The Very Flat Conjecture

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Posing the Problem

Let R be a commutative ring and S be a finitely presented commutative R-algebra. What can one say about S as an R-module?

This question may be too general and hard to access. Even finitely presented R-modules can be complicated. So let us restrict generality a bit.

Suppose that S is a flat R-module. What else can one say about the R-module S then?

We know that all finitely presented flat R-modules are projective. What can one say about finitely presented flat R-algebras?

Filtrations and Orthogonal Classes

Let R be an associative ring and G be a left R-module. Suppose that there is an ordinal α and a nondecreasing chain of submodules G_i in G with $0 \leqslant i \leqslant \alpha$ such that $G_0 = 0$, $G_\alpha = G$, $G_i \subset G_j$ for $i \leqslant j$, and $G_j = \bigcup_{i<j} G_i$ for all limit ordinals $j \leqslant \alpha$.

In this case we will say that G is a transfinitely iterated extension of the left R-modules G_{i+1}/G_i, $0 \leqslant i < \alpha$ (in the sense of the inductive limit). Sometimes one simply says that the R-module G is filtered by the R-modules G_{i+1}/G_i.

Filtrations and Orthogonal Classes

We denote by $R\text{-}\mathrm{mod}$ the abelian category of all left R-modules.

For any class of left R-modules $\mathcal{F} \subset R\text{-}\mathrm{mod}$, denote by $\mathcal{F}^{\perp_1} \subset R\text{-}\mathrm{mod}$ the class of all left R-modules C such that $\mathrm{Ext}^1_R(F, C) = 0$ for all $F \in \mathcal{F}$. For any class of left R-modules $\mathcal{C} \subset R\text{-}\mathrm{mod}$, denote by ${}^{\perp_1}\mathcal{C} \subset R\text{-}\mathrm{mod}$ the class of all left R-modules F such that $\mathrm{Ext}^1_R(F, C) = 0$ for all $C \in \mathcal{C}$.

The following result is due to Eklof and Trlifaj (2000). Let $\mathcal{G}$ be a set (rather than a proper class) of left R-modules. Assume that $R \in \mathcal{G}$. Then the class of left R-modules ${}^{\perp_1}(\mathcal{G}^{\perp_1})$ consists precisely of all the direct summands of transfinitely iterated extensions, in the sense of the inductive limit, of the R-modules from $\mathcal{G}$.

Formulation of the Conjecture (now Theorem)

Let R be a commutative ring and $r \in R$ be an element. We denote by $R[r^{-1}]$ the localization of the ring R with respect to the multiplicative subset $\{1, r, r^2, r^3, \dots\} \subset R$ generated by r. The projective dimension of the R-module $R[r^{-1}]$ does not exceed 1.

An R-module C is said to be *r-contraadjusted* if $\mathrm{Ext}^1_R(R[r^{-1}], C) = 0$. An R-module C is *contraadjusted* if it is r-contraadjusted for all $r \in R$.

An R-module F is said to be *very flat* if $\mathrm{Ext}^1_R(F, C) = 0$ for all contraadjusted R-modules C. So an R-module is very flat if and only if it is a direct summand of a transfinitely iterated extension (in the sense of the inductive limit) of R-modules, each of which is isomorphic to $R[r^{-1}]$ for some element $r \in R$.

More precisely, let $\mathbf{r}$ denote a finite set of elements $r_1, \ldots, r_m \in R$. An R-module C is said to be **r**-contraadjusted if it is r_j-contraadjusted for all $j = 1, \ldots, m$. An R-module F is said to be **r**-very flat if $\operatorname{Ext}^1_R(F, C) = 0$ for all $\mathbf{r}$-contraadjusted R-modules C. So an R-module is $\mathbf{r}$-very flat if and only if it is a direct summand of a transfinitely iterated extension, in the sense of the inductive limit, of R-modules isomorphic to R or $R[r_j^{-1}]$, $j = 1, \ldots, m$.

In other words, an R-module F is $\mathbf{r}$-very flat if and only if there exists an R-module G and an (arbitrarily large) ordinal α such that there is a increasing chain of submodules $G_i \subset G$, $0 \leqslant i \leqslant \alpha$ as above and for every $0 \leqslant i < \alpha$ one has either $G_{i+1}/G_i \simeq R$ or $G_{i+1}/G_i \simeq R[r_{j(i)}^{-1}]$ for some $1 \leqslant j(i) \leqslant m$.

An R-module F is said to be finitely very flat if there exists a finite subset $\mathbf{r} \subset R$ such that F is $\mathbf{r}$-very flat.

Formulation of Main Theorem

Finitely Very Flat Main Theorem

Let R be a commutative ring, S be a finitely presented commutative R-algebra, and F be a finitely presented S-module. Assume that the R-module F is flat. Then there exists a finite set of elements $\mathbf{r} = \{r_1, \ldots, r_m\} \subset R$ such that the R-module F is $\mathbf{r}$-very flat.

In other words, F is a finitely very flat R-module. Consequently, F is a very flat R-module.

Notice that the R-module F in the Main Theorem is always countably presented. Hence it follows that one can choose an R-module G as above such that the filtration G_i on it is indexed by a countable ordinal α.

Examples

Let R be a commutative ring and $f \in R[x]$ be a polynomial in one variable x with the coefficients in R. Then $R[x][f^{-1}]$ is a finitely presented R-algebra and a flat R-module. According to the Main Theorem, it follows that $R[x][f^{-1}]$ is a finitely very flat R-module. This is not at all obvious, as f is an element of $R[x]$ and not of R.

Let R be a commutative ring and $g(x) = x^n + g_{n-1}x^n + \ldots + g_0 \in R[x]$ be a unital polynomial. Then the ring $R[x]/gR[x]$ is a free R-module of rank n. Let $f \in R[x]$ be another polynomial. Then $(R[x]/gR[x])[f^{-1}]$ is a finitely presented R-algebra and a flat R-module. According to the Main Theorem, it follows that $(R[x]/gR[x])[f^{-1}]$ is a finitely very flat R-module.

Corollaries

Corollary (of the Main Theorem)

Let R be a commutative ring and S be a flat, finitely presented commutative R-algebra. Then any very flat S-module is a very flat R-module. Any finitely very flat S-module is a finitely very flat R-module.

Corollary (of the Main Theorem)

Let R be a commutative ring and P be a finitely generated projective R-module. Let $x\colon P \longrightarrow P$ be an R-linear map. Then $P[x^{-1}] = \varinjlim (P \xrightarrow{x} P \xrightarrow{x} P \xrightarrow{x} P \longrightarrow \cdots)$ is a finitely very flat R-module.

Corollary (of the Main Theorem)

Let k be a field and R, S be two commutative k-algebras. Let $t \in R \otimes_k S$ be an element. Then the ring $(R \otimes_k S)[t^{-1}]$ is a very flat R-module.

Applications

The applications of the Very Flat Conjecture (Main Theorem) known so far belong to Algebraic Geometry. They concern the properties of the push-forwards of quasi-coherent sheaves and pull-backs of contraherent cosheaves.

More precisely, the Main Theorem allows to

- push forward very flat quasi-coherent sheaves with respect to affine flat morphisms of finite presentation;
- pull back arbitrary ($=$ locally contraadjusted) locally contraherent cosheaves with respect to flat morphisms of finite presentation (between schemes).

The point is that there are many flat morphisms of finite presentation known in Algebraic Geometry. The Very Flat Conjecture tells that all of them are very flat.

Formulation of Main Theorem

The following version of the Main Theorem requires stronger assumptions and provides a weaker conclusion. However, it has a separate, quite different proof which may be of interest on its own.

Noetherian Main Theorem

Let R be a Noetherian commutative ring, S be a finitely generated commutative R-algebra, and F be a finitely generated S-module. Assume that the R-module F is flat. Then the R-module F is very flat.

The proof of Finitely Very Flat Main Theorem has a combinatorial flavor, in that it uses a complicated induction process. The proof of Noetherian Main Theorem has an abstract homological flavor, in that it uses completeness of certain cotorsion pairs in contramodule categories.

Proof of Noetherian Main Theorem

The proof of Noetherian Main Theorem is based on two lemmas.

Generic Freeness Lemma

Let R be a Noetherian commutative integral domain, S be a finitely generated commutative R-algebra, and F be a finitely generated S-module. Then there exists a nonzero element $a \in R$ such that the $R[a^{-1}]$-module $F[a^{-1}] = R[a^{-1}] \otimes_R F$ is free.

This is a textbook theorem in commutative algebra [H. Matsumura, Commutative ring theory, Theorem 24.1].

Proof of Noetherian Main Theorem

Noetherian Very Flat Main Lemma

Let R be a Noetherian commutative ring, $r \in R$ be an element, and F be a flat R-module. Then the R-module F is very flat if and only if the R/rR-module F/rF is very flat and the $R[r^{-1}]$-module $F[r^{-1}]$ is very flat.

To deduce the Noetherian Main Theorem from the Generic Freeness Lemma and the Noetherian Very Flat Main Lemma, one uses the Noetherian induction.

Noetherian Induction Principle

Let R be a Noetherian commutative ring. Then there cannot exist a sequence of rings R_n, $n \geqslant 0$, and nonzero elements $r_n \in R_n$ such that $R_0 = R$ and $R_{n+1} = R_n/r_nR_n$ for all $n \geqslant 0$.

Proof of Finitely Very Flat Main Theorem

The proof of Finitely Very Flat Main Theorem is based on the above Generic Freeness Lemma together with the following Finitely Very Flat Main Lemma.

Finitely Very Flat Main Lemma

Let R be a commutative ring, $r \in R$ be an element, and F be a flat R-module. Then the R-module F is finitely very flat if and only if the R/rR-module F/rF is finitely very flat and the $R[r^{-1}]$-module $F[r^{-1}]$ is finitely very flat.

Just as the proof of the Noetherian Main Theorem, the proof of the Finitely Very Flat Main Theorem uses Noetherian induction. In order to apply Noetherian induction to non-Noetherian rings one observes that any finitely presented algebra over a commutative ring is actually defined over its finitely generated subring.

Proof of Finitely Very Flat Main Theorem

Let R be a commutative ring, S be a finitely presented commutative R-algebra, and F be a finitely presented S-module.

Then there exists a subring $\overline{R} \subset R$ finitely generated over $\mathbb{Z}$, a finitely generated $\overline{R}$-algebra $\overline{S}$, and a finitely generated $\overline{S}$-module $\overline{F}$ such that $S = R \otimes_{\overline{R}} \overline{S}$ and $F = S \otimes_{\overline{S}} \overline{F} = R \otimes_{\overline{R}} \overline{F}$.

The subring $\overline{R} \subset R$ is generated by the coefficients in the finite sets of relations defining S as a finitely presented R-algebra and F as a finitely presented S-module.

Notice that when the R-module F is flat, there is no claim that the $\overline{R}$-module $\overline{F}$ will be also flat. This is not a problem for us, as the Generic Freeness Lemma does not require any flatness assumptions.

In order to prove the Finitely Very Flat Main Theorem for R, S, and F, one applies Noetherian induction to the ring $\overline{R}$ and the Finitely Very Flat Main Lemma to quotient rings of R by ideals generated by finite sets of elements from $\overline{R}$.

Toy Main Lemmas

Now let us discuss the proofs of the Main Lemmas. The following two results can be formulated for comparison with the Noetherian Very Flat and Finitely Very Flat Main Lemmas.

Let R be a commutative ring and $r \in R$ be an element.
An R-module F is said to be r-very flat if $\mathrm{Ext}^1_R(F, C) = 0$ for all r-contraadjusted R-modules C.

An R-module F is r-very flat if and only if it is a direct summand of an R-module G for which there exists a short exact sequence of R-modules

$$0 \longrightarrow U \longrightarrow G \longrightarrow V \longrightarrow 0,$$

where U is a free R-module and V is a free $R[r^{-1}]$-module.

Toy Main Lemmas

Toy Main Lemma

Let R be a commutative ring and $r \in R$ be an element. Then a flat R-module F is r-very flat if and only if the R/rR-module F/rF is projective and the $R[r^{-1}]$-module $F[r^{-1}]$ is projective.

Flat Main Lemma

Let R be a commutative ring and $r \in R$ be an element. Then an R-module F is flat and only if the R/rR-module F/rF is flat, the $R[r^{-1}]$-module $F[r^{-1}]$ is flat, and
$\operatorname{Tor}_1^R(R/rR, F) = 0 = \operatorname{Tor}_2^R(R/rR, F)$.

The Flat Main Lemma and Toy Main Lemma are much easier to prove than the Noetherian Very Flat Main Lemma and the Finitely Very Flat Main Lemma.

Bounded Torsion Very Flat Main Lemma

Notice that the Noetherian Very Flat Main Lemma is not a particular case of the Finitely Very Flat Main Lemma. One would like to generalize the Noetherian Very Flat Main Lemma by weakening (ideally, removing) the Noetherianity assumption. Here is the result that we can prove.

Let R be a commutative ring, $r \in R$ be an element, and M be an R-module. One says that the r-torsion in M is bounded if there exists $m \geqslant 1$ such that $r^n x = 0$ for $n \geqslant 1$ and $x \in M$ implies $r^m x = 0$ in M.

Bounded Torsion Very Flat Main Lemma

Let R be a commutative ring and $r \in R$ be an element. Assume that the r-torsion in R is bounded (or, more generally, the r-torsion in R is a sum of bounded r-torsion and r-divisible r-torsion). Then a flat R-module F is very flat if and only if the R/rR-module F/rF is very flat and the $R[r^{-1}]$-module $F[r^{-1}]$ is very flat.

Proofs of the Main Lemmas

The proofs of the Main Lemmas are based on a combination of three techniques: obtainable modules, contramodules, and a derived category version of the Nunke–Matlis exact sequence. Let us start with obtainable modules.

Let R be an associative ring and H be a left R-module. Suppose that there is an ordinal α and a projective system of R-modules $H_i \longleftarrow H_j$, $0 \leqslant i < j \leqslant \alpha$, such that $H_0 = 0$, $H_\alpha = H$, the triangle diagrams $H_i \longleftarrow H_j \longleftarrow H_k$ are commutative for all $i < j < k \leqslant \alpha$, $H_j = \varprojlim_{i<j} H_i$ for all limit ordinals $j \leqslant \alpha$, and the maps $H_i \longleftarrow H_{i+1}$ are surjective with the kernels L_i.

Then we say that the left R-module H is a transfinitely iterated extension of the left R-modules L_i, $0 \leqslant i < \alpha$ in the sense of the projective limit.

Let $\mathcal{F}$ and $\mathcal{C} \subset R\text{-}\mathrm{mod}$ be two classes of left R-modules. Then the class of left R-modules ${}^{\perp_1}\mathcal{C}$ is closed under transfinitely iterated extensions in the sense of the inductive limit and direct summands, while the class of left R-modules $\mathcal{F}^{\perp_1}$ is closed under transfinitely iterated extensions in the sense of the projective limit and direct summands.

Denote by $\mathcal{F}^{\perp_{\geqslant 1}} \subset R\text{-}\mathrm{mod}$ the class of all left R-modules C such that $\mathrm{Ext}^i_R(F, C) = 0$ for all $F \in \mathcal{F}$ and $i \geqslant 1$, and by ${}^{\perp_{\geqslant 1}}\mathcal{C}$ the class of all left R-modules F such that $\mathrm{Ext}^i_R(F, C) = 0$ for all $C \in \mathcal{C}$ and $i \geqslant 1$.

Then the class of modules $\mathcal{F}^{\perp_{\geqslant 1}} \subset R\text{-}\mathrm{mod}$ is closed under transfinitely iterated extensions in the sense of the projective limit, direct summands, and the cokernels of injective morphisms. The class of modules ${}^{\perp_{\geqslant 1}}\mathcal{C} \subset R\text{-}\mathrm{mod}$ is closed under transfinitely iterated extensions in the sense of the inductive limit, direct summands, and the kernels of surjective morphisms.

Obtainable Modules

Definition

Let $\mathcal{E} \subset R\text{-}\mathrm{mod}$ be a class of left R-modules. A left R-module is said to be simply right obtainable from the class $\mathcal{E}$ if it belongs to the minimal class of left R-modules containing $\mathcal{E}$ and closed under transfinitely iterated extensions in the sense of the projective limit, direct summands, and the cokernels of injective morphisms.

Definition

Let $\mathcal{M} \subset R\text{-}\mathrm{mod}$ be a class of left R-modules. A left R-module is said to be simply left obtainable from the class $\mathcal{M}$ if it belongs to the minimal class of left R-modules containing $\mathcal{M}$ and closed under transfinitely iterated extensions in the sense of the inductive limit, direct summands, and the kernels of surjective morphisms.

Obtainable Modules

Notice that transfinitely iterated extensions in the sense of the projective limit include extensions and infinite products. Similarly, transfinitely iterated extensions in the sense of the inductive limit include extensions and infinite direct sums.

For any class of modules $\mathcal{F} \subset R\text{-}\mathrm{mod}$, all modules simply right obtainable from the class of modules $\mathcal{F}^{\perp_{\geqslant 1}} \subset R\text{-}\mathrm{mod}$ belong to the class $\mathcal{F}^{\perp_{\geqslant 1}}$.

For any class of modules $\mathcal{C} \subset R\text{-}\mathrm{mod}$, all modules simply left obtainable from the class of modules ${}^{\perp_{\geqslant 1}}\mathcal{C} \subset R\text{-}\mathrm{mod}$ belong to the class ${}^{\perp_{\geqslant 1}}\mathcal{C}$.

Let $\mathcal{E} \subset R\text{-}\mathrm{mod}$ be a class of modules. Our aim is to describe the class $\mathcal{C} = ({}^{\perp_{\geqslant 1}}\mathcal{E})^{\perp_{\geqslant 1}}$. We already know that all R-modules simply right obtainable from $\mathcal{E}$ belong to $\mathcal{C}$.

But we will need a more powerful, two-sorted right obtainability procedure.

Let $\mathcal{F} \subset R\text{-}\mathrm{mod}$ be a class of modules. Denote by $\mathcal{F}^{\perp_{\geqslant 2}}$ the class of all left R-modules C such that $\operatorname{Ext}^i_R(F, C) = 0$ for all $F \in \mathcal{F}$ and $i \geqslant 2$. Set $\mathcal{C}_1 = \mathcal{F}^{\perp_{\geqslant 1}}$ and $\mathcal{C}_2 = \mathcal{F}^{\perp_{\geqslant 2}}$.

The two classes of modules $\mathcal{C}_1$ and $\mathcal{C}_2 \subset R\text{-}\mathrm{mod}$ have the following properties:

- $\mathcal{C}_1 \subset \mathcal{C}_2$;
- all modules simply right obtainable from $\mathcal{C}_1$ belong to $\mathcal{C}_1$;
- all modules simply right obtainable from $\mathcal{C}_2$ belong to $\mathcal{C}_2$;
- the kernel of any surjective morphism from a module from $\mathcal{C}_2$ to a module from $\mathcal{C}_1$ belongs to $\mathcal{C}_2$;
- the cokernel of any injective morphism from a module from $\mathcal{C}_2$ to a module from $\mathcal{C}_1$ belongs to $\mathcal{C}_1$.

Definition

Let $\mathcal{E} \subset R\text{-mod}$ be a class of modules. The pair of classes of left R-modules right 1-obtainable from $\mathcal{E}$ and right 2-obtainable from $\mathcal{E}$ is defined as the (obviously, unique) minimal pair of classes of left R-modules satisfying the following conditions:

- all modules from $\mathcal{E}$ are right 1-obtainable from $\mathcal{E}$; all modules right 1-obtainable from $\mathcal{E}$ are right 2-obtainable from $\mathcal{E}$;
- all modules simply right obtainable from right 1-obtainable modules are right 1-obtainable;
- all modules simply right obtainable from right 2-obtainable modules are right 2-obtainable;
- the kernel of any surjective morphism from a right 2-obtainable module to a right 1-obtainable module is right 2-obtainable;
- the cokernel of any injective morphism from a right 2-obtainable module to a right 1-obtainable module is right 1-obtainable.

Obtainable Modules

Let R be an associative ring and $\mathcal{E} \subset R\text{-}\mathrm{mod}$ be a class of modules. Set $\mathcal{F} = {}^{\perp_{\geqslant 1}}\mathcal{E}$, $\mathcal{C}_1 = \mathcal{F}^{\perp_{\geqslant 1}}$, and $\mathcal{C}_2 = \mathcal{F}^{\perp_{\geqslant 2}}$.

Then all modules right 1-obtainable from $\mathcal{E}$ belong to $\mathcal{C}_1$, and all modules right 2-obtainable from $\mathcal{E}$ belong to $\mathcal{C}_2$.

The proofs of the Main Lemmas are based on results describing the right classes of modules in certain cotorsion pairs as the classes of all modules right 1-obtainable from certain “seed” classes.

Proof of Toy Main Lemma

The Toy Main Lemma follows easily from the following Toy Main Proposition.

Toy Main Proposition

Let R be a commutative ring and $r \in R$ be an element. Then an R-module is r-contraadjusted if and only if is it simply right obtainable from R/rR-modules and $R[r^{-1}]$-modules.

For comparison, the following proposition is closely related to the Flat Lemma.

Proposition

Let R be a commutative ring and $r \in R$ be an element. Then all R-modules are simply left obtainable from R/rR-modules and $R[r^{-1}]$-modules.

Proof of Noetherian Very Flat Main Lemma

The Noetherian Very Flat Main Lemma follows easily from the following Noetherian Contraadjusted Main Proposition.

Noetherian Contraadjused Main Proposition

Let R be a Noetherian commutative ring and $r \in R$ be an element. Then an R-module is contraadjusted if and only if it is right 1-obtainable from contraadjusted R/rR-modules and contraadjusted $R[r^{-1}]$-modules.

Notice that an R/rR-module is contraadjusted if and only if it is contraadjusted as an R-module; and an $R[r^{-1}]$-module is contraadjusted if and only if it is contraadjusted as an R-module. So the formulation of the Main Proposition is unambiguous.

Let us also mention that it follows from the Main Proposition that all R-modules are right 2-obtainable from contraadjusted R/rR-modules and contraadjusted $R[r^{-1}]$-modules.

Proof of Bounded Torsion Very Flat Main Lemma

The Noetherianity assumption in the Main Proposition can be weakened to the assumption on the r-torsion in R: the r-torsion in R should be a sum of bounded r-torsion and r-divisible r-torsion. This allows to deduce the Bounded Torsion Very Flat Main Lemma.

Proof of Finitely Very Flat Main Lemma

Let R be a commutative ring and $\mathbf{r} = \{r_1, \ldots, r_m\}$ be a finite set of elements in R. Recall that an R-module is said to be $\mathbf{r}$-contraadjusted if it is r_j-contraadjusted for all $j = 1, \ldots, m$. An R-module F is said to be $\mathbf{r}$-very flat if $\mathrm{Ext}^1_R(F, C) = 0$ for all $\mathbf{r}$-contraadjusted R-modules C.

In the proof of the Finitely Very Flat Main Lemma, we work with finite subsets $\mathbf{r} \subset R$ of certain specific form. For any subset of indices $J \subset \{1, \ldots, m\}$ denote by $r_J \in R$ the product $\prod_{j \in J} r_j$. Set

$$\mathbf{r}^\times = \{r_J \mid J \subset \{1, \ldots, m\}\}.$$

Put $K = \{1, \ldots, m\} \setminus J$. Denote by R_J the ring

$$R[r_J^{-1}]/(\textstyle\sum_{k \in K} r_k R[r_J^{-1}]).$$

So the ring R_J is obtained by inverting all the elements r_j, $j \in J$, and annihilating all the elements r_k, $k \in K$ in the ring R.

Proof of Finitely Very Flat Main Lemma

The Finitely Very Flat Main Lemma is deduced from the following theorem.

$\mathbf{r}^\times$-Very Flat Theorem

Let R be a commutative ring and $\mathbf{r} = \{r_1, \dots, r_m\} \subset R$ be a finite subset of its elements. Let F be a flat R-module. Then the R-module F is $\mathbf{r}^\times$-very flat if and only if for every subset of indices $J \subset \{1, \dots, m\}$ the R_J-module $R_J \otimes_R F$ is projective.

The $\mathbf{r}^\times$-Very Flat Theorem is deduced from the following Main Proposition.

$\mathbf{r}^\times$-Contraadjusted Main Proposition

Let R be a commutative ring and $\mathbf{r} = \{r_1, \dots, r_m\} \subset R$ be a finite subset of its elements. Then an R-module is $\mathbf{r}^\times$-contraadjusted if and only if it is right 1-obtainable from R_J-modules, where J runs over the all the subsets in the set of indices $\{1, \dots, m\}$.

Contramodules

Let R be a commutative ring and $r \in R$ be an element.
An R-module C is said to be an *r-contramodule* if
$\mathrm{Hom}_R(R[r^{-1}], C) = 0 = \mathrm{Ext}^1_R(R[r^{-1}], C)$.

The full subcategory of r-contramodule R-modules $R\text{-}\mathrm{mod}_{r\text{-}\mathrm{ctra}}$ is closed under the kernels, cokernels, extensions, and infinite products in $R\text{-}\mathrm{mod}$. So $R\text{-}\mathrm{mod}_{r\text{-}\mathrm{ctra}}$ is an abelian category and the embedding functor $R\text{-}\mathrm{mod}_{r\text{-}\mathrm{ctra}} \longrightarrow R\text{-}\mathrm{mod}$ is exact.

An R-module C is said to be *r-complete* if the natural map from it to its r-adic completion $C \longrightarrow \varprojlim_{n\geqslant 1} C/r^nC$ is surjective.
An R-module C is said to be *r-separated* if this map is injective.

All r-contramodule R-modules (and more generally, all r-contraadjusted R-modules) are r-complete. All r-separated r-complete R-modules are r-contramodules. But the converse assertions do not hold.

Torsion Module and Contramodule Lemmas

The notion of an r-contramodule R-module is the dual version of the simpler notion of an r-torsion R-module. An R-module M is said to be r-torsion if $R[r^{-1}] \otimes_R M = 0$.

Torsion Module Lemma

Let R be a commutative ring and $r \in R$ be an element. Then an R-module is r-torsion if and only if it is a transfinitely iterated extension, in the sense of the inductive limit, of R/rR-modules.

It follows that an R-module is r-torsion if and only if is it simply left obtainable from R/rR-modules.

Contramodule Lemma

Let R be a commutative ring and $r \in R$ be an element. Then an R-module is an r-contramodule if and only if it is simply right obtainable from R/rR-modules.

Proof of Toy Main Proposition

Let R be a commutative ring and $r \in R$ be an element. Let us first prove that all R-modules are simply left obtainable from R/rR-modules and $R[r^{-1}]$-modules.

Let N be an R-module. Consider the exact sequence

$$0 \longrightarrow \Gamma_r(N) \longrightarrow N \longrightarrow N[r^{-1}] \longrightarrow N[r^{-1}]/N \longrightarrow 0,$$

where $\Gamma_r(N)$ denotes the submodule of all r-torsion elements in N.

Then $\Gamma_r(N)$ and $N[r^{-1}]/N$ are r-torsion R-modules, while $N[r^{-1}]$ is an $R[r^{-1}]$-module. Hence N is simply left obtainable from two r-torsion R-modules and one $R[r^{-1}]$-module. By the Torsion Module Lemma, all r-torsion R-modules are simply left obtainable from R/rR-modules.

Proof of Toy Main Proposition

Now let us prove that all r-contraadjusted R-modules are simply right obtainable from R/rR-modules and $R[r^{-1}]$-modules.

Denote by $K^\bullet$ the two-term complex $R \longrightarrow R[r^{-1}]$, where the term R sits in the cohomological degree -1 and the term $R[r^{-1}]$ sits in the cohomological degree 0. When r is a regular element in R, one can use the quotient module $K = R[r^{-1}]/R$ in lieu of the complex $K^\bullet$.

For any R-module A, let us denote by $\mathrm{Ext}^i_R(K^\bullet, A)$ the modules of morphisms $\mathrm{Hom}_{\mathrm{D}(R\text{-mod})}(K^\bullet, A[i])$ in the derived category of R-modules. Applying the functor $\mathrm{Hom}_{\mathrm{D}(R\text{-mod})}(-, A[*])$ to the distinguished triangle

$$R \longrightarrow R[r^{-1}] \longrightarrow K^\bullet \longrightarrow R[1],$$

Proof of Toy Main Proposition

we obtain an exact sequence of R-modules

$$0 \longrightarrow \mathrm{Ext}^0_R(K^\bullet, A) \longrightarrow \mathrm{Hom}_R(R[r^{-1}], A) \longrightarrow A \longrightarrow \mathrm{Ext}^1_R(K^\bullet, A) \longrightarrow \mathrm{Ext}^1_R(R[r^{-1}], A) \longrightarrow 0.$$

This is what can be called the derived category version of the Nunke–Matlis exact sequence in our context.

In particular, for an r-contraadjusted R-module C, we get an exact sequence

$$0 \longrightarrow \mathrm{Ext}^0_R(K^\bullet, C) \longrightarrow \mathrm{Hom}_R(R[r^{-1}], C) \longrightarrow C \longrightarrow \mathrm{Ext}^1_R(K^\bullet, C) \longrightarrow 0.$$

Proof of Toy Main Proposition

For any R-module A, the R-modules $\mathrm{Ext}^i_R(K^\bullet, A)$ are r-contramodules, while the R-module $\mathrm{Hom}_R(R[r^{-1}], A)$ is an $R[r^{-1}]$-module.

Thus any r-contraajusted R-module C is simply right obtainable from two r-contramodule R-modules and one $R[r^{-1}]$-module. By the Contramodule Lemma, all r-contramodule R-modules are simply right obtainable from R/rR-modules.

Conversely, all R-modules simply right obtainable (or even right 1-obtainable) from R/rR-modules and $R[r^{-1}]$-modules are r-contraadjusted, because all the R/rR-modules and $R[r^{-1}]$-modules are r-contraadjusted.

Proof of Contramodule Lemma

The functor $A \longmapsto \Delta_r(A) = \mathrm{Ext}^1_R(K^\bullet, A)$ is left adjoint to the embedding functor $R\text{-}\mathrm{mod}_{r\text{-ctra}} \longrightarrow R\text{-}\mathrm{mod}$. Hence it suffices to show that the R-module $\Delta_r(A)$ is simply right obtainable from R/rR-modules for any R-module A.

The complex $K^\bullet = (R \to R[r^{-1}])$ is the inductive limit of the complexes $R \xrightarrow{r^n} R$ mapping one into another as follows

$$\begin{array}{ccc} R & \xrightarrow{\ r^n\ } & R \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle r} \\ R & \xrightarrow{\ r^{n+1}\ } & R \end{array}$$

Proof of Contramodule Lemma

Therefore, for any R-module A there is a natural short exact sequence of R-modules

$$0 \longrightarrow \varprojlim^1_{n\geqslant 1} {}_{r^n}A \longrightarrow \Delta_r(A) \longrightarrow \varprojlim_{n\geqslant 1} A/r^nA \longrightarrow 0,$$

where ${}_tA \subset A$ denotes the submodule of all elements annihilated by an element $t \in R$. The projective system $(A/r^nA)_{n\geqslant 1}$ is formed by the natural projection maps $A/r^{n+1}A \longrightarrow A/r^nA$, while the projective system $({}_{r^n}A)_{n\geqslant 1}$ is formed by the multiplication maps $r\colon {}_{r^{n+1}}A \longrightarrow {}_{r^n}A$.

The r-adic completion module $\varprojlim_{n\geqslant 1} A/r^nA$ is obviously a transfinitely iterated extension (in the sense of the projective limit) of R/rR-modules. The derived projective limit module $\varprojlim^1_{n\geqslant 1} {}_{r^n}A$ is simply right obtainable from R/rR-modules according to the following lemma.

Proof of Contramodule Lemma

Lemma

Let R be a commutative ring and $r \in R$ be an element. Let $D_1 \longleftarrow D_2 \longleftarrow D_3 \longleftarrow \cdots$ be a projective system of R-modules such that the R-module D_n is annihilated by r^n. Then the R-modules (a) $\varprojlim_n D_n$ and (b) $\varprojlim_n^1 D_n$ are simply right obtainable from R/rR-modules.

Proof of part (a).

First of all, all R/r^nR-modules are finitely iterated extensions of R/rR-modules, so they are simply right obtainable. Set $D'_n = \operatorname{im}(\varprojlim_m D_m \to D_n) \subset D_n$. Then $D = \varprojlim_n D'_n$ and the maps $D'_{n+1} \longrightarrow D'_n$ are surjective, so D is a transfinitely iterated extension of D'_1 and $\ker(D'_{n+1} \to D'_n)$. The latter are R/r^nR-modules. □

Proof of part (b).

By the definition of $\varprojlim^1_{n\geqslant 1}$, there is an exact sequence of R-modules

$$0 \longrightarrow \varprojlim_n D_n \longrightarrow \prod_n D_n \xrightarrow{\mathrm{id}-\mathit{shift}} \prod_n D_n \longrightarrow \varprojlim^1_n D_n \longrightarrow 0.$$

Hence the R-module $\varprojlim^1_n D_n$ can be obtained from the R-modules $\varprojlim_n D_n$ and $\prod_n D_n$ by two passages to the cokernel of an injective morphism.

Since the projective limit $\varprojlim_n D_n$ and the product $\prod_n D_n$ are simply right obtainable from R/rR-modules, so is the derived projective limit $\varprojlim^1_n D_n$. □

Proof of Noetherian Contraadjusted Main Proposition

Given a Noetherian commutative ring R with an element $r \in R$, we need to prove that all contraadjusted R-modules C are right 1-obtainable from contraadjusted R/rR-modules and contraadjusted $R[r^{-1}]$-modules.

We have an exact sequence of R-modules

$$0 \longrightarrow \mathrm{Ext}^0_R(K^\bullet, C) \longrightarrow \mathrm{Hom}_R(R[r^{-1}], C) \\ \longrightarrow C \longrightarrow \mathrm{Ext}^1_R(K^\bullet, C) \longrightarrow 0.$$

In order to show that C is right 1-obtainable, it suffices to check that $\mathrm{Hom}_R(R[r^{-1}], C)$ and $\mathrm{Ext}^1_R(K^\bullet, C)$ are right 1-obtainable and that $\mathrm{Ext}^0_R(K^\bullet, C)$ is right 2-obtainable.

Proof of Noetherian Contraadjusted Main Proposition

The R-module $\mathrm{Hom}_R(R[r^{-1}], C)$ is already a contraadjusted $R[r^{-1}]$-module.

The R-module $\mathrm{Ext}^0_R(K^\bullet, C)$ is an r-contramodule. According to the Contramodule Lemma, it is simply right obtainable from R/rR-modules. Any R/rR-module can be embedded into a contraadjusted (e.g., injective) R/rR-module, and the quotient R/rR-module is also contraadjusted. Hence any R/rR-module is right 2-obtainable from contraadjusted R/rR-modules.

The R-module $\mathrm{Ext}^1_R(K^\bullet, C)$ is a contraadjusted r-contramodule R-module. It remains to show that all contraadjusted r-contramodule R-modules are simply right obtainable from contraadjusted R/rR-modules.

Firstly we notice that all contraadjusted r-separated r-complete R-modules are transfinitely iterated extensions (in the sense of the projective limit) of contraadjusted R/rR-modules. Indeed, if C is a contraadjusted R-module and the map $C \longrightarrow \varprojlim_n C/r^nC$ is an isomorphism then C is a transfinitely iterated extension of the R-modules $r^nC/r^{n+1}C$, which are all contraadjusted as quotient R-modules of a contraadjusted R-module C.

Secondly, one shows that any contraadjusted r-contramodule R-module C is the cokernel of an injective morphism of contraadjusted r-separated r-complete R-modules. For this purpose, one uses a special precover sequence $0 \longrightarrow K \longrightarrow F \longrightarrow C \longrightarrow 0$ in the flat or very flat cotorsion pair in the abelian category $R\text{-}\mathrm{mod}_{r\text{-ctra}}$.

All flat r-contramodule R-modules are r-separated, and all submodules of r-separated R-modules are r-separated, so both K and F are r-separated r-complete R-modules. The R-module F is contraadjusted as an extension of two contraadjusted R-modules.

Many results mentioned in this talk have their versions and generalizations applicable to multiplicative subsets S in commutative ring R more complicated than $\{1, r, r^2, r^3, \dots\}$. Using the related techniques, we can prove, in particular, the following theorems.

Theorem 1

Let R be a Noetherian commutative ring. Then all R-modules are simply left obtainable from vector spaces over the residue fields $k_R(\mathfrak{p})$ of the prime ideals $\mathfrak{p} \subset R$.

Theorem 2

Let R be a Noetherian commutative ring with countable spectrum. Then an R-module is Enochs cotorsion if and only if it is right 1-obtainable from vector spaces over the residue fields $k_R(\mathfrak{p})$ of the prime ideals $\mathfrak{p} \subset R$.

Beyond Very Flat and Contraadjusted Modules

Theorem 3

Let R be a Noetherian commutative ring with countable spectrum. Then there exists a countable collection of countable multiplicative subsets $S_1, S_2, S_3, \ldots \subset R$ such that every flat R-module is a direct summand of a transfinitely iterated extension (in the sense of the inductive limit) of R-modules isomorphic to $S_j^{-1}R$, $j = 1, 2, 3, \ldots$

Theorem 4

Let R be a Noetherian commutative ring of finite Krull dimension d with countable spectrum. Then there exists a finite collection of at most $m = 2^{(d+1)^2/4}$ countable multiplicative subsets $S_1, \ldots, S_m \subset R$ such that every flat R-module is a direct summand of a transfinitely iterated extension (in the sense of the inductive limit) of R-modules isomorphic to $S_j^{-1}R$, $j = 1, \ldots, m$.