# Semi-Infinite Algebraic Geometry

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We will come to more specific definitions shortly.

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More generally, if G is an affine algebraic group over a field k, then the fibration  $G(k((z))) \longrightarrow G(k((z)))/G(k[[z]])$  can be viewed as a semi-infinite object in algebraic geometry.

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Classical homological algebra: two hypercohomology spectral sequences

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Then there are two spectral sequences converging to the same limit

$${}^{\prime}E_{2}^{pq} = R^{p}F(H^{q}C^{\bullet}) \Longrightarrow \mathbb{H}^{p+q}(C^{\bullet});$$
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Hence differential derived functors of the first and the second kind [Eilenberg–Moore '62 — Husemoller–Moore–Stasheff '74].

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Let  $\mathcal A$  be an abelian category with enough projectives and injectives. Then the derived category of complexes over  $\mathcal A$  bounded above or below can be alternatively described as

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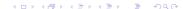
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   derived category of the first kind (conventional)
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[Hinich, Lefèvre-Hasegawa, Krause, L.P., H. Becker, ... '98 – ]



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Classical homological algebra is the realm in which there is no difference between the theories of the first and the second kind.

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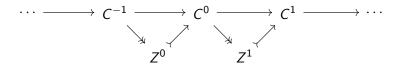
- a derived category of the second kind (the coderived or the contraderived category) along the variables from A and
- the derived category of the first kind (the conventional derived category) along the complementary variables from *R*.

Let  ${\mathcal E}$  be an exact category (in the sense of Quillen).

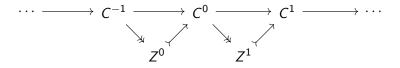
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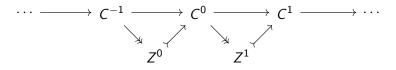


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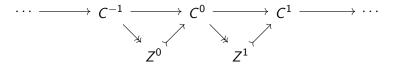
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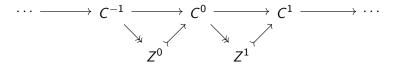
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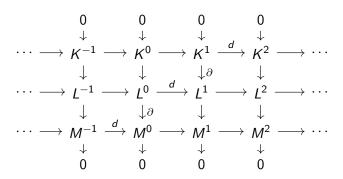
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$$0 \qquad 0 \qquad 0 \qquad 0$$

$$\cdots \longrightarrow K^{-1} \longrightarrow K^{0} \longrightarrow K^{1} \xrightarrow{d} K^{2} \longrightarrow \cdots$$

$$\downarrow \qquad \downarrow \qquad \downarrow \partial \qquad \downarrow$$

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Form the total complex  $\operatorname{Tot}(K^{\bullet} \to L^{\bullet} \to M^{\bullet})$  by taking direct sums along the diagonals, with the differential  $D = \partial \pm d$ .

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Example: the acyclic complex  $\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$  of modules over the algebra of dual numbers  $\Lambda = k[\varepsilon]/(\varepsilon^2)$  is neither coacyclic, nor contraacyclic.

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## Semi-infinite algebraic varieties

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Can be viewed as the "ind-spectrum" of the pro-Noetherian topological ring

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This definition works well enough to provide an exact category of contraherent cosheaves X- $\operatorname{ctrh}$  on X.



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The abelian category X-qcoh always has enough injective objects.

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In particular, if A is a Noetherian commutative ring, then a dualizing complex of A-modules  $D_A^{\bullet}$  is the same thing as a dualizing complex of quasi-coherent sheaves on Spec A.

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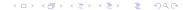
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[Jørgensen, Krause, Iyengar-Krause '05-'06]



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[Neeman, Murfet '07-'08]



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Contramodules are module-like objects endowed with infinite summation (or, occasionally, integration) operations, understood algebraically as infinitary (linear) operations subject to natural axioms. Contramodules carry no underlying topologies on them, but feel like being in some sense "complete". For about every class of "discrete" or "torsion" modules, there is an much less familiar, but no less interesting accompanying class of contramodules.

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Contramodule categories have exact functors of infinite product, and typically enough projective objects, but nonexact functors of infinite direct sum and no injectives.

The historical obscurity/neglect of contramodules seems to be the reason why many people believe that projectives are much less common than injectives in "naturally appearing" abelian categories.

Fancy definition of (conventional) modules over a discrete ring R:

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The composition of the contraaction map  $\pi \colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$  with the obvious embedding  $\mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$  defines the underlying left  $\Re$ -module structure on every left  $\Re$ -contramodule.

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For any discrete right  $\mathfrak{R}$ -module  $\mathcal{N}$  and any abelian group U, the left  $\mathfrak{R}$ -module  $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{N},U)$  has a natural left  $\mathfrak{R}$ -contramodule structure.

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[Schenzel, Porta–Shaul–Yekutieli, '03–'14]



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[MGM Duality: Matlis, Greenlees–May, Dwyer–Greenlees, Porta–Shaul–Yekutieli, L.P., '78–'15]

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The class  $\mathfrak{A}\text{-}\mathrm{contra}_{\mathrm{flat}}$  of flat  $\mathfrak{A}\text{-}\mathrm{contra}$  modules is closed under extensions, infinite products, and the passage to the kernels of surjective morphisms in  $\mathfrak{A}\text{-}\mathrm{contra}$ , so in particular  $\mathfrak{A}\text{-}\mathrm{contra}_{\mathrm{flat}}$  inherits an exact category structure from  $\mathfrak{A}\text{-}\mathrm{contra}$ .

Denote by  $\mathfrak{A}\text{-}\mathrm{discr}$  the abelian category of discrete  $\mathfrak{A}\text{-}\mathrm{modules}.$ 



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#### Theorem

Any compatible system  $\mathcal{D}_{\mathfrak{A}_n}^{\bullet}$  of choices of dualizing complexes  $D_{A_n}^{\bullet}$  for the Noetherian rings  $A_n$ ,  $n \geqslant 0$ , induces an equivalence of triangulated categories

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The reason for the base and the fiber being this way comes from the definition of the semiderived category, which turns out to be the co/contraderived category along the subring and the conventional derived category in the complementary direction of the ambient ring relative to the subring.

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The choice of a dualizing complex  $D_A^{\bullet}$  for the coherent ring A

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In particular, for any dualizing complex  $\mathcal{D}_Y^{\bullet}$  on Y, the complex  $f^!\mathcal{D}_Y^{\bullet}$  is a dualizing complex on X. So we can set  $\mathcal{D}_X^{\bullet} = p^!\mathcal{O}_{\operatorname{Spec} k}$ .

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In particular, for any dualizing complex  $\mathcal{D}_{\mathbf{v}}^{\bullet}$  on Y, the complex  $f^!\mathcal{D}_{Y}^{\bullet}$  is a dualizing complex on X. So we can set  $\mathcal{D}_{X}^{\bullet} = \rho^!\mathcal{O}_{\operatorname{Spec} k}$ .

Then for any two complexes of quasi-coherent sheaves  $\mathcal{M}^{\bullet}$  and  $\mathcal{N}^{\bullet}$  on X one has

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where  $\boxtimes_k$  denotes the external tensor product functor, so  $\mathcal{M}^{\bullet} \boxtimes_{\iota} \mathcal{N}^{\bullet}$  is a complex of quasi-coherent sheaves on  $X \times_{k} X$ .

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The double-sided derived semitensor product operation should provide a tensor structure on the semiderived category of quasi-coherent torsion sheaves  $D^{\rm sico}_{\mathfrak{X}}(\mathfrak{Y}\text{-qcoh})$  with the unit object  $\pi^*\mathcal{D}^{\bullet}_{\mathfrak{X}}$ 

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The example of the fibration  $k((z)) \longrightarrow k((z))/k[[z]]$ 



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The example of the fibration  $k((z)) \longrightarrow k((z))/k[[z]]$  comes as an afterthought.

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