Semi-infinite algebraic geometry

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Let me emphasize that we are interested in the surface of the torus and not in its interior.

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When a cycle has several connected components, one takes the sum of the related pairs of integers in the group of pairs \mathbb{Z}^2 .

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Introduction: Homology of two-dimensional torus

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$$H_3(T^2) = H_4(T^2) = \cdots = 0.$$

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Feigin also shared with us his vision of semi-infinite geometry and topology. This made a deep impression on me.

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One of the main results of [Pos10], contained in an appendix written jointly with Arkhipov, was a comparison theorem

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One of the main results of [Pos10], contained in an appendix written jointly with Arkhipov, was a comparison theorem establishing that my version of the semi-infinite homology of associative algebras agrees with Feigin's and Beilinson–Drinfeld's semi-infinite homology of Lie algebras in a suitable setting.

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This program was partially realized in my 2021-23 book [Pos23].

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So the cardinality of a semi-infinite subset $S^{+\prime} \subset S$ is an expression of the form $\infty/2 + n$, with $n \in \mathbb{Z}$.

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The vector space V_{S,S^+} remains unchanged when one removes a finite number of elements from S^+ or adjoins to S^+ a finite number of elements from S. The set S is a topological basis in the natural topology on V_{S,S^+} .

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Affine algebraic varities over an algebraically closed field \Bbbk

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where ℓ and $m \in \mathbb{Z}_{\geq 0}$ depend on W. So all the coordinates with large negative numbers are equated to zero, and in addition a finite number of polynomial equations are imposed on the coordinates with intermediate numbers.

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A complex M_{\bullet} such that $H_n(M_{\bullet}) = 0$ (i. e., $\text{Ker}(d_n) = \text{Im}(d_{n+1})$) for all $n \in \mathbb{Z}$ is said to be acyclic.

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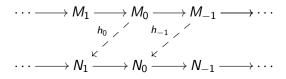
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3

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This is an acyclic complex of injective *R*-modules, but its identity endomorphism is not homotopic to zero.

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