

A coring over an associative ring R is comonoid object in the monoidal category of R-R-bimodules. In other words, an R-coring $\mathcal{C}$ is an R-R-bimodule endowed with R-R-bimodule maps of comultiplication $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_R \mathcal{C}$ and counit $\epsilon\colon \mathcal{C} \longrightarrow R$. The usual coassociativity and counitality axioms are imposed.

Let A be a left flat Koszul graded ring over R. The quadratic dual graded coring $\mathcal{C} = A^!$ is constructed as follows.

Quadratic dual coring

A coring over an associative ring R is comonoid object in the monoidal category of R-R-bimodules. In other words, an R-coring $\mathcal{C}$ is an R-R-bimodule endowed with R-R-bimodule maps of comultiplication $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_R \mathcal{C}$ and counit $\epsilon\colon \mathcal{C} \longrightarrow R$. The usual coassociativity and counitality axioms are imposed.

Let A be a left flat Koszul graded ring over R. The quadratic dual graded coring $\mathcal{C} = A^!$ is constructed as follows.

Recall the notation $I_A = \ker(A_1 \otimes_R A_1 \to A_2)$.

Quadratic dual coring

A coring over an associative ring R is comonoid object in the monoidal category of R-R-bimodules. In other words, an R-coring $\mathcal{C}$ is an R-R-bimodule endowed with R-R-bimodule maps of comultiplication $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_R \mathcal{C}$ and counit $\epsilon\colon \mathcal{C} \longrightarrow R$. The usual coassociativity and counitality axioms are imposed.

Let A be a left flat Koszul graded ring over R. The quadratic dual graded coring $\mathcal{C} = A^!$ is constructed as follows.

Recall the notation $I_A = \ker(A_1 \otimes_R A_1 \to A_2)$. Then one has $A = T_R(A_1)/(I_A)$

Quadratic dual coring

A coring over an associative ring R is comonoid object in the monoidal category of R-R-bimodules. In other words, an R-coring $\mathcal{C}$ is an R-R-bimodule endowed with R-R-bimodule maps of comultiplication $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_R \mathcal{C}$ and counit $\epsilon\colon \mathcal{C} \longrightarrow R$. The usual coassociativity and counitality axioms are imposed.

Let A be a left flat Koszul graded ring over R. The quadratic dual graded coring $\mathcal{C} = A^!$ is constructed as follows.

Recall the notation $I_A = \ker(A_1 \otimes_R A_1 \to A_2)$. Then one has $A = T_R(A_1)/(I_A)$, where $T_R(A_1) = \bigoplus_{n=0}^{\infty} A_1^{\otimes_R n}$ is the tensor ring of the R-R-bimodule A_1

Quadratic dual coring

A coring over an associative ring R is comonoid object in the monoidal category of R-R-bimodules. In other words, an R-coring $\mathcal{C}$ is an R-R-bimodule endowed with R-R-bimodule maps of comultiplication $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_R \mathcal{C}$ and counit $\epsilon\colon \mathcal{C} \longrightarrow R$. The usual coassociativity and counitality axioms are imposed.

Let A be a left flat Koszul graded ring over R. The quadratic dual graded coring $\mathcal{C} = A^!$ is constructed as follows.

Recall the notation $I_A = \ker(A_1 \otimes_R A_1 \to A_2)$. Then one has $A = T_R(A_1)/(I_A)$, where $T_R(A_1) = \bigoplus_{n=0}^{\infty} A_1^{\otimes_R n}$ is the tensor ring of the R-R-bimodule A_1 and $(I_A) \subset T_R(A_1)$ is the ideal spanned by the subbimodule $I_A \subset A_1 \otimes_R A_1$.

Quadratic dual coring

A coring over an associative ring R is comonoid object in the monoidal category of R-R-bimodules. In other words, an R-coring $\mathcal{C}$ is an R-R-bimodule endowed with R-R-bimodule maps of comultiplication $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_R \mathcal{C}$ and counit $\epsilon\colon \mathcal{C} \longrightarrow R$. The usual coassociativity and counitality axioms are imposed.

Let A be a left flat Koszul graded ring over R. The quadratic dual graded coring $\mathcal{C} = A^!$ is constructed as follows.

Recall the notation $I_A = \ker(A_1 \otimes_R A_1 \to A_2)$. Then one has $A = T_R(A_1)/(I_A)$, where $T_R(A_1) = \bigoplus_{n=0}^{\infty} A_1^{\otimes_R n}$ is the tensor ring of the R-R-bimodule A_1 and $(I_A) \subset T_R(A_1)$ is the ideal spanned by the subbimodule $I_A \subset A_1 \otimes_R A_1$. Put

- $\mathcal{C}_0 = R$, $\mathcal{C}_1 = A_1$, $\mathcal{C}_2 = I_A$,

Quadratic dual coring

A coring over an associative ring R is comonoid object in the monoidal category of R-R-bimodules. In other words, an R-coring $\mathcal{C}$ is an R-R-bimodule endowed with R-R-bimodule maps of comultiplication $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_R \mathcal{C}$ and counit $\epsilon\colon \mathcal{C} \longrightarrow R$. The usual coassociativity and counitality axioms are imposed.

Let A be a left flat Koszul graded ring over R. The quadratic dual graded coring $\mathcal{C} = A^!$ is constructed as follows.

Recall the notation $I_A = \ker(A_1 \otimes_R A_1 \to A_2)$. Then one has $A = T_R(A_1)/(I_A)$, where $T_R(A_1) = \bigoplus_{n=0}^{\infty} A_1^{\otimes_R n}$ is the tensor ring of the R-R-bimodule A_1 and $(I_A) \subset T_R(A_1)$ is the ideal spanned by the subbimodule $I_A \subset A_1 \otimes_R A_1$. Put

- $\mathcal{C}_0 = R$, $\mathcal{C}_1 = A_1$, $\mathcal{C}_2 = I_A$,
- $\mathcal{C}_3 = (I_A \otimes_R A_1) \cap (A_1 \otimes_R I_A) \subset A_1 \otimes_R A_1 \otimes_R A_1$;

Quadratic dual coring

A coring over an associative ring R is comonoid object in the monoidal category of R-R-bimodules. In other words, an R-coring $\mathcal{C}$ is an R-R-bimodule endowed with R-R-bimodule maps of comultiplication $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_R \mathcal{C}$ and counit $\epsilon\colon \mathcal{C} \longrightarrow R$. The usual coassociativity and counitality axioms are imposed.

Let A be a left flat Koszul graded ring over R. The quadratic dual graded coring $\mathcal{C} = A^!$ is constructed as follows.

Recall the notation $I_A = \ker(A_1 \otimes_R A_1 \to A_2)$. Then one has $A = T_R(A_1)/(I_A)$, where $T_R(A_1) = \bigoplus_{n=0}^\infty A_1^{\otimes_R n}$ is the tensor ring of the R-R-bimodule A_1 and $(I_A) \subset T_R(A_1)$ is the ideal spanned by the subbimodule $I_A \subset A_1 \otimes_R A_1$. Put

- $\mathcal{C}_0 = R$, $\mathcal{C}_1 = A_1$, $\mathcal{C}_2 = I_A$,
- $\mathcal{C}_3 = (I_A \otimes_R A_1) \cap (A_1 \otimes_R I_A) \subset A_1 \otimes_R A_1 \otimes_R A_1$;
- $\mathcal{C}_n = \bigcap_{k=1}^{n-1} (A_1^{\otimes_R k-1} \otimes_R I_A \otimes_R A_1^{\otimes_R n-k-1}) \subset A_1^{\otimes_R n}$, $n \geqslant 2$.

Quadratic dual coring

A coring over an associative ring R is comonoid object in the monoidal category of R-R-bimodules. In other words, an R-coring $\mathcal{C}$ is an R-R-bimodule endowed with R-R-bimodule maps of comultiplication $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_R \mathcal{C}$ and counit $\epsilon\colon \mathcal{C} \longrightarrow R$. The usual coassociativity and counitality axioms are imposed.

Let A be a left flat Koszul graded ring over R. The quadratic dual graded coring $\mathcal{C} = A^!$ is constructed as follows.

Recall the notation $I_A = \ker(A_1 \otimes_R A_1 \to A_2)$. Then one has $A = T_R(A_1)/(I_A)$, where $T_R(A_1) = \bigoplus_{n=0}^{\infty} A_1^{\otimes_R n}$ is the tensor ring of the R-R-bimodule A_1 and $(I_A) \subset T_R(A_1)$ is the ideal spanned by the subbimodule $I_A \subset A_1 \otimes_R A_1$. Put

- $\mathcal{C}_0 = R,\ \ \mathcal{C}_1 = A_1,\ \ \mathcal{C}_2 = I_A,$
- $\mathcal{C}_3 = (I_A \otimes_R A_1) \cap (A_1 \otimes_R I_A) \subset A_1 \otimes_R A_1 \otimes_R A_1;$
- $\mathcal{C}_n = \bigcap_{k=1}^{n-1} (A_1^{\otimes_R k-1} \otimes_R I_A \otimes_R A_1^{\otimes_R n-k-1}) \subset A_1^{\otimes_R n},\ \ n \geqslant 2.$

So $\mathcal{C}$ is a graded subcoring of the tensor coring $\bigoplus_{n=0}^{\infty} A_1^{\otimes_R n}$.

Quasi-differential corings

Quasi-differential corings

A quasi-differential coring $(\mathcal{D}, \partial)$ over a ring R

Quasi-differential corings

A quasi-differential coring $(\mathcal{D}, \partial)$ over a ring R is a graded R-coring $\mathcal{D} = \bigoplus_{n\in\mathbb{Z}} \mathcal{D}_n$ endowed with an R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$

Quasi-differential corings

A quasi-differential coring $(\mathcal{D}, \partial)$ over a ring R is a graded R-coring $\mathcal{D} = \bigoplus_{n\in\mathbb{Z}} \mathcal{D}_n$ endowed with an R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$ such that $H^{\partial}_*(\mathcal{D}) = 0$.

Quasi-differential corings

A quasi-differential coring $(\mathcal{D}, \partial)$ over a ring R is a graded R-coring $\mathcal{D} = \bigoplus_{n\in\mathbb{Z}} \mathcal{D}_n$ endowed with an R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$ such that $H^{\partial}_*(\mathcal{D}) = 0$.

Let us say that a graded R-coring $\mathcal{C} = \bigoplus_{n=0}^{\infty} \mathcal{C}_n$ is left flat Koszul

Quasi-differential corings

A quasi-differential coring $(\mathcal{D}, \partial)$ over a ring R is a graded R-coring $\mathcal{D} = \bigoplus_{n\in\mathbb{Z}} \mathcal{D}_n$ endowed with an R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$ such that $H^{\partial}_*(\mathcal{D}) = 0$.

Let us say that a graded R-coring $\mathcal{C} = \bigoplus_{n=0}^{\infty} \mathcal{C}_n$ is left flat Koszul if it is quadratic dual to a left flat Koszul graded ring.

Quasi-differential corings

A quasi-differential coring $(\mathcal{D}, \partial)$ over a ring R is a graded R-coring $\mathcal{D} = \bigoplus_{n\in\mathbb{Z}} \mathcal{D}_n$ endowed with an R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$ such that $H^{\partial}_{*}(\mathcal{D}) = 0$.

Let us say that a graded R-coring $\mathcal{C} = \bigoplus_{n=0}^{\infty} \mathcal{C}_n$ is left flat Koszul if it is quadratic dual to a left flat Koszul graded ring. A more substantial definition can be given in terms of the derived functor of cotensor product of $\mathcal{C}$-comodules.

Quasi-differential corings

A quasi-differential coring $(\mathcal{D}, \partial)$ over a ring R is a graded R-coring $\mathcal{D} = \bigoplus_{n\in\mathbb{Z}} \mathcal{D}_n$ endowed with an R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$ such that $H^{\partial}_{*}(\mathcal{D}) = 0$.

Let us say that a graded R-coring $\mathcal{C} = \bigoplus_{n=0}^{\infty} \mathcal{C}_n$ is left flat Koszul if it is quadratic dual to a left flat Koszul graded ring. A more substantial definition can be given in terms of the derived functor of cotensor product of $\mathcal{C}$-comodules.

A quasi-differential coring $(\mathcal{D}, \partial)$ is called left flat Koszul

Quasi-differential corings

A quasi-differential coring $(\mathcal{D}, \partial)$ over a ring R is a graded R-coring $\mathcal{D} = \bigoplus_{n\in\mathbb{Z}} \mathcal{D}_n$ endowed with an R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$ such that $H_*^\partial(\mathcal{D}) = 0$.

Let us say that a graded R-coring $\mathcal{C} = \bigoplus_{n=0}^\infty \mathcal{C}_n$ is left flat Koszul if it is quadratic dual to a left flat Koszul graded ring. A more substantial definition can be given in terms of the derived functor of cotensor product of $\mathcal{C}$-comodules.

A quasi-differential coring $(\mathcal{D}, \partial)$ is called left flat Koszul if $\mathcal{D} = \bigoplus_{n=0}^\infty \mathcal{D}_n$ is nonnegatively graded with $\mathcal{D}_0 = R$

Quasi-differential corings

A quasi-differential coring $(\mathcal{D}, \partial)$ over a ring R is a graded R-coring $\mathcal{D} = \bigoplus_{n\in\mathbb{Z}} \mathcal{D}_n$ endowed with an R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$ such that $H^\partial_*(\mathcal{D}) = 0$.

Let us say that a graded R-coring $\mathcal{C} = \bigoplus_{n=0}^\infty \mathcal{C}_n$ is left flat Koszul if it is quadratic dual to a left flat Koszul graded ring. A more substantial definition can be given in terms of the derived functor of cotensor product of $\mathcal{C}$-comodules.

A quasi-differential coring $(\mathcal{D}, \partial)$ is called left flat Koszul if $\mathcal{D} = \bigoplus_{n=0}^\infty \mathcal{D}_n$ is nonnegatively graded with $\mathcal{D}_0 = R$, the graded R-R-bimodule $\operatorname{coker}(\partial\colon \mathcal{D} \to \mathcal{D})$ is flat as a left R-module

Quasi-differential corings

A quasi-differential coring $(\mathcal{D}, \partial)$ over a ring R is a graded R-coring $\mathcal{D} = \bigoplus_{n\in\mathbb{Z}} \mathcal{D}_n$ endowed with an R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$ such that $H^{\partial}_*(\mathcal{D}) = 0$.

Let us say that a graded R-coring $\mathcal{C} = \bigoplus_{n=0}^{\infty} \mathcal{C}_n$ is left flat Koszul if it is quadratic dual to a left flat Koszul graded ring. A more substantial definition can be given in terms of the derived functor of cotensor product of $\mathcal{C}$-comodules.

A quasi-differential coring $(\mathcal{D}, \partial)$ is called left flat Koszul if $\mathcal{D} = \bigoplus_{n=0}^{\infty} \mathcal{D}_n$ is nonnegatively graded with $\mathcal{D}_0 = R$, the graded R-R-bimodule $\operatorname{coker}(\partial\colon \mathcal{D} \to \mathcal{D})$ is flat as a left R-module, and the induced coring structure on $\mathcal{C} = \operatorname{coker}(\partial)$ makes $\mathcal{C}$ a left flat Koszul graded coring.

Quasi-differential corings

A quasi-differential coring $(\mathcal{D}, \partial)$ over a ring R is a graded R-coring $\mathcal{D} = \bigoplus_{n\in\mathbb{Z}} \mathcal{D}_n$ endowed with an R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$ such that $H^\partial_*(\mathcal{D}) = 0$.

Let us say that a graded R-coring $\mathcal{C} = \bigoplus_{n=0}^\infty \mathcal{C}_n$ is left flat Koszul if it is quadratic dual to a left flat Koszul graded ring. A more substantial definition can be given in terms of the derived functor of cotensor product of $\mathcal{C}$-comodules.

A quasi-differential coring $(\mathcal{D}, \partial)$ is called left flat Koszul if $\mathcal{D} = \bigoplus_{n=0}^\infty \mathcal{D}_n$ is nonnegatively graded with $\mathcal{D}_0 = R$, the graded R-R-bimodule $\operatorname{coker}(\partial\colon \mathcal{D} \to \mathcal{D})$ is flat as a left R-module, and the induced coring structure on $\mathcal{C} = \operatorname{coker}(\partial)$ makes $\mathcal{C}$ a left flat Koszul graded coring.

Theorem (Poincaré–Birkhoff–Witt theorem over a base ring II)

Quasi-differential corings

A quasi-differential coring $(\mathcal{D}, \partial)$ over a ring R is a graded R-coring $\mathcal{D} = \bigoplus_{n\in\mathbb{Z}} \mathcal{D}_n$ endowed with an R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$ such that $H^\partial_*(\mathcal{D}) = 0$.

Let us say that a graded R-coring $\mathcal{C} = \bigoplus_{n=0}^\infty \mathcal{C}_n$ is left flat Koszul if it is quadratic dual to a left flat Koszul graded ring. A more substantial definition can be given in terms of the derived functor of cotensor product of $\mathcal{C}$-comodules.

A quasi-differential coring $(\mathcal{D}, \partial)$ is called left flat Koszul if $\mathcal{D} = \bigoplus_{n=0}^\infty \mathcal{D}_n$ is nonnegatively graded with $\mathcal{D}_0 = R$, the graded R-R-bimodule $\mathrm{coker}(\partial\colon \mathcal{D} \to \mathcal{D})$ is flat as a left R-module, and the induced coring structure on $\mathcal{C} = \mathrm{coker}(\partial)$ makes $\mathcal{C}$ a left flat Koszul graded coring.

Theorem (Poincaré–Birkhoff–Witt theorem over a base ring II)

There is a natural equivalence between the category of left flat nonhomogeneous Koszul rings $(\widetilde{A}, F)$

Quasi-differential corings

A quasi-differential coring $(\mathcal{D}, \partial)$ over a ring R is a graded R-coring $\mathcal{D} = \bigoplus_{n\in\mathbb{Z}} \mathcal{D}_n$ endowed with an R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n+1}$ such that $H^{\partial}_{*}(\mathcal{D}) = 0$.

Let us say that a graded R-coring $\mathcal{C} = \bigoplus_{n=0}^{\infty} \mathcal{C}_n$ is left flat Koszul if it is quadratic dual to a left flat Koszul graded ring. A more substantial definition can be given in terms of the derived functor of cotensor product of $\mathcal{C}$-comodules.

A quasi-differential coring $(\mathcal{D}, \partial)$ is called left flat Koszul if $\mathcal{D} = \bigoplus_{n=0}^{\infty} \mathcal{D}_n$ is nonnegatively graded with $\mathcal{D}_0 = R$, the graded R-R-bimodule $\operatorname{coker}(\partial\colon \mathcal{D} \to \mathcal{D})$ is flat as a left R-module, and the induced coring structure on $\mathcal{C} = \operatorname{coker}(\partial)$ makes $\mathcal{C}$ a left flat Koszul graded coring.

Theorem (Poincaré–Birkhoff–Witt theorem over a base ring II)

There is a natural equivalence between the category of left flat nonhomogeneous Koszul rings $(\widetilde{A}, F)$ and the category of left flat Koszul quasi-differential corings $(\mathcal{D}, d)$.

Construction of the category equivalence

Construction of the category equivalence

Let $(\widetilde{A}, F)$ be a left flat nonhomogeneous Koszul ring.

Construction of the category equivalence

Let $(\widetilde{A}, F)$ be a left flat nonhomogeneous Koszul ring. Consider the Rees ring $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$.

Construction of the category equivalence

Let $(\widetilde{A}, F)$ be a left flat nonhomogeneous Koszul ring. Consider the Rees ring $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$. Then L is a left flat Koszul graded ring over R.

Construction of the category equivalence

Let $(\widetilde{A}, F)$ be a left flat nonhomogeneous Koszul ring. Consider the Rees ring $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$. Then L is a left flat Koszul graded ring over R. Let $\mathcal{D} = L^!$ be the quadratic dual coring to L.

Construction of the category equivalence

Let $(\widetilde{A}, F)$ be a left flat nonhomogeneous Koszul ring. Consider the Rees ring $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$. Then L is a left flat Koszul graded ring over R. Let $\mathcal{D} = L^!$ be the quadratic dual coring to L.

We have the natural inclusion map
$t\colon L_0 = R = F_0\widetilde{A} \longrightarrow F_1\widetilde{A} = L_1$.

Construction of the category equivalence

Let $(\widetilde{A}, F)$ be a left flat nonhomogeneous Koszul ring. Consider the Rees ring $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$. Then L is a left flat Koszul graded ring over R. Let $\mathcal{D} = L^!$ be the quadratic dual coring to L.

We have the natural inclusion map $t\colon L_0 = R = F_0\widetilde{A} \longrightarrow F_1\widetilde{A} = L_1$. There exists a unique R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n-1}$

Construction of the category equivalence

Let $(\widetilde{A}, F)$ be a left flat nonhomogeneous Koszul ring. Consider the Rees ring $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$. Then L is a left flat Koszul graded ring over R. Let $\mathcal{D} = L^!$ be the quadratic dual coring to L.

We have the natural inclusion map
$t\colon L_0 = R = F_0\widetilde{A} \longrightarrow F_1\widetilde{A} = L_1$. There exists a unique R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n-1}$ acting on $\mathcal{D}_0 = R$ by the map $t\colon \mathcal{D}_0 \longrightarrow \mathcal{D}_1 = F_1\widetilde{A}$.

Construction of the category equivalence

Let $(\widetilde{A}, F)$ be a left flat nonhomogeneous Koszul ring. Consider the Rees ring $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$. Then L is a left flat Koszul graded ring over R. Let $\mathcal{D} = L^!$ be the quadratic dual coring to L.

We have the natural inclusion map
$t\colon L_0 = R = F_0\widetilde{A} \longrightarrow F_1\widetilde{A} = L_1$. There exists a unique R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n-1}$ acting on $\mathcal{D}_0 = R$ by the map $t\colon \mathcal{D}_0 \longrightarrow \mathcal{D}_1 = F_1\widetilde{A}$.

The pair $(\mathcal{D}, \partial)$ is the left flat Koszul quasi-differential coring corresponding to $(\widetilde{A}, F)$.

Construction of the category equivalence

Let $(\widetilde{A}, F)$ be a left flat nonhomogeneous Koszul ring. Consider the Rees ring $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$. Then L is a left flat Koszul graded ring over R. Let $\mathcal{D} = L^!$ be the quadratic dual coring to L.

We have the natural inclusion map $t\colon L_0 = R = F_0\widetilde{A} \longrightarrow F_1\widetilde{A} = L_1$. There exists a unique R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n-1}$ acting on $\mathcal{D}_0 = R$ by the map $t\colon \mathcal{D}_0 \longrightarrow \mathcal{D}_1 = F_1\widetilde{A}$.

The pair $(\mathcal{D}, \partial)$ is the left flat Koszul quasi-differential coring corresponding to $(\widetilde{A}, F)$. The coring $\mathcal{C} = \mathrm{coker}(\partial)$ is quadratic dual to the left flat Koszul graded ring $A = \mathrm{gr}^F \widetilde{A}$.

Construction of the category equivalence

Let $(\widetilde{A}, F)$ be a left flat nonhomogeneous Koszul ring. Consider the Rees ring $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$. Then L is a left flat Koszul graded ring over R. Let $\mathcal{D} = L^!$ be the quadratic dual coring to L.

We have the natural inclusion map $t\colon L_0 = R = F_0\widetilde{A} \longrightarrow F_1\widetilde{A} = L_1$. There exists a unique R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n-1}$ acting on $\mathcal{D}_0 = R$ by the map $t\colon \mathcal{D}_0 \longrightarrow \mathcal{D}_1 = F_1\widetilde{A}$.

The pair $(\mathcal{D}, \partial)$ is the left flat Koszul quasi-differential coring corresponding to $(\widetilde{A}, F)$. The coring $\mathcal{C} = \operatorname{coker}(\partial)$ is quadratic dual to the left flat Koszul graded ring $A = \operatorname{gr}^F \widetilde{A}$.

For any R-coring $\mathcal{D}$, there is a natural associative ring structure on the R-R-bimodule $\operatorname{Hom}_R(\mathcal{D}, R)$

Construction of the category equivalence

Let $(\widetilde{A}, F)$ be a left flat nonhomogeneous Koszul ring. Consider the Rees ring $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$. Then L is a left flat Koszul graded ring over R. Let $\mathcal{D} = L^!$ be the quadratic dual coring to L.

We have the natural inclusion map $t\colon L_0 = R = F_0\widetilde{A} \longrightarrow F_1\widetilde{A} = L_1$. There exists a unique R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n-1}$ acting on $\mathcal{D}_0 = R$ by the map $t\colon \mathcal{D}_0 \longrightarrow \mathcal{D}_1 = F_1\widetilde{A}$.

The pair $(\mathcal{D}, \partial)$ is the left flat Koszul quasi-differential coring corresponding to $(\widetilde{A}, F)$. The coring $\mathcal{C} = \operatorname{coker}(\partial)$ is quadratic dual to the left flat Koszul graded ring $A = \operatorname{gr}^F\widetilde{A}$.

For any R-coring $\mathcal{D}$, there is a natural associative ring structure on the R-R-bimodule $\operatorname{Hom}_R(\mathcal{D}, R)$, together with a ring homomorphism $R \longrightarrow \operatorname{Hom}_R(\mathcal{D}, R)$.

Construction of the category equivalence

Let $(\widetilde{A}, F)$ be a left flat nonhomogeneous Koszul ring. Consider the Rees ring $L = \bigoplus_{n=0}^{\infty} F_n\widetilde{A}$. Then L is a left flat Koszul graded ring over R. Let $\mathcal{D} = L^!$ be the quadratic dual coring to L.

We have the natural inclusion map $t\colon L_0 = R = F_0\widetilde{A} \longrightarrow F_1\widetilde{A} = L_1$. There exists a unique R-R-bilinear odd coderivation $\partial\colon \mathcal{D}_n \longrightarrow \mathcal{D}_{n-1}$ acting on $\mathcal{D}_0 = R$ by the map $t\colon \mathcal{D}_0 \longrightarrow \mathcal{D}_1 = F_1\widetilde{A}$.

The pair $(\mathcal{D}, \partial)$ is the left flat Koszul quasi-differential coring corresponding to $(\widetilde{A}, F)$. The coring $\mathcal{C} = \operatorname{coker}(\partial)$ is quadratic dual to the left flat Koszul graded ring $A = \operatorname{gr}^F\widetilde{A}$.

For any R-coring $\mathcal{D}$, there is a natural associative ring structure on the R-R-bimodule $\operatorname{Hom}_R(\mathcal{D}, R)$, together with a ring homomorphism $R \longrightarrow \operatorname{Hom}_R(\mathcal{D}, R)$. The R-R-bimodule structure on $\operatorname{Hom}_R(\mathcal{D}, R)$ is induced by the latter ring homomorphism.

Construction of the category anti-equivalence

Construction of the category anti-equivalence

Now assume that $(\widetilde{A}, F)$ is a left finitely projective nonhomogeneous Koszul ring.

Construction of the category anti-equivalence

Now assume that $(\widetilde{A}, F)$ is a left finitely projective nonhomogeneous Koszul ring. Then the ring $\mathrm{Hom}_R(\mathcal{D}, R)$ with the odd derivation $\mathrm{Hom}_R(\partial, R)$ is an acyclic DG-ring.

Construction of the category anti-equivalence

Now assume that $(\widetilde{A}, F)$ is a left finitely projective nonhomogeneous Koszul ring. Then the ring $\mathrm{Hom}_R(\mathcal{D}, R)$ with the odd derivation $\mathrm{Hom}_R(\partial, R)$ is an acyclic DG-ring.

Applying the inverse hat construction to $(\mathrm{Hom}_R(\mathcal{D}, R), \mathrm{Hom}_R(\partial, R))$, we obtain a CDG-ring (B, d, h).

Construction of the category anti-equivalence

Now assume that $(\widetilde{A}, F)$ is a left finitely projective nonhomogeneous Koszul ring. Then the ring $\mathrm{Hom}_R(\mathcal{D}, R)$ with the odd derivation $\mathrm{Hom}_R(\partial, R)$ is an acyclic DG-ring.

Applying the inverse hat construction to $(\mathrm{Hom}_R(\mathcal{D}, R), \mathrm{Hom}_R(\partial, R))$, we obtain a CDG-ring (B, d, h). So $(\mathrm{Hom}_R(\mathcal{D}, R), \mathrm{Hom}_R(\partial, R)) = \widehat{B}^\bullet$.

Construction of the category anti-equivalence

Now assume that $(\widetilde{A}, F)$ is a left finitely projective nonhomogeneous Koszul ring. Then the ring $\mathrm{Hom}_R(\mathcal{D}, R)$ with the odd derivation $\mathrm{Hom}_R(\partial, R)$ is an acyclic DG-ring.

Applying the inverse hat construction to $(\mathrm{Hom}_R(\mathcal{D}, R), \mathrm{Hom}_R(\partial, R))$, we obtain a CDG-ring (B, d, h). So $(\mathrm{Hom}_R(\mathcal{D}, R), \mathrm{Hom}_R(\partial, R)) = \widehat{B}^\bullet$.

Then (B, d, h) is the right finitely projective Koszul CDG-ring

Construction of the category anti-equivalence

Now assume that $(\widetilde{A}, F)$ is a left finitely projective nonhomogeneous Koszul ring. Then the ring $\mathrm{Hom}_R(\mathcal{D}, R)$ with the odd derivation $\mathrm{Hom}_R(\partial, R)$ is an acyclic DG-ring.

Applying the inverse hat construction to $(\mathrm{Hom}_R(\mathcal{D}, R), \mathrm{Hom}_R(\partial, R))$, we obtain a CDG-ring (B, d, h). So $(\mathrm{Hom}_R(\mathcal{D}, R), \mathrm{Hom}_R(\partial, R)) = \widehat{B}^\bullet$.

Then (B, d, h) is the right finitely projective Koszul CDG-ring corresponding to the left finitely projective nonhomogeneous Koszul ring $(\widetilde{A}, F)$

Construction of the category anti-equivalence

Now assume that $(\widetilde{A}, F)$ is a left finitely projective nonhomogeneous Koszul ring. Then the ring $\mathrm{Hom}_R(\mathcal{D}, R)$ with the odd derivation $\mathrm{Hom}_R(\partial, R)$ is an acyclic DG-ring.

Applying the inverse hat construction to $(\mathrm{Hom}_R(\mathcal{D}, R), \mathrm{Hom}_R(\partial, R))$, we obtain a CDG-ring (B, d, h). So $(\mathrm{Hom}_R(\mathcal{D}, R), \mathrm{Hom}_R(\partial, R)) = \widehat{B}^{\bullet}$.

Then (B, d, h) is the right finitely projective Koszul CDG-ring corresponding to the left finitely projective nonhomogeneous Koszul ring $(\widetilde{A}, F)$ under the anti-equivalence of categories claimed in the Poincaré–Birkhoff–Witt theorem.

Construction of the category anti-equivalence

Now assume that $(\widetilde{A}, F)$ is a left finitely projective nonhomogeneous Koszul ring. Then the ring $\mathrm{Hom}_R(\mathcal{D}, R)$ with the odd derivation $\mathrm{Hom}_R(\partial, R)$ is an acyclic DG-ring.

Applying the inverse hat construction to $(\mathrm{Hom}_R(\mathcal{D}, R), \mathrm{Hom}_R(\partial, R))$, we obtain a CDG-ring (B, d, h). So $(\mathrm{Hom}_R(\mathcal{D}, R), \mathrm{Hom}_R(\partial, R)) = \widehat{B}^\bullet$.

Then (B, d, h) is the right finitely projective Koszul CDG-ring corresponding to the left finitely projective nonhomogeneous Koszul ring $(\widetilde{A}, F)$ under the anti-equivalence of categories claimed in the Poincaré–Birkhoff–Witt theorem.

Notice that the differential ∂ on $\widehat{B}^\bullet$ annihilates R, so it is (both left and right) R-linear.

Construction of the category anti-equivalence

Now assume that $(\widetilde{A}, F)$ is a left finitely projective nonhomogeneous Koszul ring. Then the ring $\mathrm{Hom}_R(\mathcal{D}, R)$ with the odd derivation $\mathrm{Hom}_R(\partial, R)$ is an acyclic DG-ring.

Applying the inverse hat construction to $(\mathrm{Hom}_R(\mathcal{D}, R), \mathrm{Hom}_R(\partial, R))$, we obtain a CDG-ring (B, d, h). So $(\mathrm{Hom}_R(\mathcal{D}, R), \mathrm{Hom}_R(\partial, R)) = \widehat{B}^\bullet$.

Then (B, d, h) is the right finitely projective Koszul CDG-ring corresponding to the left finitely projective nonhomogeneous Koszul ring $(\widetilde{A}, F)$ under the anti-equivalence of categories claimed in the Poincaré–Birkhoff–Witt theorem.

Notice that the differential ∂ on $\widehat{B}^\bullet$ annihilates R, so it is (both left and right) R-linear. The differential d on B does not annihilate R, so it is not R-linear.

Questions and explanations cont'd

Questions and explanations cont'd

Let us return to the example of the ring of differential operators $\widetilde{A} = \mathrm{Diff}(M)$

Questions and explanations cont'd

Let us return to the example of the ring of differential operators $\widetilde{A} = \mathrm{Diff}(M)$ with the filtration F by the orders of diff. operators.

Let us return to the example of the ring of differential operators $\widetilde{A} = \mathrm{Diff}(M)$ with the filtration F by the orders of diff. operators.

The nonhomogeneous quadratic dual (C)DG-ring to $(\widetilde{A}, F)$ is $B^\bullet = (B, d) = (\Omega(M), d_{dR})$.

Questions and explanations cont'd

Let us return to the example of the ring of differential operators $\widetilde{A} = \mathrm{Diff}(M)$ with the filtration F by the orders of diff. operators.

The nonhomogeneous quadratic dual (C)DG-ring to $(\widetilde{A}, F)$ is $B^\bullet = (B, d) = (\Omega(M), d_{dR})$. Our philosophy suggests that $B^\bullet$ "really wants to be a coring".

Questions and explanations cont'd

Let us return to the example of the ring of differential operators $\widetilde{A} = \mathrm{Diff}(M)$ with the filtration F by the orders of diff. operators.

The nonhomogeneous quadratic dual (C)DG-ring to $(\widetilde{A}, F)$ is $B^\bullet = (B, d) = (\Omega(M), d_{dR})$. Our philosophy suggests that $B^\bullet$ "really wants to be a coring". This means the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ over the ring of functions $R = O(M)$.

Questions and explanations cont'd

Let us return to the example of the ring of differential operators $\widetilde{A} = \mathrm{Diff}(M)$ with the filtration F by the orders of diff. operators.

The nonhomogeneous quadratic dual (C)DG-ring to $(\widetilde{A}, F)$ is $B^\bullet = (B, d) = (\Omega(M), d_{dR})$. Our philosophy suggests that $B^\bullet$ "really wants to be a coring". This means the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ over the ring of functions $R = O(M)$.

To implement this point of view, one needs some structure on the coring of polyvector fields $\mathcal{C} = \mathrm{Hom}_R(B, R)$

Questions and explanations cont'd

Let us return to the example of the ring of differential operators $\widetilde{A} = \mathrm{Diff}(M)$ with the filtration F by the orders of diff. operators.

The nonhomogeneous quadratic dual (C)DG-ring to $(\widetilde{A}, F)$ is $B^\bullet = (B, d) = (\Omega(M), d_{dR})$. Our philosophy suggests that $B^\bullet$ "really wants to be a coring". This means the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ over the ring of functions $R = O(M)$.

To implement this point of view, one needs some structure on the coring of polyvector fields $\mathcal{C} = \mathrm{Hom}_R(B, R)$ corresponding to the de Rham differential on the ring of differential forms B.

Questions and explanations cont'd

Let us return to the example of the ring of differential operators $\widetilde{A} = \mathrm{Diff}(M)$ with the filtration F by the orders of diff. operators.

The nonhomogeneous quadratic dual (C)DG-ring to $(\widetilde{A}, F)$ is $B^\bullet = (B, d) = (\Omega(M), d_{dR})$. Our philosophy suggests that $B^\bullet$ "really wants to be a coring". This means the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ over the ring of functions $R = O(M)$.

To implement this point of view, one needs some structure on the coring of polyvector fields $\mathcal{C} = \mathrm{Hom}_R(B, R)$ corresponding to the de Rham differential on the ring of differential forms B.
But the de Rham differential is not R-linear

Questions and explanations cont'd

Let us return to the example of the ring of differential operators $\widetilde{A} = \mathrm{Diff}(M)$ with the filtration F by the orders of diff. operators.

The nonhomogeneous quadratic dual (C)DG-ring to $(\widetilde{A}, F)$ is $B^\bullet = (B, d) = (\Omega(M), d_{dR})$. Our philosophy suggests that $B^\bullet$ "really wants to be a coring". This means the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ over the ring of functions $R = O(M)$.

To implement this point of view, one needs some structure on the coring of polyvector fields $\mathcal{C} = \mathrm{Hom}_R(B, R)$ corresponding to the de Rham differential on the ring of differential forms B. But the de Rham differential is not R-linear, so the functor $\mathrm{Hom}_R(-, R)$ cannot be applied to it.

Questions and explanations cont'd

Let us return to the example of the ring of differential operators $\widetilde{A} = \mathrm{Diff}(M)$ with the filtration F by the orders of diff. operators.

The nonhomogeneous quadratic dual (C)DG-ring to $(\widetilde{A}, F)$ is $B^\bullet = (B, d) = (\Omega(M), d_{dR})$. Our philosophy suggests that $B^\bullet$ "really wants to be a coring". This means the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ over the ring of functions $R = O(M)$.

To implement this point of view, one needs some structure on the coring of polyvector fields $\mathcal{C} = \mathrm{Hom}_R(B, R)$ corresponding to the de Rham differential on the ring of differential forms B. But the de Rham differential is not R-linear, so the functor $\mathrm{Hom}_R(-, R)$ cannot be applied to it.

The solution is to apply the hat construction first

Questions and explanations cont'd

Let us return to the example of the ring of differential operators $\widetilde{A} = \mathrm{Diff}(M)$ with the filtration F by the orders of diff. operators.

The nonhomogeneous quadratic dual (C)DG-ring to $(\widetilde{A}, F)$ is $B^\bullet = (B, d) = (\Omega(M), d_{dR})$. Our philosophy suggests that $B^\bullet$ "really wants to be a coring". This means the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ over the ring of functions $R = O(M)$.

To implement this point of view, one needs some structure on the coring of polyvector fields $\mathcal{C} = \mathrm{Hom}_R(B, R)$ corresponding to the de Rham differential on the ring of differential forms B. But the de Rham differential is not R-linear, so the functor $\mathrm{Hom}_R(-, R)$ cannot be applied to it.

The solution is to apply the hat construction first, producing an acyclic DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ from the de Rham DG-ring $B^\bullet = (B, d)$.

Let us return to the example of the ring of differential operators $\widetilde{A} = \mathrm{Diff}(M)$ with the filtration F by the orders of diff. operators.

The nonhomogeneous quadratic dual (C)DG-ring to $(\widetilde{A}, F)$ is $B^\bullet = (B, d) = (\Omega(M), d_{dR})$. Our philosophy suggests that $B^\bullet$ "really wants to be a coring". This means the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ over the ring of functions $R = O(M)$.

To implement this point of view, one needs some structure on the coring of polyvector fields $\mathcal{C} = \mathrm{Hom}_R(B, R)$ corresponding to the de Rham differential on the ring of differential forms B. But the de Rham differential is not R-linear, so the functor $\mathrm{Hom}_R(-, R)$ cannot be applied to it.

The solution is to apply the hat construction first, producing an acyclic DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ from the de Rham DG-ring $B^\bullet = (B, d)$. The new, acyclic differential ∂ is $O(M)$-linear

Questions and explanations cont'd

Let us return to the example of the ring of differential operators $\widetilde{A} = \mathrm{Diff}(M)$ with the filtration F by the orders of diff. operators.

The nonhomogeneous quadratic dual (C)DG-ring to $(\widetilde{A}, F)$ is $B^\bullet = (B, d) = (\Omega(M), d_{dR})$. Our philosophy suggests that $B^\bullet$ "really wants to be a coring". This means the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ over the ring of functions $R = O(M)$.

To implement this point of view, one needs some structure on the coring of polyvector fields $\mathcal{C} = \mathrm{Hom}_R(B, R)$ corresponding to the de Rham differential on the ring of differential forms B. But the de Rham differential is not R-linear, so the functor $\mathrm{Hom}_R(-, R)$ cannot be applied to it.

The solution is to apply the hat construction first, producing an acyclic DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ from the de Rham DG-ring $B^\bullet = (B, d)$. The new, acyclic differential ∂ is $O(M)$-linear, so the functor $\mathrm{Hom}_{O(M)}(-, O(M))$ can be applied to it

Questions and explanations cont'd

Let us return to the example of the ring of differential operators $\widetilde{A} = \mathrm{Diff}(M)$ with the filtration F by the orders of diff. operators.

The nonhomogeneous quadratic dual (C)DG-ring to $(\widetilde{A}, F)$ is $B^\bullet = (B, d) = (\Omega(M), d_{dR})$. Our philosophy suggests that $B^\bullet$ "really wants to be a coring". This means the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ over the ring of functions $R = O(M)$.

To implement this point of view, one needs some structure on the coring of polyvector fields $\mathcal{C} = \mathrm{Hom}_R(B, R)$ corresponding to the de Rham differential on the ring of differential forms B. But the de Rham differential is not R-linear, so the functor $\mathrm{Hom}_R(-, R)$ cannot be applied to it.

The solution is to apply the hat construction first, producing an acyclic DG-ring $\widehat{B}^\bullet = (\widehat{B}, \partial)$ from the de Rham DG-ring $B^\bullet = (B, d)$. The new, acyclic differential ∂ is $O(M)$-linear, so the functor $\mathrm{Hom}_{O(M)}(-, O(M))$ can be applied to it, producing a quasi-differential coring $(\mathcal{D}, \partial)$ with $\mathcal{D} = \mathrm{Hom}_R(\widehat{B}, R)$.

Questions and explanations fin'd

Questions and explanations fin'd

The quasi-differential coring $(\mathcal{D}, \partial)$ is viewed as a quasi-differential structure

Questions and explanations fin'd

The quasi-differential coring $(\mathcal{D}, \partial)$ is viewed as a quasi-differential structure on the graded coring $\mathcal{C} = \operatorname{coker}(\partial\colon \mathcal{D} \to \mathcal{D}) = \bigwedge(TM)$.

Questions and explanations fin'd

The quasi-differential coring $(\mathcal{D}, \partial)$ is viewed as a quasi-differential structure on the graded coring $\mathcal{C} = \mathrm{coker}(\partial\colon \mathcal{D} \to \mathcal{D}) = \bigwedge(TM)$.

Thus the structure on the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$

Questions and explanations fin'd

The quasi-differential coring $(\mathcal{D}, \partial)$ is viewed as a quasi-differential structure on the graded coring $\mathcal{C} = \operatorname{coker}(\partial\colon \mathcal{D} \to \mathcal{D}) = \bigwedge(TM)$.

Thus the structure on the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ corresponding to the de Rham DG-algebra structure on the ring of differential forms $B = \Omega(M)$

Questions and explanations fin'd

The quasi-differential coring $(\mathcal{D}, \partial)$ is viewed as a quasi-differential structure on the graded coring $\mathcal{C} = \mathrm{coker}(\partial\colon \mathcal{D} \to \mathcal{D}) = \bigwedge(TM)$.

Thus the structure on the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ corresponding to the de Rham DG-algebra structure on the ring of differential forms $B = \Omega(M)$ is the quasi-diffential structure $(\mathcal{D}, \partial)$.

Questions and explanations fin'd

The quasi-differential coring $(\mathcal{D}, \partial)$ is viewed as a quasi-differential structure on the graded coring $\mathcal{C} = \operatorname{coker}(\partial\colon \mathcal{D} \to \mathcal{D}) = \bigwedge(TM)$.

Thus the structure on the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ corresponding to the de Rham DG-algebra structure on the ring of differential forms $B = \Omega(M)$ is the quasi-diffential structure $(\mathcal{D}, \partial)$.

The problem is that quasi-differential structures are very counterintutive to work with

Questions and explanations fin'd

The quasi-differential coring $(\mathcal{D}, \partial)$ is viewed as a quasi-differential structure on the graded coring $\mathcal{C} = \operatorname{coker}(\partial\colon \mathcal{D} \to \mathcal{D}) = \bigwedge(TM)$.

Thus the structure on the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ corresponding to the de Rham DG-algebra structure on the ring of differential forms $B = \Omega(M)$ is the quasi-diffential structure $(\mathcal{D}, \partial)$.

The problem is that quasi-differential structures are very counterintutive to work with, particularly in the context of the triangulated categories of modules.

The quasi-differential coring $(\mathcal{D}, \partial)$ is viewed as a quasi-differential structure on the graded coring $\mathcal{C} = \mathrm{coker}(\partial\colon \mathcal{D} \to \mathcal{D}) = \bigwedge(TM)$.

Thus the structure on the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ corresponding to the de Rham DG-algebra structure on the ring of differential forms $B = \Omega(M)$ is the quasi-diffential structure $(\mathcal{D}, \partial)$.

The problem is that quasi-differential structures are very counterintutive to work with, particularly in the context of the triangulated categories of modules. It is never clear what formulas one should write for specific maps, like the differentials etc.

Questions and explanations fin'd

The quasi-differential coring $(\mathcal{D}, \partial)$ is viewed as a quasi-differential structure on the graded coring $\mathcal{C} = \operatorname{coker}(\partial\colon \mathcal{D} \to \mathcal{D}) = \bigwedge(TM)$.

Thus the structure on the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ corresponding to the de Rham DG-algebra structure on the ring of differential forms $B = \Omega(M)$ is the quasi-diffential structure $(\mathcal{D}, \partial)$.

The problem is that quasi-differential structures are very counterintutive to work with, particularly in the context of the triangulated categories of modules. It is never clear what formulas one should write for specific maps, like the differentials etc. It is much easier to write formulas for DG-modules over the de Rham DG-algebra $(\Omega(M), d_{dR})$

Questions and explanations fin'd

The quasi-differential coring $(\mathcal{D}, \partial)$ is viewed as a quasi-differential structure on the graded coring $\mathcal{C} = \operatorname{coker}(\partial \colon \mathcal{D} \to \mathcal{D}) = \bigwedge(TM)$.

Thus the structure on the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ corresponding to the de Rham DG-algebra structure on the ring of differential forms $B = \Omega(M)$ is the quasi-diffential structure $(\mathcal{D}, \partial)$.

The problem is that quasi-differential structures are very counterintutive to work with, particularly in the context of the triangulated categories of modules. It is never clear what formulas one should write for specific maps, like the differentials etc. It is much easier to write formulas for DG-modules over the de Rham DG-algebra $(\Omega(M), d_{dR})$, or for curved DG-modules if there is a curvature present.

Questions and explanations fin'd

The quasi-differential coring $(\mathcal{D}, \partial)$ is viewed as a quasi-differential structure on the graded coring $\mathcal{C} = \operatorname{coker}(\partial\colon \mathcal{D} \to \mathcal{D}) = \bigwedge(TM)$.

Thus the structure on the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ corresponding to the de Rham DG-algebra structure on the ring of differential forms $B = \Omega(M)$ is the quasi-diffential structure $(\mathcal{D}, \partial)$.

The problem is that quasi-differential structures are very counterintutive to work with, particularly in the context of the triangulated categories of modules. It is never clear what formulas one should write for specific maps, like the differentials etc. It is much easier to write formulas for DG-modules over the de Rham DG-algebra $(\Omega(M), d_{dR})$, or for curved DG-modules if there is a curvature present.

For this reason, in relative nonhomogeneous Koszul duality over a base ring

Questions and explanations fin'd

The quasi-differential coring $(\mathcal{D}, \partial)$ is viewed as a quasi-differential structure on the graded coring $\mathcal{C} = \mathrm{coker}(\partial\colon \mathcal{D} \to \mathcal{D}) = \bigwedge(TM)$.

Thus the structure on the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ corresponding to the de Rham DG-algebra structure on the ring of differential forms $B = \Omega(M)$ is the quasi-diffential structure $(\mathcal{D}, \partial)$.

The problem is that quasi-differential structures are very counterintutive to work with, particularly in the context of the triangulated categories of modules. It is never clear what formulas one should write for specific maps, like the differentials etc. It is much easier to write formulas for DG-modules over the de Rham DG-algebra $(\Omega(M), d_{dR})$, or for curved DG-modules if there is a curvature present.

For this reason, in relative nonhomogeneous Koszul duality over a base ring, one feels forever stuck between the two points of view.

The quasi-differential coring $(\mathcal{D}, \partial)$ is viewed as a quasi-differential structure on the graded coring $\mathcal{C} = \operatorname{coker}(\partial\colon \mathcal{D} \to \mathcal{D}) = \bigwedge(TM)$.

Thus the structure on the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ corresponding to the de Rham DG-algebra structure on the ring of differential forms $B = \Omega(M)$ is the quasi-diffential structure $(\mathcal{D}, \partial)$.

The problem is that quasi-differential structures are very counterintutive to work with, particularly in the context of the triangulated categories of modules. It is never clear what formulas one should write for specific maps, like the differentials etc. It is much easier to write formulas for DG-modules over the de Rham DG-algebra $(\Omega(M), d_{dR})$, or for curved DG-modules if there is a curvature present.

For this reason, in relative nonhomogeneous Koszul duality over a base ring, one feels forever stuck between the two points of view. Relative nonhomogeneous Koszul duality over a base coalgebra over a field is easier in this respect.

Questions and explanations fin'd

The quasi-differential coring $(\mathcal{D}, \partial)$ is viewed as a quasi-differential structure on the graded coring $\mathcal{C} = \operatorname{coker}(\partial\colon \mathcal{D} \to \mathcal{D}) = \bigwedge(TM)$.

Thus the structure on the coring of polyvector fields $\mathcal{C} = \bigwedge(TM)$ corresponding to the de Rham DG-algebra structure on the ring of differential forms $B = \Omega(M)$ is the quasi-diffential structure $(\mathcal{D}, \partial)$.

The problem is that quasi-differential structures are very counterintutive to work with, particularly in the context of the triangulated categories of modules. It is never clear what formulas one should write for specific maps, like the differentials etc. It is much easier to write formulas for DG-modules over the de Rham DG-algebra $(\Omega(M), d_{dR})$, or for curved DG-modules if there is a curvature present.

For this reason, in relative nonhomogeneous Koszul duality over a base ring, one feels forever stuck between the two points of view. Relative nonhomogeneous Koszul duality over a base coalgebra over a field is easier in this respect. But this is a separate story.

E. Getzler, J. D. S. Jones. A_∞-algebras and the cyclic bar complex. *Illinois Journ. of Math.* **34**, #2, 1990.

L. Positselski. Nonhomogeneous quadratic duality and curvature. *Functional Analysis and its Appl.* **27**, #3, p. 197–204, 1993. arXiv:1411.1982 [math.RA]

L. Positselski. Homological algebra of semimodules and semicontramodules: Semi-infinite homological algebra of associative algebraic structures. Appendix C in collaboration with D. Rumynin; Appendix D in collaboration with S. Arkhipov. Monografie Matematyczne vol. 70, Birkhäuser/Springer Basel, 2010. xxiv+349 pp. arXiv:0708.3398 [math.CT]

L. Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Memoirs of the American Math. Society* **212**, #996, 2011. vi+133 pp. arXiv:0905.2621 [math.CT]

L. Positselski. Relative nonhomogeneous Koszul duality. Frontiers in Mathematics, Birkhäuser/Springer Nature, Cham, Switzerland, 2021. xxix+278 pp. arXiv:1911.07402 [math.RA]

L. Positselski. Exact DG-categories and fully faithful triangulated inclusion functors. Electronic preprint arXiv:2110.08237 [math.CT], 2021–25, 154 pp.

L. Positselski. Differential graded Koszul duality: An introductory survey. *Bulletin of the London Math. Society* **55**, #4, p. 1551–1640, 2023. arXiv:2207.07063 [math.CT]

L. Positselski, J. Šťovíček. Coderived and contraderived categories of locally presentable abelian DG-categories. *Math. Zeitschrift* **308**, #1, Paper No. 14, 70 pp., 2024. arXiv:2210.08237 [math.CT]