Semi-infinite algebraic geometry

Leonid Positselski – MÚ AV ČR

IM off-site meeting, Kubova Huť

October 21–23, 2024

Introduction

Homological algebra is the technical heart of the algebraic part of contemporary mathematics (including ring and module theory, algebraic and differential geometry, algebraic number theory, etc.)

In algebraic topology, the homology groups formalize the idea of measuring how many holes of various dimensions does a given topological space have.

To illustrate how it works, I will consider the example of the two-dimensional torus T^2. Topologically, it is the Cartesian product of two one-dimensional circles $S^1 \times S^1$.

Let me emphasize that we are interested in the surface of the torus and not in its interior.

Introduction: Homology of two-dimensional torus

Let me sketch a simplified definition of the dimension 1 homology group of the torus, avoiding all technical details.

Elements of the homology group $H_1(T^2)$ are equivalence classes of oriented closed curves on the torus (possibly consisting of several connected components). Such oriented closed curves are called 1-cycles. Draw the picture ...

Up to equivalence, a 1-cycle on the torus is determined by a pair of integers (m, n): the number of rotations in the lateral direction (i. e., along the equator) and the number of rotations in the longitudinal direction (i. e., along the meridian). The sign of the integer depends on the direction of the rotation.

When a cycle has several connected components, one takes the sum of the related pairs of integers in the group of pairs $\mathbb{Z}^2$.

Introduction: Homology of two-dimensional torus

I have vagulely explained the assertion that the degree 1 homology group of the torus is isomorphic to the group of pairs of integers,

$$H_1(T^2) \simeq \mathbb{Z}^2.$$

If b and c are two 1-cycles on the torus, one can deform them slightly so as to make them intersect properly. Then one can compute the number of the intersection points in $b \cap c$ with signs depending on the orientations. Draw the picture ...

This produces a nondegenerate skew-symmetric bilinear form (symplectic form)

$$H_1(T^2) \times H_1(T^2) \longrightarrow \mathbb{Z}$$

called the intersection pairing.

Introduction: Homology of two-dimensional torus

A 0-cycle on T^2 is a formal linear combination of points (with integer coefficients). Since the torus is arcwise connected, two 0-cycles b and c on T^2 are equivalent if and only if the sum of the coefficients in a is equal to the sum of the coefficients in b. So the degree 0 homology group of the torus is isomorphic to $\mathbb{Z}$,

$$H_0(T^2) \simeq \mathbb{Z}.$$

The degree 2 homology group of T^2 is also isomorphic to $\mathbb{Z}$,

$$H_2(T^2) \simeq \mathbb{Z}.$$

The whole torus is a 2-cycle representing a generator of $H_2(T^2)$.

As the torus is two-dimensional, its homology groups in degrees ≥ 3 vanish,

$$H_3(T^2) = H_4(T^2) = \cdots = 0.$$

Introduction: Posing the Problem (of Semi-Infinite Algebraic Topology)

We would like to replace the 2-dimensional torus T^2 with an infinite-dimensional, and moreover, a "doubly infinite-dimensional" topological space/manifold Y. So the directions along Y would be indexed by an infinite set S roughly divided into the "positive" and "negative" halves.

Then we would like to consider cycles in Y represented by singular subvarieties $W \subset Y$ spread along the "positive" directions. Such cycles up to a suitable equivalence would form a doubly unbounded sequence of semi-infinite homology groups denoted by

$$H_{\infty/2+n}(Y), \quad n \in \mathbb{Z}.$$

Here the notation presumes that if the set $\mathbb{Z}$ has ∞ elements and its subset $\mathbb{Z}_{\geq 0}$ has $\infty/2$ elements, then the set $\mathbb{Z}_{\geq 1}$ has $\infty/2 - 1$ elements, while the set $\mathbb{Z}_{\geq -1}$ has $\infty/2 + 1$ elements. The set $\{-2, -1\} \cup \mathbb{Z}_{\geq 1}$ also has $\infty/2 + 1$ elements, etc.

Introduction: Posing the Problem (of Semi-Infinite Algebraic Topology)

One might also want to have a sequence of homology groups formed by cycles spread in the “negative” directions along Y. These might be denoted $H_{\infty-\infty/2+n}(Y)$, $n \in \mathbb{Z}$. The homology groups formed by cycles spread in both the positive and negative directions would be then denoted $H_{\infty-m}(Y)$, $m \geq 0$.

Intersecting the “positive” and “negative” semi-infinite cycles, one would construct a pairing

$$H_{\infty/2+n}(Y) \times H_{\infty-\infty/2-n}(Y) \longrightarrow \mathbb{Z}.$$

Intersecting the semi-infinite cycles with cycles going in both kinds of directions might provide multiplication maps

$$H_{\infty/2+n}(Y) \times H_{\infty-m}(Y) \longrightarrow H_{\infty/2+n-m}(Y),$$

etc.

Time to Admit It: Fake Introduction

The preceding part of this talk might be titled "Fake introduction" or "Introduction to a Dream". Its aim is to provide some bits of relevant geometric intuition, not to introduce a real, presently existing subject.

Semi-infinite algebraic topology does not exist. To the best of my knowledge, noone was able to develop such a theory yet.

The actual content of my book [Pos23] on which this talk is based is quite different. Still, my work is inspired by the kind of geometric intuition described above.

Actual History

The term “semi-infinite” was introduced in a 1984 paper of Boris Feigin, where he defined the semi-infinite homology of certain “doubly infinite-dimensional” Lie algebras (specifically, Kac–Moody and Virasoro). These are Lie algebras with an unbounded grading by (both the positive and negative) integers, $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$, with finite-dimensional grading components $\mathfrak{g}_n$.

The definitive treatment of the semi-infinite homology of Lie algebras was given by Beilinson and Drinfeld in a 2000 manuscript. It was published in their 2004 book.

Around 1993-95, I participated in a seminar run by Feigin for his graduate students. The problem of defining the semi-infinite (co)homology of associative algebras was posed there.

Feigin also shared with us his vision of semi-infinite geometry and topology. This made a deep impression on me.

Actual History

The first definition of semi-infinite homology and cohomology of associative algebras was proposed by Arkhipov in 1996–97. A version of Arkhipov's definition was suggested by Sevostyanov in 2000–01. A completely different approach was initiated by Bezrukavnikov in 2000.

My work on semi-infinite (co)homology of associative algebras started in 2000 and took many years to ripen. I wrote some letters to Arkhipov and Bezrukavnikov outlining my point of view in 2000, and then a second series of letters in 2002. My `arXiv` preprint on the topic only appeared in 2007. In 2010, it was published as a book [Pos10].

One of the main results of [Pos10], contained in an appendix written jointly with Arkhipov, was a comparison theorem establishing that my version of the semi-infinite homology of associative algebras agrees with Feigin's and Beilinson–Drinfeld's semi-infinite homology of Lie algebras in a suitable setting.

Actual History

In a more geometric context, the semi-infinite flag manifolds and semi-infinite Grassmannians were introduced by Feigin and Frenkel in 1990. These are infinite-dimensional algebraic varieties acted upon by infinite-dimensional algebraic groups with orbits of infinite dimension and infinite codimension in the ambient variety. They are natural objects of interest in geometric representation theory.

In my work, a program of Semi-Infinite Algebraic Geometry was outlined in 2015–17 in the introduction to the paper [Pos17]. The aim was to suggest a setting in infinite-dimensional algebraic geometry where a semi-infinite homology theory (and other features of semi-infinite homological algebra known from the book [Pos10]) could be constructed.

This program was partially realized in my 2021–23 book [Pos23].

Introduction Cont'd: Semi-Infinite Set Theory

A semi-infinite structure on a set S is the datum of a subset $S^+ \subset S$ defined up to adjoining or removing a finite number of elements from S.

The subset $S^+ \subset S$, as well any other subset $S^{+\prime} \subset S$ for which the symmetric difference $(S^+ \cup S^{+\prime}) \setminus (S^+ \cap S^{+\prime})$ is finite, is called a semi-infinite subset in S. The complements to semi-infinite subsets, like $S^- = S \setminus S^+$, are called co-semi-infinite subsets.

The cardinalities of semi-infinite subsets in S are defined by the rules $|S^+| = \infty/2$ and

$$|S^{+\prime}| = \infty/2 + |S^{+\prime} \setminus S^+| - |S^+ \setminus S^{+\prime}|.$$

So the cardinality of a semi-infinite subset $S^{+\prime} \subset S$ is an expression of the form $\infty/2 + n$, with $n \in \mathbb{Z}$.

Given a field $\Bbbk$ and a set S with a semi-infinite structure, one can construct a topological $\Bbbk$-vector space

$$V_{S,S^+} = \bigoplus\nolimits_{t\in S\setminus S^+} \Bbbk t \oplus \prod\nolimits_{s\in S^+} \Bbbk s.$$

Specifically, the elements of V_{S,S^+} are the formal expressions

$$v = \sum\nolimits_{s\in S} a_s s, \quad a_s \in \Bbbk$$

where the coefficient a_s is arbitrary for elements $s \in S^+$, but a_t may be only nonzero for a finite number of elements $t \in S \setminus S^+$.

The vector space V_{S,S^+} remains unchanged when one removes a finite number of elements from S^+ or adjoins to S^+ a finite number of elements from S. The set S is a topological basis in the natural topology on V_{S,S^+}.

Introduction Cont'd: Semi-Infinite Linear Algebra

Topological vector spaces of the form V_{S,S^+} are called locally linearly compact, or Tate vector spaces.

More precisely, a complete, separated topological $\mathbb{k}$-vector space is called linearly compact (or "pseudocompact", or "pro-finite-dimensional") if it has a base of neighborhoods of zero consisting of vector subspaces of finite codimension. A topological vector space is called locally linearly compact if it has a linearly compact open subspace.

The standard example of a set with a semi-infinite structure is the set of all integers $S = \mathbb{Z}$ with the semi-infinite subset of nonnegative integers $S^+ = \mathbb{Z}_{\geq 0}$. The related topological vector space V_{S,S^+} is the vector space of Laurent formal power series $V_{S,S^+} = \mathbb{k}((t))$.

Introduction Cont'd: Semi-Infinite Geometry

Semi-infinite geometry can be informally defined as a study of geometric shapes with local coordinates indexed by sets with semi-infinite structure. For a semi-infinite variety Y with local coordinates y_s indexed by a set S with a semi-infinite subset $S^+ \subset S$, it makes sense to assume that, for every point $p \in Y$, the set of all indices $s \in S$ such that $y_s(p) \neq 0$ is contained in some semi-infinite subset $S^{+\prime} \subset S$ (depending on the point p).

The standard example of a semi-infinite algebraic variety is the underlying affine algebraic variety Y of the vector space of Laurent formal power series $\Bbbk((t))$. Let us write $f(t) = \sum_{n\in\mathbb{Z}} y_n t^n$ for a generic element $f(t) \in \Bbbk((t))$. Then y_n, $n \in \mathbb{Z}$ is a global coordinate system on Y, indexed by the set $S = \mathbb{Z}$ with the standard semi-infinite structure. The condition above is satisfied: for every $f \in \Bbbk((t))$, the set of all $n \in \mathbb{Z}$ such that $y_n \neq 0$ is at most semi-infinite, i. e., it is contained in the union of $S^+ = \mathbb{Z}_{\geq 0}$ with a finite set of negative integers.

Introduction Cont'd: Semi-Infinite Geometry

Affine algebraic varities over an algebraically closed field $\Bbbk$ are defined as the zero sets of systems of polynomial equations $f_1(x_1, \ldots, x_m) = 0, \ \ldots, \ f_n(x_1, \ldots, x_m) = 0$. These are the closed subvarieties in the m-dimensional affine space $\mathbb{A}_k^m$.

One can define a semi-infinite closed subvariety or a semi-infinite algebraic cycle W in the space of Laurent series $Y = \Bbbk((t))$ as the zero sets of systems of polynomial equations

$$\begin{aligned} &y_{-\ell-1} = y_{-\ell-2} = y_{-\ell-3} = \cdots = 0, \\ &f_1(y_{-\ell}, y_{-\ell+1}, \ldots, y_0, y_1, \ldots, y_m) = 0, \ \cdots, \\ &f_n(y_{-\ell}, y_{-\ell+1}, \ldots, y_0, y_1, \ldots, y_m) = 0, \ n \geq 0, \end{aligned}$$

where ℓ and $m \in \mathbb{Z}_{\geq 0}$ depend on W. So all the coordinates with large negative numbers are equated to zero, and in addition a finite number of polynomial equations are imposed on the coordinates with intermediate numbers. No equations are imposed on the coordinates with large positive numbers.

Introduction Fin'd: Semi-Infinite Geometry

Considering semi-infinite algebraic cycles in Y up to a suitable equivalence, and intersecting them with algebraic cycles of finite codimension in Y (also viewed up to a suitable equivalence), one might have a kind of semi-infinite intersection theory in infinite-dimensional algebraic geometry. This would be a (presently nonexistent) algebro-geometric counterpart of the nonexistent algebraic topology of semi-infinite cycles discussed above.

The actually existing theory developed in [Pos23] considers a flat affine morphism of infinite-dimensional varieties $\pi\colon Y \longrightarrow X$. The typical/standard example is

$$\pi\colon \Bbbk((t)) \longrightarrow \Bbbk((t))/\Bbbk[[t]],$$

where $\Bbbk[[t]]$ is the vector subspace of formal Taylor power series in the vector space $\Bbbk((t))$ of formal Laurent power series. The quotient space $\Bbbk((t))/\Bbbk[[t]]$ is the space of "Laurent tails".

Summary for the Standard Example

For the rest of this talk, I switch to the algebraic language.

Continuing with the standard example, consider the ring of polynomials $B = \mathbb{k}[\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots]$ in the doubly infinite sequence of variables y_n, $n \in \mathbb{Z}$. Let us say that a B-module M is torsion if for every $m \in M$ there exists $\ell \geq 0$ such that $y_n m = 0$ for all $n < -\ell$. No restrictions are imposed on the action of the variables y_n, $n \geq 0$, in the module M.

Alternatively, one can think of torsion B-modules as of discrete modules over the complete topological ring

$$\mathfrak{B} = \varprojlim_{\ell<0} \mathbb{k}[y_\ell, \dots, y_{-1}, y_0, y_1, y_2, \dots]$$

of functions on $\mathbb{k}((t)) = \{\, \sum_{n\in\mathbb{Z}} y_n t^n \mid y_n = 0 \text{ for } n \ll 0 \,\}$.

Here the ring $\mathfrak{B}$ can be also constructed as the completion of the polynomial ring B with respect to the ring topology with a base of neighborhoods of zero formed by the ideals J_ℓ generated by $y_{-\ell-1}$, $y_{-\ell-2}$, …, where $\ell \geq 0$.

Summary for the Standard Example

The category $\mathcal{B} = B\text{-}\mathcal{M}od_{tors}$ of torsion B-modules is abelian. In the rest of this talk, my aim is to explain how to define a certain exotic derived category of the abelian category $\mathcal{B}$, called the semiderived category of torsion B-modules and denoted $\mathcal{D}^{si}(\mathcal{B})$.

The semiderived category, first defined in the book [Pos10], is the key technical concept and the main innovation in my approach to semi-infinite algebra and geometry. It plays a central role in both the books [Pos10] and [Pos23].

The main result of [Pos23] is the construction of a tensor (monoidal) structure on the semiderived category $\mathcal{D}^{si}_X(Y)$ assigned to a flat affine morphism $\pi\colon Y \longrightarrow X$. The tensor product operation $\lozenge$ on $\mathcal{D}^{si}_X(Y)$ is called the semitensor product. So, in particular, in our example $\mathcal{D}^{si}(\mathcal{B})$ is naturally a tensor triangulated category.

Summary for the Standard Example

The triangulated category $\mathcal{D}^{si}(\mathcal{B})$ is constructed as the triangulated Verdier quotient category of the chain homotopy category $\mathcal{K}(\mathcal{B})$ of unbounded complexes in $\mathcal{B}$ by a certain thick subcategory $\mathcal{A}c^{si}(\mathcal{B})$; so $\mathcal{D}^{si}(\mathcal{B}) = \mathcal{K}(\mathcal{B})/\mathcal{A}c^{si}(\mathcal{B})$.

The thick subcategory of complexes to be killed $\mathcal{A}c^{si}(\mathcal{B}) \subset \mathcal{K}(\mathcal{B})$ is properly contained in the full subcategory $\mathcal{A}c(\mathcal{B}) \subset \mathcal{K}(\mathcal{B})$ of acyclic complexes in $\mathcal{B}$, that is $\mathcal{A}c^{si}(\mathcal{B}) \subsetneq \mathcal{A}c(\mathcal{B})$.

So some acyclic complexes in the abelian category $\mathcal{B}$ represent nonzero objects in the semiderived category. In particular, the unit object of the tensor structure on $\mathcal{D}^{si}(\mathcal{B})$ turns out to be an acyclic complex in this example.

Consider the one-dimensional B-module $\mathbb{k}$ with the zero action of all the variables y_i. Then the semitensor product $\mathbb{k} \lozenge \mathbb{k}$ in $\mathcal{D}^{si}(\mathcal{B})$ is a complex of B-modules with a doubly unbounded sequence of nonzero homology modules. In this sense, the semitensor product $\lozenge$ can be viewed as a semi-infinite homology theory.

What is the semiderived category?

Let me start with the conventional derived category. Given a ring R, one considers sequences of R-modules and homomorphisms of R-modules

$$\cdots \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \longrightarrow \cdots$$

such that the composition of every pair of successive maps is zero, $d_n d_{n+1} = 0$ for every $n \in \mathbb{Z}$. Such sequences of modules and homomorphisms $M_\bullet$ are called complexes of R-modules.

The equation $d_n d_{n+1} = 0$ is equivalent to the condition that the image of the map d_{n+1} is contained in the kernel of the map d_n. The quotient R-module $H_n(M_\bullet) = \operatorname{Ker}(d_n)/\operatorname{Im}(d_{n+1})$ is called the degree n homology module of $M_\bullet$.

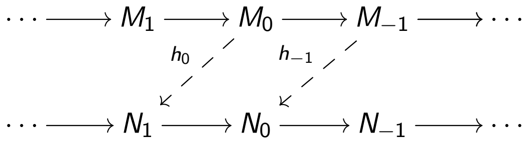
A complex $M_\bullet$ such that $H_n(M_\bullet) = 0$ (i. e., $\operatorname{Ker}(d_n) = \operatorname{Im}(d_{n+1})$) for all $n \in \mathbb{Z}$ is said to be acyclic.

What is the semiderived category?

In homological algebra, one considers complexes and homomorphisms of complexes up to various equivalence relations. A (homo)morphism of complexes $f_\bullet\colon M_\bullet \longrightarrow N_\bullet$ is a commutative diagram of homomorphisms of modules

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M_1 & \xrightarrow{d_1^M} & M_0 & \xrightarrow{d_0^M} & M_{-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} & & \\
\cdots & \longrightarrow & N_1 & \xrightarrow{d_1^N} & N_0 & \xrightarrow{d_0^N} & N_{-1} & \longrightarrow & \cdots
\end{array}$$

A morphism of complexes $f_\bullet\colon M_\bullet \longrightarrow N_\bullet$ is said to be homotopic to zero if there exists a sequence of module maps $h_n\colon M_n \longrightarrow N_{n+1}$ such that $f_n = d_{n+1}^N h_n + h_{n-1} d_n^M$ for all $n \in \mathbb{Z}$,



What is the semiderived category?

Two homomorphisms of complexes $f_\bullet, g_\bullet \colon M_\bullet \longrightarrow N_\bullet$ are said to be (chain) homotopic if their difference $f_\bullet - g_\bullet$ is homotopic to zero. The chain homotopy is the most elementary type of equivalence relation on complexes and morphisms of complexes. The category of complexes of R-modules with morphisms up to chain homotopy is called the homotopy category $\mathcal{K}(R\text{-}\mathcal{M}od)$.

More powerful equivalence relations on complexes are classified by the full subcategories in $\mathcal{K}(R\text{-}\mathcal{M}od)$ consisting of all the complexes which this equivalence relation turns into zero objects. For example, a morphism of complexes $f_\bullet \colon M_\bullet \longrightarrow N_\bullet$ is called a quasi-isomorphism if the induced morphism of the homology modules $H_n(f_\bullet) \colon H_n(M_\bullet) \longrightarrow H_n(N_\bullet)$ is an isomorphism for all n.

The category of complexes up to quasi-isomorphism is called the derived category $\mathcal{D}(R\text{-}\mathcal{M}od)$. The related class of annihilated objects $\mathcal{A}c(R\text{-}\mathcal{M}od)$ consists of all the acyclic complexes.

What is the semiderived category?

An R-module J is called injective if, for every R-module M and a submodule $N \subset M$, every R-module homomorphism $N \longrightarrow J$ can be extended to an R-module homomorphism $M \longrightarrow J$.

A complex $M_\bullet$ is called bounded below if $M_n = 0$ for all $n \gg 0$,

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M_m \longrightarrow M_{m-1} \longrightarrow M_{m-2} \longrightarrow \cdots$$

A bounded below complex of R-modules $M_\bullet$ is acyclic if and only if for any (arbitrary or bounded below) complex of injective R-modules $J_\bullet$, every morphism of complexes $M_\bullet \longrightarrow J_\bullet$ is homotopic to zero.

For unbounded complexes of R-modules $M_\bullet$ this is no longer true. The thematic example is the ring of dual numbers $R = \Bbbk[\epsilon]/(\epsilon^2)$ and the complex of R-modules

$$\cdots \longrightarrow R \xrightarrow{\epsilon *} R \xrightarrow{\epsilon *} R \longrightarrow \cdots$$

This is an acyclic complex of injective R-modules, but its identity endomorphism is not homotopic to zero.

What is the semiderived category?

An (unbounded) complex of R-modules $M_\bullet$ is called coacyclic if, for every complex of injective R-modules $J_\bullet$, any morphism of complexes $M_\bullet \longrightarrow J_\bullet$ is homotopic to zero. Any coacyclic complex of modules is acyclic, but the converse need not be the case. For example, the complex of modules over the ring of dual numbers written above is acyclic, but not coacyclic.

Notice that the property of a complex of R-modules $M_\bullet$ to be acyclic only depends on $M_\bullet$ as a complex of abelian groups. But the property of $M_\bullet$ to be coacyclic does depend of the R-module structures on the components of $M_\bullet$.

The class of all coacyclic complexes is denoted by $\mathcal{A}c^{co}(R\text{-}\mathcal{M}od) \subset \mathcal{K}(R\text{-}\mathcal{M}od)$. The equivalence relation on complexes of R-modules killing all the coacyclic complexes and only them is called the co-quasi-isomorphism. The category of complexes of R-modules up to co-quasi-isomorphism is called the coderived category $\mathcal{D}^{co}(R\text{-}\mathcal{M}od)$.

What is the semiderived category?

Finally we return to the torsion modules over the ring $B = \Bbbk[\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots]$. Consider the subring $A = \Bbbk[\dots, y_{-2}, y_{-1}] \subset B$ spanned by the variables y_n with negative numbers $n < 0$. Let $\mathcal{A}$ denote the abelian category of torsion A-modules. Then there is a forgetful functor $\mathcal{B} \longrightarrow \mathcal{A}$ (forgetting the action of y_n with $n \geq 0$).

A complex of torsion B-modules $M_\bullet$ is called semicoacyclic if $M_\bullet$ is coacyclic as a complex of torsion A-modules. This means, for every complex of injective objects $J_\bullet$ in the category $\mathcal{A}$, every morphism of complexes of A-modules $M_\bullet \longrightarrow J_\bullet$ must be homotopic to zero.

The equivalence relation on the complexes of torsion B-modules killing all the semicoacyclic complexes and only them is called the semi-co-quasi-isomorphism. The category of complexes of torsion B-modules up to semi-co-quasi-isomorphism is called the semiderived category $\mathcal{D}^{si}(\mathcal{B}) = \mathcal{D}^{si}_{\mathcal{A}}(\mathcal{B})$. This is the main object of study in the book [Pos23] (for this standard example).

S. M. Arkhipov. Semi-infinite cohomology of associative algebras and bar-duality. *Internat. Math. Research Notices* **1997**, #17, p. 833–863. arXiv:q-alg/9602013

S. Arkhipov. Semi-infinite cohomology of quantum groups II. Topics in quantum groups and finite-type invariants, p. 3–42, *Amer. Math. Soc. Translations, Ser. 2* **185**, AMS, Providence, RI, 1998. arXiv:q-alg/9610020

A. Beilinson, V. Drinfeld. Quantization of Hitchin's integrable system and Hecke eigensheaves. February 2000. Available from http://www.math.utexas.edu/~benzvi/Langlands.html or http://math.uchicago.edu/~drinfeld/langlands.html

A. Beilinson, V. Drinfeld. Chiral algebras. AMS Colloquium Publications, 51. AMS, Providence, RI, 2004.

R. Bezrukavnikov, L. Positselski. On semi-infinite cohomology of finite-dimensional graded algebras. *Compositio Math.* **146**, #2, pp. 480–496, 2010. arXiv:0803.3252 [math.QA]

B. L. Feigin. The semi-infinite homology of Kac-Moody and Virasoro Lie algebras. *Russian Math. Surveys* **39**, #2, p. 155–156, 1984.

B. L. Feigin, E. V. Frenkel. Affine Kac-Moody algebras and semi-infinite flag manifolds. *Comm. Math. Phys.* **128**, #1, p. 161–189, 1990.

A. Sevostyanov. Semi-infinite cohomology and Hecke algebras. *Advances Math.* **159**, #1, p. 83–141, 2001. arXiv:math.RA/0004139

L. Positselski. Homological algebra of semimodules and semicontramodules: Semi-infinite homological algebra of associative algebraic structures. Appendix C in collaboration with D. Rumynin; Appendix D in collaboration with S. Arkhipov. Monografie Matematyczne vol. 70, Birkhäuser/Springer Basel, 2010. xxiv+349 pp. arXiv:0708.3398 [math.CT]

L. Positselski. Coherent rings, fp-injective modules, dualizing complexes, and covariant Serre–Grothendieck duality. *Selecta Math. (New Ser.)* **23**, #2, p. 1279–1307, 2017. arXiv:1504.00700 [math.CT]

L. Positselski. Semi-infinite algebraic geometry. Slides of the presentation at the conference "Some Trends in Algebra", Prague, September 2015. Available from http://positselski.narod.ru/semi-inf-nopause.pdf or http://math.cas.cz/~positselski/semi-inf-nopause.pdf

L. Positselski. Semi-infinite algebraic geometry of quasi-coherent sheaves on ind-schemes: Quasi-coherent torsion sheaves, the semiderived category, and the semitensor product. Birkhäuser/Springer Nature, Cham, Switzerland, 2023. xix+216 pp. arXiv:2104.05517 [math.AG]

L. Positselski. Semi-infinite algebraic geometry of quasi-coherent torsion sheaves. Slides of the presentation at the seminar "New Directions in Group Theory and Triangulated Categories", February 2024. Available from http://positselski.narod.ru/semi-inf-alg-geom-nopause.pdf or http://math.cas.cz/~positselski/semi-inf-alg-geom-nopause.pdf