Semi-infinite algebraic geometry

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Let me emphasize that we are interested in the surface of the torus and not in its interior.

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When a cycle has several connected components, one takes the sum of the related pairs of integers in the group of pairs $\mathbb{Z}^2$.

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$$H_3(T^2) = H_4(T^2) = \cdots = 0.$$

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Intersecting the semi-infinite cycles with cycles going in both kinds of directions might provide multiplication maps

$$H_{\infty/2+n}(Y) \times H_{\infty-m}(Y) \longrightarrow H_{\infty/2+n-m}(Y),$$

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My work on semi-infinite (co)homology of associative algebras started in 2000

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This program was partially realized in my 2021–23 book [Pos23].

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So the cardinality of a semi-infinite subset $S^{+\prime} \subset S$ is an expression of the form $\infty/2 + n$, with $n \in \mathbb{Z}$.

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Introduction Cont'd: Semi-Infinite Geometry

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Introduction Cont'd: Semi-Infinite Geometry

Affine algebraic varities over an algebraically closed field $\Bbbk$

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Introduction Cont'd: Semi-Infinite Geometry

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Introduction Fin'd: Semi-Infinite Geometry

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An R-module J is called injective if, for every R-module M and a submodule $N \subset M$, every R-module homomorphism $N \longrightarrow J$ can be extended to an R-module homomorphism $M \longrightarrow J$.

A complex $M_\bullet$ is called bounded below if $M_n = 0$ for all $n \gg 0$,

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M_m \longrightarrow M_{m-1} \longrightarrow M_{m-2} \longrightarrow \cdots$$

A bounded below complex of R-modules $M_\bullet$ is acyclic if and only if for any (arbitrary or bounded below) complex of injective R-modules $J_\bullet$, every morphism of complexes $M_\bullet \longrightarrow J_\bullet$ is homotopic to zero.

For unbounded complexes of R-modules $M_\bullet$ this is no longer true. The thematic example is the ring of dual numbers $R = \Bbbk[\epsilon]/(\epsilon^2)$ and the complex of R-modules

$$\cdots \longrightarrow R \xrightarrow{\epsilon *} R \xrightarrow{\epsilon *} R \longrightarrow \cdots$$

This is an acyclic complex of injective R-modules, but its identity endomorphism is not homotopic to zero.

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The equivalence relation on the complexes of torsion B-modules killing all the semicoacyclic complexes and only them is called the semi-co-quasi-isomorphism. The category of complexes of torsion B-modules up to semi-co-quasi-isomorphism is called the semiderived category $\mathcal{D}^{si}(\mathcal{B}) = \mathcal{D}^{si}_{\mathcal{A}}(\mathcal{B})$. This is the main object of study in the book [Pos23] (for this standard example).

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