Semi-Infinite Algebraic Geometry

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We will come to more specific definitions shortly.

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More generally, if G is an affine algebraic group over a field k, then the fibration $G(k((z))) \longrightarrow G(k((z)))/G(k[[z]])$ can be viewed as a semi-infinite object in algebraic geometry.

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Classical homological algebra:

two hypercohomology spectral sequences

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Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor between abelian categories (assume that $\mathcal{A}$ has enough injectives).

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Then there are two spectral sequences converging to the same limit

$$'E_2^{pq} = R^pF(H^qC^\bullet) \Longrightarrow \mathbb{H}^{p+q}(C^\bullet);$$
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- "projective" and/or "injective" (adjusted for computing the derived functors),

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[Hinich, Lefèvre-Hasegawa, Krause, L.P., H. Becker, ... '98 –]

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\cdots & \longrightarrow & C^{-1} & & \longrightarrow & & C^{0} & & \longrightarrow & & C^{1} & \longrightarrow & \cdots \\
& & & \searrow\!\!\!\!\twoheadrightarrow & & \nearrow\!\!\!\!\rightarrowtail & & \searrow\!\!\!\!\twoheadrightarrow & & \nearrow\!\!\!\!\rightarrowtail & & & \\
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 & & \downarrow & & \downarrow & & \downarrow\partial & & \downarrow & & \\
\cdots & \longrightarrow & L^{-1} & \longrightarrow & L^0 & \xrightarrow{d} & L^1 & \longrightarrow & L^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow\partial & & \downarrow & & \downarrow & & \\
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 & & 0 & & 0 & & 0 & & 0 & &
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Form the total complex $\mathrm{Tot}(K^\bullet \to L^\bullet \to M^\bullet)$ by taking direct sums along the diagonals, with the differential $D = \partial \pm d$.

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A complex $C^\bullet$ in $\mathcal{E}$ is called absolutely acyclic if it belongs to the minimal thick subcategory of the homotopy category $\mathrm{Hot}(\mathcal{E})$ containing the complexes $\mathsf{Tot}(K^\bullet \to L^\bullet \to M^\bullet)$ for all the short exact sequences $0 \longrightarrow K^\bullet \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow 0$ of complexes in $\mathcal{E}$:

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- an $\mathcal{O}_X(U)$-module $\mathcal{M}(U)$ to every affine open subscheme $U \subset X$
- and an isomorphism of $\mathcal{O}_X(V)$-modules $\mathcal{M}(V) \simeq \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{M}(U)$ to every pair of embedded affine open subschemes $V \subset U \subset X$
- satisfying a compatibility equation for every triple of embedded affine open subschemes $W \subset V \subset U \subset X$.

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This definition works well enough to provide an exact category of contraherent cosheaves $X\text{-}\mathrm{ctrh}$ on X.

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The abelian category $X\text{-qcoh}$ always has enough injective objects. When the scheme X is quasi-compact and semi-separated, or Noetherian of finite Krull dimension, the exact category $X\text{-ctrh}$ has enough projective objects.

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[Jørgensen, Krause, Iyengar–Krause '05–'06]

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Example: equivalence on the base

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[Neeman, Murfet '07–'08]

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The historical obscurity/neglect of contramodules seems to be the reason why many people believe that projectives are much less common than injectives in "naturally appearing" abelian categories.

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The composition of the contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ with the obvious embedding $\mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$

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The composition of the contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ with the obvious embedding $\mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$ defines the underlying left $\mathfrak{R}$-module structure on every left $\mathfrak{R}$-contramodule.

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For any discrete right $\mathfrak{R}$-module $\mathcal{N}$ and any abelian group U, the left $\mathfrak{R}$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}, U)$ has a natural left $\mathfrak{R}$-contramodule structure.

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Conversely, given an additive, associative, and unital s-power infinite summation operation

$$(p_n)_{n=0}^{\infty} \longmapsto \sum_{n=0}^{\infty} s^n p_n$$

in P one can uniquely solve the system of equations $q_n = sq_{n+1} + p_n$ by setting

$$q_n = \sum_{i=0}^{\infty} s^i p_{n+i}.$$

A module P over a commutative ring R with an element $s \in R$ is an s-contramodule (i.e., a contramodule with respect to the operator of multiplication with s) if and only if $\mathrm{Ext}^i_R(R[s^{-1}], P) = 0$ for $i = 0$ and 1. (Notice that the R-module $R[s^{-1}]$ has projective dimension at most 1.)

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The weak proregularity property of a sequence of generators $\mathbf{s}$ of a finitely generated ideal $I \subset R$ only depends on the ideal $\sqrt{I} \subset R$ and not on the chosen generators. Hence the notion of a weakly proregular finitely generated ideal $I \subset R$.

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[Schenzel, Porta–Shaul–Yekutieli, '03–'14]

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[MGM Duality: Matlis, Greenlees–May, Dwyer–Greenlees, Porta–Shaul–Yekutieli, L.P., '78–'15]

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Denote by $\mathfrak{A}\text{-}\mathrm{discr}$ the abelian category of discrete $\mathfrak{A}$-modules.

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The example of the fibration $k((z)) \longrightarrow k((z))/k[[z]]$

To conclude:

- The fibers are glued in a simple way from large affine pieces.
- The base is glued in a complicated way from small affine pieces (and endowed with a dualizing complex).

The reason for the base and the fibers being like that is because

- The conventional derived category is well-behaved for modules over an arbitrary ring, and by the way of generalization for (co)sheaves on infinite-dimensional quasi-compact semi-separated schemes.
- The co/contraderived category is well-behaved for co/contramodules over a coalgebra, and by the way of generalization for (co)sheaves on finite-dimensional stacks and ind-Noetherian ind-schemes. (A coalgebra is a dualizing complex over itself.)

The example of the fibration $k((z)) \longrightarrow k((z))/k[[z]]$ comes as an afterthought.

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