Semi-infinite algebraic geometry of quasi-coherent torsion sheaves

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Introduction: Semi-Infinite Set Theory

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which remains unchanged when one removes a finite number of elements from S^+ or adjoins to S^+ a finite number of elements from S. The set S is a topological basis in the natural topology on V_{S,S^+}. Topological vector spaces of this form are called locally linearly compact, or Tate vector spaces.

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So some acyclic complexes in the abelian category A represent nonzero objects in the semiderived category.

Introduction: Summary for the Standard Example

The triangulated category $\mathrm{D}^{\mathrm{si}}(\mathrm{A})$ is constructed as the triangulated Verdier quotient category of the cochain homotopy category $\mathrm{K}(\mathrm{A})$ of unbounded complexes in A by a certain thick subcategory $\mathrm{Ac}^{\mathrm{si}}(\mathrm{A})$; so $\mathrm{D}^{\mathrm{si}}(\mathrm{A}) = \mathrm{K}(\mathrm{A})/\mathrm{Ac}^{\mathrm{si}}(\mathrm{A})$.

The thick (in fact, localizing) subcategory of complexes to be killed $\mathrm{Ac}^{\mathrm{si}}(\mathrm{A}) \subset \mathrm{K}(\mathrm{A})$ is properly contained in the full subcategory $\mathrm{Ac}(\mathrm{A}) \subset \mathrm{K}(\mathrm{A})$ of acyclic complexes in A, that is $\mathrm{Ac}^{\mathrm{si}}(\mathrm{A}) \subsetneq \mathrm{Ac}(\mathrm{A})$.

So some acyclic complexes in the abelian category A represent nonzero objects in the semiderived category. In particular, the unit object of the tensor structure on $\mathrm{D}^{\mathrm{si}}(\mathrm{A})$ turns out to be an acyclic complex in this example.

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Let the basis vector $(*)$ have grading $\infty/2+0$. Then deleting m vectors e_i from the wedge product $(*)$ and adjoining n new vectors e_j instead produces an infinite wedge product representing a basis vector of $\bigwedge^{\infty/2+*}(V)$ having grading $\infty/2+n-m$.

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where the direct limit is taken over all open subspaces $U \subset W$.

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The dimensional shifts and determinantal twists arise from certain choices one has to make when defining the semitensor product operation on $\mathrm{D}^{\mathrm{si}}(R\text{-}\mathrm{Mod}_{\mathrm{tors}})$.

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Examples: Derived Category of Torsion Abelian Groups

Example 2a

Let $\mathsf{A} = \mathbb{Z}\text{-}\mathrm{Mod}_{\mathrm{tors}}$ be the abelian category of torsion abelian groups.

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The torsion product is a left exact functor. The right derived functor of torsion product $\circledcirc_{\mathbb{Q}/\mathbb{Z}}^{\mathbb{R}}$, constructed using injective coresolutions, makes the derived category of torsion abelian groups $\mathrm{D}(\mathrm{A})$ a tensor triangulated category.

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The torsion product is a left exact functor. The right derived functor of torsion product $\circledT^{\mathbb{R}}_{\mathbb{Q}/\mathbb{Z}}$, constructed using injective coresolutions, makes the derived category of torsion abelian groups $\mathrm{D}(\mathrm{A})$ a tensor triangulated category.

The tensor triangulated category $\mathrm{D}(\mathrm{A})$ can be embedded into the derived category of abelian groups $\mathrm{D}(\mathbb{Z}\text{-}\mathrm{Mod})$ with the shifted tensor product operation $(A, B) \longmapsto (A \otimes^{\mathbb{L}}_{\mathbb{Z}} B)[-1]$, as a tensor triangulated subcategory with its own unit object. The unit object $\mathbb{Z}[1]$ of the monoidal structure $(A, B) \longmapsto (A \otimes^{\mathbb{L}}_{\mathbb{Z}} B)[-1]$ on $\mathrm{D}(\mathbb{Z}\text{-}\mathrm{Mod})$ is different from the unit object $\mathbb{Q}/\mathbb{Z}$ of $\mathrm{D}(\mathbb{Z}\text{-}\mathrm{Mod}_{\mathrm{tors}})$.

Examples: Derived Category of Torsion $\mathbb{k}[x]$-Modules

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Examples: Coderived Category of Comodules

Example 3 cont'd

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When the coalgebra C is cocommutative, one can define the right derived cotensor product operation on the coderived category $\mathrm{D}^{\mathrm{co}}(C\text{-}\mathrm{Comod})$ by taking the cotensor products of complexes of injective C-comodules. In fact, for any complex of injective C-comodules $J^\bullet$ and any coacyclic complex of C-comodules $A^\bullet$, the complex of C-comodules $C^\bullet \mathbin{\square_C} A^\bullet$ is coacyclic. So it suffices to replace only one of the two complexes of C-comodules by a complex of injectives before taking their cotensor product.

Examples: Coderived Category of Comodules

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Examples: Noetherian Scheme with a Dualizing Complex

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Example 4

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Examples: Noetherian Scheme with a Dualizing Complex

Example 4 cont'd

One can show that the composition of triangulated functors
$\mathrm{K}(X\text{-}\mathrm{Qcoh}_{\mathrm{inj}}) \longrightarrow \mathrm{K}(X\text{-}\mathrm{Qcoh}) \longrightarrow \mathrm{K}(X\text{-}\mathrm{Qcoh})/\mathrm{Ac}^{\mathrm{co}}(X\text{-}\mathrm{Qcoh})$

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Examples: Noetherian Scheme with a Dualizing Complex

Example 4 cont'd

On any scheme X, the derived category of flat quasi-coherent sheaves $\mathrm{D}(X\text{-}\mathrm{Qcoh}_{\mathrm{fl}})$ is a tensor triangulated category

Examples: Noetherian Scheme with a Dualizing Complex

Example 4 cont'd

On any scheme X, the derived category of flat quasi-coherent sheaves $\mathrm{D}(X\text{-}\mathrm{Qcoh_{fl}})$ is a tensor triangulated category with respect to the tensor product functor $\otimes_{\mathcal{O}_X}$.

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The resulting operation on $\mathrm{D}^{\mathrm{co}}(X\text{-}\mathrm{Qcoh})$ is called the cotensor product

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The resulting operation on $\mathrm{D}^{\mathrm{co}}(X\text{-}\mathrm{Qcoh})$ is called the cotensor product and denoted by $\square_{\mathcal{D}^\bullet}$. So $\mathrm{D}^{\mathrm{co}}(X\text{-}\mathrm{Qcoh})$ is a tensor triangulated category with respect to the cotensor product over $\mathcal{D}^\bullet$.

Examples: Noetherian Scheme with a Dualizing Complex

Example 4 fin'd

Explicitly, let $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ be two complexes in X-Qcoh,

Examples: Noetherian Scheme with a Dualizing Complex

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Explicitly, let $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ be two complexes in $X\text{-}\mathrm{Qcoh}$, and let $\mathcal{M}^\bullet \longrightarrow \mathcal{J}^\bullet$ and $\mathcal{N}^\bullet \longrightarrow \mathcal{K}^\bullet$ be two morphisms with coacyclic cones,

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The three complexes above represent the coderived category object $\mathcal{M}^\bullet \,\square_{\mathcal{D}^\bullet}\, \mathcal{N}^\bullet \in \mathrm{D}^{\mathrm{co}}(X\text{-}\mathrm{Qcoh})$.

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Posing the Problem

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Ind-Schemes

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So one has to develop some kind of "sheafification" theory, embedding the desired category of "sheaves" into a larger ambient category of "presheaves". The cokernel of a morphism of "sheaves" is then constructed as the "sheafification" of the cokernel of the same morphism taken in the category of "presheaves".

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Due to the isomorphism condition (iii), the category $\mathfrak{X}\text{-Tors}$ of quasi-coherent torsion sheaves on $\mathfrak{X}$ does not depend on the choice of a direct system of reasonable closed immersions $(X_\gamma)_{\gamma\in\Gamma}$ representing a given reasonable ind-scheme $\mathfrak{X}$.

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Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be a reasonable ind-scheme and $i_\gamma\colon X_\gamma \longrightarrow \mathfrak{X}$ be the natural closed immersion. Then one has $i_\gamma^*\mathfrak{P} = \mathfrak{P}^\gamma$ and $i_\gamma^!\mathscr{M} = \mathscr{M}_\gamma$ for all $\mathfrak{P} \in \mathfrak{X}\text{-}\mathrm{Pro}$ and $\mathscr{M} \in \mathfrak{X}\text{-}\mathrm{Tors}$.

Dualizing Complexes on Ind-Noetherian Ind-Schemes

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One can check that this condition on a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}$ does not depend on the choice of a direct system $(X_\gamma)_{\gamma\in\Gamma}$ representing $\mathfrak{X}$.

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Notice that, by definition, a dualizing complex $\mathscr{D}^\bullet$ is an object of the homotopy category of unbounded complexes of injective quasi-coherent torsion sheaves on $\mathfrak{X}$. Another name for this category is the coderived category of quasi-coherent torsion sheaves, $\mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-}\mathrm{Tors}) = \mathrm{K}(\mathfrak{X}\text{-}\mathrm{Tors}_{\mathrm{inj}})$. A dualizing complex on an ind-scheme cannot be viewed as an object of the conventional derived category $\mathrm{D}(\mathfrak{X}\text{-}\mathrm{Tors})$.

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As ℓ tends to $-\infty$, the only cohomology sheaf of the complex $\mathcal{D}_\ell^\bullet$ moves to ever higher negative cohomological degrees and, in the direct limit, disappears at the cohomological degree $-\infty$.

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As ℓ tends to $-\infty$, the only cohomology sheaf of the complex $\mathcal{D}_\ell^\bullet$ moves to ever higher negative cohomological degrees and, in the direct limit, disappears at the cohomological degree $-\infty$. Consequently, the dualizing complex of quasi-coherent torsion sheaves $\mathscr{D}^\bullet$ on the ind-scheme $\mathfrak{X}$ is acyclic.

Coderived Category of Quasi-Coherent Torsion Sheaves

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$$\mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-}\mathrm{Tors}) = \mathrm{K}(\mathfrak{X}\text{-}\mathrm{Tors})/\mathrm{Ac}^{\mathrm{co}}(\mathfrak{X}\text{-}\mathrm{Tors})$$

of the homotopy category $\mathrm{K}(\mathfrak{X}\text{-}\mathrm{Tors})$

Coderived Category of Quasi-Coherent Torsion Sheaves

Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme. As above, a complex of quasi-coherent torsion sheaves $\mathscr{A}^\bullet \in \mathrm{K}(\mathfrak{X}\text{-}\mathrm{Tors})$ is said to be coacyclic if, for every complex of injective quasi-coherent torsion sheaves $\mathscr{J}^\bullet \in \mathrm{K}(\mathfrak{X}\text{-}\mathrm{Tors}_{\mathrm{inj}})$, the complex of abelian groups $\mathrm{Hom}_{\mathfrak{X}}(\mathscr{A}^\bullet, \mathscr{J}^\bullet)$ is acyclic. The coderived category $\mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-}\mathrm{Tors})$ is defined as the triangulated Verdier quotient category

$$\mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-}\mathrm{Tors}) = \mathrm{K}(\mathfrak{X}\text{-}\mathrm{Tors})/\mathrm{Ac}^{\mathrm{co}}(\mathfrak{X}\text{-}\mathrm{Tors})$$

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One can show that the composition of triangulated functors $\mathrm{K}(\mathfrak{X}\text{-}\mathrm{Tors}_{\mathrm{inj}}) \longrightarrow \mathrm{K}(\mathfrak{X}\text{-}\mathrm{Tors}) \longrightarrow \mathrm{D^{co}}(\mathfrak{X}\text{-}\mathrm{Tors})$ is a triangulated equivalence. So, in particular, a dualizing complex $\mathscr{D}^\bullet$ on $\mathfrak{X}$ is naturally viewed as an object of $\mathrm{D^{co}}(\mathfrak{X}\text{-}\mathrm{Tors})$.

Flat Pro-Quasi-Coherent Pro-Sheaves

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The full subcategory of flat pro-quasi-coherent pro-sheaves $\mathfrak{X}\text{-}\mathrm{Flat}$

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The full subcategory of flat pro-quasi-coherent pro-sheaves $\mathfrak{X}\text{-}\mathrm{Flat}$ is a monoidal subcategory in the monoidal category of pro-quasi-coherent pro-sheaves $\mathfrak{X}\text{-}\mathrm{Pro}$.

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The full subcategory of flat pro-quasi-coherent pro-sheaves $\mathfrak{X}\text{-}\mathrm{Flat}$ is a monoidal subcategory in the monoidal category of pro-quasi-coherent pro-sheaves $\mathfrak{X}\text{-}\mathrm{Pro}$. The tensor product functor $\otimes^{\mathfrak{X}}$ is exact in $\mathfrak{X}\text{-}\mathrm{Flat}$.

Absolute Covariant Serre-Grothendieck Duality

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where $\mathrm{D}(\mathfrak{X}\text{-Flat})$ is the conventional unbounded derived category of the exact category $\mathfrak{X}\text{-Flat}$ of flat pro-quasi-coherent pro-sheaves.

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The inverse functor $\mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-}\mathrm{Tors}) \longrightarrow \mathrm{D}(\mathfrak{X}\text{-}\mathrm{Flat})$ is denoted by $\mathbb{R}\,\mathfrak{Hom}_{\mathfrak{X}\text{-}\mathrm{qc}}(\mathscr{D}^\bullet, -)$.

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The Cotensor Product

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For an ind-Noetherian ind-scheme $\mathfrak{X}$, the coderived category $\mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors})$ is a triangulated module category over the triangulated tensor category $\mathrm{D}(\mathfrak{X}\text{-Flat})$,

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Now, given an ind-semi-separated ind-Noetherian ind-scheme $\mathfrak{X}$ with a dualizing complex $\mathscr{D}^\bullet$, one can transfer the tensor triangulated structure of $\mathrm{D}(\mathfrak{X}\text{-Flat})$ along the triangulated equivalence $\mathrm{D}(\mathfrak{X}\text{-Flat}) \simeq \mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors})$.

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For an ind-Noetherian ind-scheme $\mathfrak{X}$, the coderived category $\mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors})$ is a triangulated module category over the triangulated tensor category $\mathrm{D}(\mathfrak{X}\text{-Flat})$,

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Now, given an ind-semi-separated ind-Noetherian ind-scheme $\mathfrak{X}$ with a dualizing complex $\mathscr{D}^\bullet$, one can transfer the tensor triangulated structure of $\mathrm{D}(\mathfrak{X}\text{-Flat})$ along the triangulated equivalence $\mathrm{D}(\mathfrak{X}\text{-Flat}) \simeq \mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors})$. The resulting operation on coderived category is called the cotensor product

The Cotensor Product

For any ind-scheme $\mathfrak{X}$, the tensor product of flat pro-quasi-coherent pro-sheaves induces a well-defined tensor product functor on the derived category $\mathrm{D}(\mathfrak{X}\text{-Flat})$,

$$\otimes^{\mathfrak{X}}\colon \mathrm{D}(\mathfrak{X}\text{-Flat}) \times \mathrm{D}(\mathfrak{X}\text{-Flat}) \longrightarrow \mathrm{D}(\mathfrak{X}\text{-Flat}).$$

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$$\square_{\mathscr{D}^\bullet}\colon \mathrm{D^{co}}(\mathfrak{X}\text{-Tors}) \times \mathrm{D^{co}}(\mathfrak{X}\text{-Tors}) \longrightarrow \mathrm{D^{co}}(\mathfrak{X}\text{-Tors}).$$

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$$\mathscr{M}^\bullet \otimes_{\mathfrak{X}} \mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet, \mathscr{K}^\bullet),$$
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These three complexes represent the coderived category object $\mathscr{M}^\bullet \,\square_{\mathscr{D}^\bullet}\, \mathscr{N}^\bullet \in \mathrm{D^{co}}(\mathfrak{X}\text{-}\mathrm{Tors})$. The coderived category $\mathrm{D^{co}}(\mathfrak{X}\text{-}\mathrm{Tors})$ endowed with the cotensor product operation $\square_{\mathscr{D}^\bullet}$ becomes a tensor triangulated category. The dualizing complex $\mathscr{D}^\bullet \in \mathrm{D^{co}}(\mathfrak{X}\text{-}\mathrm{Tors})$ is the unit object of this tensor structure.

Semiderived Category

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Assume that $\mathfrak{X}$ is ind-Noetherian. The semiderived category $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-Tors})$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ relative to $\mathfrak{X}$

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Informally, the semiderived category $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$ is a mixture of the coderived category along the base ind-scheme $\mathfrak{X}$ and the conventional unbounded derived category along the fibers of the morphism $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$.

Semiderived Category

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If the homological dimension of the category $\mathfrak{X}\text{-Tors}$ is finite (e. g., $\mathfrak{X} = X$ is a regular scheme of finite Krull dimension),

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If the homological dimension of the category $\mathfrak{X}\text{-Tors}$ is finite (e. g., $\mathfrak{X} = X$ is a regular scheme of finite Krull dimension), then the semiderived category coincides with the derived category: $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-Tors}) = \mathrm{D}(\mathfrak{Y}\text{-Tors})$.

Semiderived Category

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For any ind-Noetherian ind-scheme $\mathfrak{X}$ and any flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the composition of triangulated functors

$$\mathrm{K}(\mathfrak{Y}\text{-}\mathrm{Tors}_{\mathfrak{X}\text{-inj}}) \longrightarrow \mathrm{K}(\mathfrak{Y}\text{-}\mathrm{Tors}) \longrightarrow \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$$

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For any ind-Noetherian ind-scheme $\mathfrak{X}$ and any flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the composition of triangulated functors

$$\mathrm{K}(\mathfrak{Y}\text{-}\mathrm{Tors}_{\mathfrak{X}\text{-inj}}) \longrightarrow \mathrm{K}(\mathfrak{Y}\text{-}\mathrm{Tors}) \longrightarrow \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$$

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For any ind-Noetherian ind-scheme $\mathfrak{X}$ and any flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the composition of triangulated functors

$$\mathrm{K}(\mathfrak{Y}\text{-}\mathrm{Tors}_{\mathfrak{X}\text{-inj}}) \longrightarrow \mathrm{K}(\mathfrak{Y}\text{-}\mathrm{Tors}) \longrightarrow \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$$

induces a triangulated equivalence $\mathrm{D}(\mathfrak{Y}\text{-}\mathrm{Tors}_{\mathfrak{X}\text{-inj}}) \simeq \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$. Here $\mathrm{D}(\mathfrak{Y}\text{-}\mathrm{Tors}_{\mathfrak{X}\text{-inj}})$ is the conventional unbounded derived category of the exact category $\mathfrak{Y}\text{-}\mathrm{Tors}_{\mathfrak{X}\text{-inj}}$.

Pro-Sheaves Flat over the Base

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and denote by $\mathfrak{Bar}^\bullet_\pi(\mathfrak{G}^\bullet)$ its totalization constructed by taking coproducts along the diagonals.

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are naturally isomorphic in $\mathrm{D}(\mathfrak{Y}_\mathfrak{X}\text{-Flat})$. These three complexes represent the desired derived tensor product object $\mathfrak{F}^\bullet \otimes^{\mathfrak{Y},\mathbb{L}} \mathfrak{G}^\bullet$ in $\mathrm{D}(\mathfrak{Y}_\mathfrak{X}\text{-Flat})$.

Similarly, assuming $\mathfrak{X}$ to be ind-Noetherian, one constructs the left derived functor of tensor product of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves and quasi-coherent torsion sheaves on $\mathfrak{Y}$,

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The Semitensor Product

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are naturally isomorphic in $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-Tors})$. These three complexes represent the derived tensor product object $\mathfrak{G}^\bullet\otimes^{\mathbb{L}}_{\mathfrak{Y}}\mathscr{M}^\bullet$ in $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-Tors})$.

The Semitensor Product

For any flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$,
the derived category $\mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat})$ is a tensor triangulated category

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For any flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the derived category $\mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat})$ is a tensor triangulated category with respect to the left derived tensor product functor $\otimes^{\mathfrak{Y},\mathbb{L}}$.

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For any flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the derived category $\mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat})$ is a tensor triangulated category with respect to the left derived tensor product functor $\otimes^{\mathfrak{Y},\mathbb{L}}$. Assuming $\mathfrak{X}$ to be ind-Noetherian, the semiderived category $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$ is a triangulated module category over $\mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat})$ with respect to the left derived tensor product functor $\otimes^{\mathbb{L}}_{\mathfrak{Y}}$.

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$$\lozenge_{\pi^*\mathscr{D}^{\bullet}_{\mathfrak{X}}}\colon \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors}) \times \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors}) \longrightarrow \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors}).$$

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Explicitly, let $\mathscr{M}^\bullet$ and $\mathscr{N}^\bullet$ be two complexes in $\mathfrak{Y}\text{-}\mathrm{Tors}$, Explicitly, let $\mathscr{M}^\bullet$ and $\mathscr{N}^\bullet$ be two complexes in $\mathfrak{Y}\text{-}\mathrm{Tors}$, and let $\mathscr{M}^\bullet \longrightarrow \mathscr{J}^\bullet$ and $\mathscr{N}^\bullet \longrightarrow \mathscr{K}^\bullet$ be two morphisms whose cones become coacyclic after applying π_*,

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The Semitensor Product: Conclusion

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$$\Bbbk \Diamond \Bbbk \simeq \bigwedge\nolimits^{\infty/2+*}(\Bbbk((t))),$$

which is a doubly unbounded, $\mathbb{Z}$-graded or $(\infty/2+\mathbb{Z})$-graded complex.

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