Curvature invariants in type-*N* spacetimes

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Abstract. Scalar curvature invariants are studied in type-*N* solutions of the vacuum Einstein equations with, in general, a non-vanishing cosmological constant Λ. Zeroth-order invariants, which include only the metric and Weyl (Riemann) tensor, either vanish or are constants depending on Λ. All higher-order invariants containing covariant derivatives of the Weyl (Riemann) tensor are also shown to be trivial if a type-*N* spacetime admits a non-expanding and non-twisting null geodesic congruence.

However, in the case of expanding type-*N* spacetimes we discover a non-vanishing scalar invariant, which is quartic in the second derivatives of the Riemann tensor.

We use this invariant to demonstrate that both the linearized and third-order type-*N* twisting solutions recently discussed in literature contain singularities at large distances and thus cannot describe radiation fields outside bounded sources.

1. Introduction

The Petrov algebraic classification of the Weyl tensor and the asymptotic forms of radiative fields of spatially bounded sources (the peeling theorem) demonstrate that solutions of type *N* play a fundamental role in the theory of gravitational radiation. (For a recent review of exact approaches to radiative spacetimes see, e.g., [1] and references therein.)

All solutions of the vacuum Einstein equations with in general non-zero cosmological constant Λ, which are of type *N* with a non-twisting null congruence, are known [3, 10, 14]. In the twisting case, only one (Hauser's) solution is available [10]. Although far-zone radiation fields of bounded sources are approximately of type *N*, no known exact type-*N* solution is asymptotically flat.

Recently, an interesting discussion appeared in the literature [5, 17, 18] in which twisting type-*N* vacuum field equations were solved approximately. Stephani [18] argued, within a linear theory, that no solutions regular outside a bounded source exist. Finley *et al* [5] iterated the solution up to the third order and concluded that their iterative procedure leads to the regular solutions, so that 'it seems that the twisting, type-*N* fields *can* describe a radiation field outside bounded sources'.

In neither of these works, however, are the arguments really compelling since singularities are not analysed properly. (For general treatments of singularities, see e.g. [2, 4].) The authors study only the behaviour of invariants with respect to gauge

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transformations which leave invariant the coordinate system and field equations for type-N metrics, because for type-N vacuum spacetimes with Λ all scalar invariants of the Weyl or Riemann tensor are trivial—they either vanish or are constants depending on Λ.

It is known that for a vacuum pp-wave, which is a very special case of a type-N metric, in fact all curvature invariants of *any* order, i.e. invariants depending also on covariant *derivatives* of the Weyl (or Riemann) tensor, vanish [8, 16]. However, nothing appears to be known about higher-order invariants in more general type-N spacetimes. The problem of such higher-order invariants is addressed in the present paper.

We define an invariant of the metric $g_{\alpha\beta}(x^\gamma)$ of order k as a non-constant scalar function $I(g_{\alpha\beta}, g_{\alpha\beta,\gamma_1}, \ldots, g_{\alpha\beta,\gamma_1\ldots\gamma_{k+2}})$ which satisfies

$$I(g_{\alpha\beta}, g_{\alpha\beta,\gamma_1}, \ldots, g_{\alpha\beta,\gamma_1\ldots\gamma_{k+2}}) = I(g'_{\alpha\beta}, g'_{\alpha\beta,\gamma_1}, \ldots, g'_{\alpha\beta,\gamma_1\ldots\gamma_{k+2}}) \tag{1.1}$$

under a spacetime diffeomorphism $x^\alpha \to x'^\alpha = x'^\alpha(x^\beta)$. It can be proved (see e.g. [9]) that any invariant of order k depends on the metric and the Riemann tensor $R_{\alpha\beta\gamma\delta}$ and its covariant derivatives of order $\leqslant k$:

$$I = I(g_{\alpha\beta}, R_{\alpha\beta\gamma\delta}, \ldots, R_{\alpha\beta\gamma\delta;\varepsilon_1\ldots\varepsilon_k}). \tag{1.2}$$

The maximal number, $I(k, n)$, of functionally independent invariants of order k in a Riemannian space of dimension n has been known since 1902. Denoting

$$D(k; n) = I(k; n) - I(k - 1; n), \tag{1.3}$$

Haskins [7] found that

$$D(k; n) = \begin{cases} \binom{n}{3}\dfrac{n + 3}{2} & \text{for } k = 0, \\[2ex] \binom{n + k + 1}{n + 3}\dfrac{n(k + 1)}{2} & \text{for } k \geqslant 1. \end{cases} \tag{1.4}$$

Hence, in a four-dimensional spacetime there exist in general 14 functionally independent invariants of order zero which depend only on $g_{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}$, 60 invariants depending on $g_{\alpha\beta}$, $R_{\alpha\beta\gamma\delta}$, $R_{\alpha\beta\gamma\delta;\rho}$, 126 invariants containing also $R_{\alpha\beta\gamma\delta;\rho\sigma}$, etc. Clearly, when the derivatives of the Riemann tensor are included, the number of independent invariants grows rapidly. All 14 invariants of order zero were explicitly given in [6], their spinor equivalents can be found in [20].

In section 2 we first briefly review basic definitions and relations of the two-component spinor formalism and Newman–Penrose formalism [11], which will be needed later. Then we demonstrate the generally known fact that in vacuum type-N spacetimes with $\Lambda = 0$ all zero-order invariants vanish. If $\Lambda \neq 0$, some invariants are non-vanishing but they are just constants depending on Λ. Using spinor formalism we prove the helpful lemma 1 on the properties of the invariants constructed from the derivatives of the Weyl tensor.

In section 3 we specialize to the type-N vacuum solutions with Λ, admitting a non-vanishing and non-twisting null geodesic congruence (called Kundt's class in [10]). We prove that in these spacetimes all invariants constructed from the Weyl tensor and its covariant derivatives of arbitrary order vanish. The invariants constructed from just the Riemann tensor are all constants depending on Λ.

Expanding and non-twisting type-N solutions are discussed in the first part of section 4. Using again extensively the spinor and Newman–Penrose formalism we find that all the first-order invariants vanish. However, the formalism indicates that a non-vanishing second-order invariant may exist. A number of attempts eventually led to the non-vanishing invariant

$$I = R^{\alpha\beta\gamma\delta;\varepsilon\phi} R_{\alpha\mu\gamma\nu;\varepsilon\phi} R^{\lambda\mu\rho\nu;\sigma\tau} R_{\lambda\beta\rho\delta;\sigma\tau}. \tag{1.5}$$

The zero- and first-order invariants also vanish in the twisting case. The invariant (1.5) remains non-vanishing.

This invariant is then, in section 5, used to analyse the nature of the approximate solutions [5, 18] we mentioned earlier. We find that Stephani's conclusion, based on the linearized theory, remains true for the third-order solution obtained by Finley *et al*: both solutions contain singularities at large r. This raises more doubts about the physical meaning of type-N twisting solutions as describing radiation fields outside bounded sources; nevertheless a definitive statement can be made only when a general, exact solution is found.

The technique by which we arrive at the non-vanishing invariant of second order could probably also be employed in more complicated cases of algebraically special spacetimes. Scalar invariants obtained might play a role not only in classical relativity but also in a quantum context.

2. Higher-order curvature invariants: general properties

In this section we prove a lemma about the properties of the curvature invariants of higher order in Petrov type-N spacetimes. The lemma will be used in the next section in which curvature invariants in the specific classes of type-N spacetimes will be analysed. First, we have to summarize some basic notation and relations which will be needed later.

2.1. Spinors and the Newman–Penrose formalism

Spinors and the Newman–Penrose formalism have been reviewed by many authors (see e.g. [10, 13, 19]). Here we give only a very brief summary.

Consider a null congruence of geodesics with the tangent null vector l^α,

$$l_\alpha l^\alpha = 0, \tag{2.1}$$

which is affinely parametrized,

$$l^\alpha l^\beta{}_{;\alpha} = 0. \tag{2.2}$$

(Hereafter, we assume (2.2) to be satisfied.) The congruence is characterized by the expansion θ, twist ω and shear $|\sigma|$ given by

$$\theta = \tfrac{1}{2} l^\alpha{}_{;\alpha}, \tag{2.3}$$

$$\omega = \sqrt{\tfrac{1}{2} l_{[\alpha;\beta]} l^{\alpha;\beta}}, \tag{2.4}$$

$$|\sigma| = \sqrt{\tfrac{1}{2} l_{(\alpha;\beta)} l^{\alpha;\beta} - \theta^2}. \tag{2.5}$$

In algebraically special spacetimes the shear vanishes. A two-component spinor field $o_A \in W$ ($A = 1, 2$ and W is a two-dimensional complex vector space) can be associated with a null vector field l^α in a standard way,

$$l^\alpha \longleftrightarrow o^A \bar{o}^{\dot{A}}, \tag{2.6}$$

where $\bar{o}^{\dot{A}} \in \bar{W}$, and another spinor field, ι^A, exists such that together with o^A it forms a spinor basis satisfying

$$o_A \iota^A = \varepsilon_{AB} o^A \iota^B = 1, \qquad o_A o^A = \iota_A \iota^A = 0, \tag{2.7}$$

where the Levi-Civita alternating symbol ε_{AB} plays the role of the metric in spinor calculus.

The complex null tetrad, $\{l^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha\}$, in which l^α is introduced by (2.6) and

$$n^\alpha \longleftrightarrow \iota^A \bar{\iota}^{\dot{A}}, \qquad m^\alpha \longleftrightarrow o^A \bar{\iota}^{\dot{A}}, \qquad \bar{m}^\alpha \longleftrightarrow \iota^A \bar{o}^{\dot{A}}, \tag{2.8}$$

satisfies the usual relations

$$l_\alpha n^\alpha = -\bar{m}_\alpha m^\alpha = 1. \tag{2.9}$$

The (covariant) derivative operator ∇_α can be expressed in the form

$$\nabla_\alpha = n_\alpha D + l_\alpha \Delta - \bar{m}_\alpha \delta - m_\alpha \bar{\delta}, \tag{2.10}$$

where

$$D = l^\alpha \nabla_\alpha, \qquad \Delta = n^\alpha \nabla_\alpha, \qquad \delta = m^\alpha \nabla_\alpha, \qquad \bar{\delta} = \bar{m}^\alpha \nabla_\alpha. \tag{2.11}$$

In terms of the covariant derivative with spinor indices,

$$\nabla_{A\dot{B}} = \sigma^\alpha_{A\dot{B}} \nabla_\alpha \tag{2.12}$$

(σ^0 and σ^i are proportional to the unit and Pauli matrices, respectively), we have, equivalent to (2.11),

$$D = o^A \bar{o}^{\dot{A}} \nabla_{A\dot{A}}, \qquad \Delta = \iota^A \bar{\iota}^{\dot{A}} \nabla_{A\dot{A}},$$

$$\delta = o^A \bar{\iota}^{\dot{A}} \nabla_{A\dot{A}}, \qquad \bar{\delta} = \iota^A \bar{o}^{\dot{A}} \nabla_{A\dot{A}}. \tag{2.13}$$

The twelve Newman–Penrose complex scalar spin coefficients $\kappa, \varepsilon, \pi, \ldots$ are defined as frame components of the covariant derivatives of the null-tetrad vectors. Our notation follows that customarily used in literature [12, 19]. Some details are given in the appendix.

The Weyl tensor $C_{\alpha\beta\gamma\delta}$ in spinor form is given by

$$C_{\alpha\beta\gamma\delta} \longleftrightarrow \Psi_{ABCD} \varepsilon_{\dot{A}\dot{B}} \varepsilon_{\dot{C}\dot{D}} + \bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} \varepsilon_{AB} \varepsilon_{CD}, \tag{2.14}$$

where $\Psi_{ABCD} = \Psi_{(ABCD)}$. In the Newman–Penrose formalism the Weyl tensor is described by five complex scalar quantities $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ given by the projections of Ψ_{ABCD} onto the basis spinors o^A, ι^A. The Riemann tensor $R_{\alpha\beta\gamma\delta}$ in spinor form is given by

$$R_{\alpha\beta\gamma\delta} \longleftrightarrow X_{ABCD} \varepsilon_{\dot{A}\dot{B}} \varepsilon_{\dot{C}\dot{D}} + \bar{X}_{\dot{A}\dot{B}\dot{C}\dot{D}} \varepsilon_{AB} \varepsilon_{CD} + \Phi_{AB\dot{C}\dot{D}} \varepsilon_{\dot{A}\dot{B}} \varepsilon_{CD} + \bar{\Phi}_{\dot{A}\dot{B}CD} \varepsilon_{AB} \varepsilon_{\dot{C}\dot{D}}, \tag{2.15}$$

where

$$X_{ABCD} = \Psi_{ABCD} + \tfrac{1}{12} R(\varepsilon_{AC} \varepsilon_{BD} + \varepsilon_{AD} \varepsilon_{BC}),$$

R is the scalar curvature, and the spinor $\Phi_{AB\dot{C}\dot{D}} = \Phi_{(AB)(\dot{C}\dot{D})} = \bar{\Phi}_{AB\dot{C}\dot{D}}$ corresponds to the traceless Ricci tensor $S_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{4} R g_{\alpha\beta}$:

$$\Phi_{AB\dot{A}\dot{B}} \longleftrightarrow -\tfrac{1}{2} S_{ab}. \tag{2.16}$$

In vacuum spacetimes with generally non-vanishing cosmological constant Λ,

$$R = 4\Lambda, \tag{2.17}$$

and

$$\Phi_{AB\dot{C}\dot{D}} = 0. \tag{2.18}$$

Bianchi identities connect the derivatives of the Ψ's with the Ψ's themselves and with the Newman–Penrose spin coefficients. All Bianchi identities, the relations giving the Riemann (Weyl) tensor in terms of the spin coefficients and the commutation relations for the derivative operators (2.11), are explicitly written down in, for example, [10].

Here we are interested in the *vacuum* spacetimes of Petrov type N with in general $\Lambda \neq 0$. Let o^A be the 4-fold principal null spinor of Ψ_{ABCD}, and ι^A satisfy (2.7). Then we have

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \tag{2.19}$$

$$\Psi_{ABCD} = \Psi_4 o_A o_B o_C o_D, \tag{2.20}$$

and R and $\Phi_{AB\dot{C}\dot{D}}$ are given by (2.17) and (2.18). The Bianchi identities reduce to

$$D\Psi_4 = (\rho - 4\varepsilon)\Psi_4, \tag{2.21}$$

$$\delta\Psi_4 = (\tau - 4\beta)\Psi_4, \tag{2.22}$$

$$\sigma = \kappa = 0. \tag{2.23}$$

Although under the conditions (2.17)–(2.20) and (2.23) the remaining Newman–Penrose equations take a much simpler form than in the general case, they still represent the set of 21 equations. These are given in the appendix.

As is well known, transformations preserving the direction of l^α can be divided into two subclasses:

(i) null rotations around the vector l^α

$$l'^\alpha = l^\alpha, \qquad m'^\alpha = m^\alpha + \bar{c}l^\alpha, \qquad n'^\alpha = n^\alpha + cm^\alpha + \bar{c}\bar{m}^\alpha + c\bar{c}l^\alpha,$$

$$o'^A = o^A, \qquad \iota'^A = \iota^A + co^A. \tag{2.24}$$

(ii) boosts in the (l^α, n^α)-plane and spacelike rotations in the $(m^\alpha, \bar{m}^\alpha)$-plane

$$l'^\alpha = a^2 l^\alpha, \qquad n'^\alpha = a^{-2} n^\alpha, \qquad m'^\alpha = e^{2i\theta} m^\alpha,$$

$$o'^A = z o^A, \qquad \iota'^A = z^{-1}\iota^A, \tag{2.25}$$

where $z = ae^{i\theta}$.

In the following we shall need, in particular, the behaviour of various quantities under the pure constant boosts in the (l^α, n^α)-plane ($a = $ constant, $\theta = 0$ in (2.25)).

If a quantity Ω transforms under the boosts as

$$\Omega' = a^q \Omega, \tag{2.26}$$

the number q is called the boost-weight of Ω. We write $b(\Omega)$ to denote the boost-weight. First notice that $b(o^A) = 1$, $b(\iota^A) = -1$, and recall that the operators D, Δ, δ and $\bar{\delta}$ are also weighted. The boost-weights of the Newman–Penrose coefficients (NP) and operators (OP) are summarized in table 1.

Table 1. Summary of the boost-weights for Newman–Penrose coefficients (NP) and operators (OP).

$x \in NP$	$x \in OP$	$b(x) = b(\bar{x})$
κ		4
$\varepsilon\ \rho\ \sigma$	D	2
$\alpha\ \beta\ \pi\ \tau$	δ	0
$\gamma\ \lambda\ \mu$	Δ	-2
ν		-4

We shall also need the behaviour of Ψ_4:

$$\Psi'_4 = a^{-4}\Psi_4 \implies b(\Psi_4) = -4. \tag{2.27}$$

2.2. Higher-order curvature invariants

As noted in the introduction, in a general spacetime there exist 14 independent invariants of order zero, i.e. invariants depending only on the metric and the Riemann tensor. In terms of the spinors determining the Riemann tensor according to (2.14), these invariants can be constructed as products of the form

$$\Phi_{\dot{A}\dot{B}MN}\Phi^{\dot{A}\dot{B}MN}, \qquad X_{ABMN}X^{ABMN}, \qquad \Phi^{GD\dot{A}\dot{B}}\Phi_{RS\dot{A}\dot{B}}X^{RSKL}\Phi_{KL\dot{M}\dot{N}}\Phi_{GD}{}^{\dot{M}\dot{N}},$$

etc. Since, however, in the vacuum Petrov type-N spacetimes with Λ, equations (2.17)–(2.20) are satisfied, it is easy to see that among the zero-order invariants nine vanish and five are dependent just on the value of Λ (as, e.g., (2.17)). Hence, we have to turn to invariants of higher order.

First, let us decompose the spinor derivative $\nabla^{C_n \dot{X}_n} \cdots \nabla^{C_1 \dot{X}_1}(\Psi_4 o^A o^B o^C o^D)$ into the spinor basis of the appropriate spinor space. Let us write

$$\mathcal{W}^{[p,k]} \equiv \underbrace{W \times W \times \cdots \times W}_{p} \times \underbrace{\overline{W} \times \overline{W} \times \cdots \times \overline{W}}_{k}.$$

Thus, $\mathcal{W}^{[p,k]}$ is a 2^{p+k}-dimensional complex vector space whose basis spinors $B_1^{[p,k]}$, $B_2^{[p,k]}, \ldots, B_{2^{p+k}}^{[p,k]}$ can be constructed from the tensorial products of o^A and ι^A of the form $\xi_1^{A_1}\xi_2^{A_2}\cdots\xi_p^{A_p}\lambda_1^{\dot{X}_1}\lambda_2^{\dot{X}_2}\cdots\lambda_k^{\dot{X}_k}$, where the $\xi_i^{A_i}$'s are o^{A_i} or ι^{A_i} and the $\lambda_j^{\dot{X}_j}$'s are $\bar{o}^{\dot{X}_j}$ or $\bar{\iota}^{\dot{X}_j}$. Then the decomposition of the nth spinor derivative of $\Psi_4 o^A o^B o^C o^D$ reads

$$\nabla^{C_n \dot{X}_n} \cdots \nabla^{C_1 \dot{X}_1}(\Psi_4 o^A o^B o^C o^D) = \sum_{i=1}^{2^{2n+4}} c_i B_i^{[n+4,n]}. \tag{2.28}$$

We shall need to know some restrictions on the coefficients c_i rather than their specific values. Let us first note that the coefficients c_i must all be the sums of terms of the form

$$X_1 X_2 \cdots X_n \Psi_4, \tag{2.29}$$

where $X_i \in NP$ or $X_i \in OP$. This can easily be seen by considering the well known decomposition of $\nabla^{A\dot{X}}$ in terms of the NP operators (2.11) and of $\nabla^{A\dot{X}}o^B$, $\nabla^{A\dot{X}}\iota^B$ in terms of the basis spinors o, ι and the NP coefficients.

It is now useful to introduce a simple notation. If a product of basis spinors has the form

$$Y \equiv \underbrace{o^{A_1}\cdots o^{A_{m_1}}}_{m_1} \underbrace{\bar{o}^{\dot{X}_1}\cdots\bar{o}^{\dot{X}_{m_2}}}_{m_2} \underbrace{\iota^{B_1}\cdots\iota^{B_{n_1}}}_{n_1} \underbrace{\bar{\iota}^{\dot{Y}_1}\cdots\bar{\iota}^{\dot{Y}_{n_2}}}_{n_2},$$

then $P_o(Y) = m_1 + m_2$ will denote the number of o^A's and $\bar{o}^{\dot{A}}$'s, and $P_\iota(Y) = n_1 + n_2$ the number of ι^A's and $\bar{\iota}^{\dot{A}}$'s which are contained in Y.

Now it is easy to see (essentially as a consequence of (2.7)), that if a spinor $S^{A_1\cdots A_m \dot{X}_1\cdots\dot{X}_k} \in \mathcal{W}^{[m,k]}$ has the form

$$S^{A_1\cdots A_m \dot{X}_1\cdots\dot{X}_k} = \sum_{i=1}^{2^{m+k}} s_i B_i^{[m,k]}, \tag{2.30}$$

then all invariants formed from the products of $S^{\cdots}$ must vanish, provided that the coefficients in (2.30) are such that $s_i = 0$ for all i for which

$$P_o(B_i^{[m,k]}) \leqslant P_\iota(B_i^{[m,k]}). \tag{2.31}$$

This observation enables us to prove the following lemma.

Lemma 1. Let an invariant constructed from the products of the spinors $\nabla^{C_n \dot{X}_n} \cdots \nabla^{C_1 \dot{X}_1}$ $(\Psi_4 o^A o^B o^C o^D)$, for fixed n, be non-vanishing. Then there exists a non-vanishing quantity $X_1 X_2 \cdots X_n \Psi_4$, $X_i \in NP \cup OP$, such that

$$b(X_1 X_2 \cdots X_n \Psi_4) = \sum_{i=1}^{n} b(X_i) + b(\Psi_4) \geqslant 0, \qquad \text{i.e.} \quad \sum_{i=1}^{n} b(X_i) \geqslant 4.$$

Proof. The nth spinor derivative is of the form (2.28), where the coefficients c_i are sums of terms of the form (2.29). According to the observation above, an invariant formed from these derivatives will be non-vanishing only if there exists $c_i \neq 0$ such that $P_o(B_i^{[n+4,n]}) \leqslant P_\iota(B_i^{[n+4,n]})$. Since the basis spinors have the boost-weight $d \equiv b(B_i^{[n+4,n]}) = P_o(B_i^{[n+4,n]}) - P_\iota(B_i^{[n+4,n]})$, and since from (2.25) and (2.27) it follows that $b(c_i) = -d$, we see that c_i must satisfy the condition $b(c_i) \geqslant 0$. Hence, there must exist a non-vanishing quantity $X_1 X_2 \cdots X_n \Psi_4$, $X_i \in NP \cup OP$, such that $b(X_1 X_2 \cdots X_n \Psi_4) \geqslant 0$.

3. Non-expanding and non-twisting solutions

Type-N vacuum solutions with Λ admitting a non-expanding and non-twisting null geodesic congruence belong to Kundt's class (see [10] ch 27 for details). Since $\theta = \omega = 0$ (cf equations (2.3), (2.4)), the NP coefficient

$$\rho = \theta + i\omega = 0. \tag{3.1}$$

In this section we shall prove that in this class all the curvature invariants of any order vanish (or are constants determined by Λ), so generalizing the results of [8, 16], where only the plane-wave metrics are considered.

Choose the null tetrad parallelly propagated along the null congruence determined by the multiple principal null direction of the Weyl tensor and parametrized by an affine parameter. Thus, only $\Psi_4 \neq 0$ (cf equation (2.19)) and the NP coefficients $\sigma = \kappa = \pi = \varepsilon = 0$. The NP equations, given in the appendix for a general type-N vacuum spacetime with Λ, then simplify considerably. From all the NP equations we shall only need those containing the operator D:

$$D\tau = 0,$$

$$D\alpha = 0,$$

$$D\beta = 0,$$

$$D\gamma = \tau\alpha + \bar{\tau}\beta - \tfrac{1}{24}R, \tag{3.2}$$

$$D\lambda = 0,$$

$$D\mu = \tfrac{1}{12}R,$$

$$D\nu = \bar{\tau}\mu + \tau\lambda,$$

and the commutators

$$(\Delta D - D\Delta) = (\gamma + \bar{\gamma})D - \tau\bar{\delta} - \bar{\tau}\delta, \tag{3.3}$$

$$(\delta D - D\delta) = (\bar{\alpha} + \beta)D. \tag{3.4}$$

We shall also need the Bianchi identity (2.21), which now simply reduces to

$$D\Psi_4 = 0. \tag{3.5}$$

The simple form of the above equations suggests the following notation: let $\mathcal{F}_k$ be the set of functions f such that $f \in \mathcal{F}_k \iff D^k f = 0$. From the NP equations (3.2) we easily find that

$$\alpha, \beta, \tau, \lambda \in \mathcal{F}_1, \qquad \gamma, \mu \in \mathcal{F}_2, \qquad \nu \in \mathcal{F}_3. \tag{3.6}$$

Using this, and employing equations (3.3) and (3.4) for the commutators, we can prove the following lemma.

Lemma 2. Let $f \in \mathcal{F}_k$. Then: (i) $\delta f \in \mathcal{F}_k$, $\bar{\delta} f \in \mathcal{F}_k$; and (ii) $\Delta f \in \mathcal{F}_{k+1}$.

Proof. (i) can easily be proven by induction. Applying (3.4) to f_1, we immediately get $D\delta f_1 = 0 \Rightarrow \delta f_1 \in \mathcal{F}_1$. Assuming then $\delta f_k \in \mathcal{F}_k$ and applying (3.4) to f_{k+1}, we find $D^{k+1}\delta f_{k+1} = 0 \Rightarrow \delta f_{k+1} \in \mathcal{F}_{k+1}$.

In order to prove (ii), we first show, by using Leibniz's formula, that $D^{k+1}(f_2 f_k) = 0$ for all $k \geqslant 1$. Then (ii) can be proven by induction similarly as in (i) (the commutator (3.3) now being used instead of (3.4)).

It will now be useful to associate the number p with any NP coefficient X which will indicate the behaviour of X under the action of the operator D. Let

$$p(X) = k - 1, \qquad \text{if} \quad X \in \mathcal{F}_k \text{ but } X \notin \mathcal{F}_{k-1}. \tag{3.7}$$

If $X \in OP$, i.e. X is one of the NP operators, we define (being motivated by lemma 2) $p(\Delta) = 1$, $p(\delta) = 0$, $p(D) = -1$.

The values of p for all relevant quantities are summarized in table 2.

Table 2. Values of p associated with NP coefficients and operators.

$X \in NP$	$X \in OP$	$p(X)$
ν		2
$\gamma\ \mu$	Δ	1
$\alpha\ \beta\ \tau\ \lambda$	δ	0
	D	-1

The indicators p enable us to formulate the following lemma in a concise form.

Lemma 3. Consider (as in lemma 1) a quantity $X_1 X_2 \cdots X_n$ where $X_i \in NP \cup OP$. If $\sum_{i=1}^n p(X_i) < 0$, then $X_1 X_2 \cdots X_n \Psi_4 = 0$.

Proof. From the Bianchi identity (3.5) we have $D\Psi_4 = 0$. Considering table 2, we observe that the condition $\sum_{i=1}^n p(X_i) < 0$ requires that for any X_i having a positive $p(X_i)$, the operator D must appear at least $p(X_i)$ times among $X_1 \cdots X_n$ since only $p(D)$ is negative (for example, with any ν, D must appear at least twice); an additional D then has to enter $X_1 \cdots X_n$ in order that $\sum_{i=1}^n p(X_i) < 0$. This results in $X_1 \cdots X_n \Psi_4 = 0$.

Combining lemma 1 and lemma 3 we can now make the following proposition.

Proposition 1. In type-N vacuum spacetimes with Λ admitting a non-expanding and non-twisting null geodesic congruence, all nth order invariants formed from the products of spinors $\nabla^{C_n \dot{X}_n} \cdots \nabla^{C_1 \dot{X}_1} (\Psi_4 o^A o^B o^C o^D)$, with n arbitrary but fixed, vanish.

Proof. According to lemmas 1 and 3, a non-vanishing invariant will exist only if there are $X_i \in NP \cup OP$, $i = 1, \ldots, n$, such that

$$\sum_{i=1}^{n} b(X_i) \geqslant 4 \qquad \text{and} \qquad \sum_{i=1}^{n} p(X_i) \geqslant 0. \tag{3.8}$$

The values of these sums depend on how many times the specific NP coefficient or NP operator enters $X_1 \cdots X_n$. Let m_1 denote the number of the coefficients ν and $\bar{\nu}$; m_2 the number of $\gamma, \bar{\gamma}, \mu, \bar{\mu}$; m_3 the number of $\lambda, \bar{\lambda}$; m_4 the number of $\alpha, \bar{\alpha}, \beta, \bar{\beta}, \tau, \bar{\tau}$; k_1 the number of operators D; k_2 the number of $\delta, \bar{\delta}$; and k_3 the number of Δ, which enter $X_1 \cdots X_n$. Then, considering tables 1 and 2, we find that the inequalities (3.8) read

$$- 4m_1 - 2m_2 - 2m_3 - 2k_3 + 2k_1 \geqslant 4 \qquad 2m_1 + m_2 + k_3 - k_1 \geqslant 0.$$

Combining the last two inequalities, we obtain $m_3 \leqslant -2$. This is impossible since all m's and k's must be non-negative.

From lemmas 1 and 3, and from the last proof it follows that all the coefficients c_i in the decomposition (2.28) which multiply basis spinors satisfying the condition (2.31), must necessarily vanish. Hence, all non-vanishing terms in the decomposition (2.28), with n arbitrary, contain a larger number of spinors o^A than of ι^A. Therefore, regarding (2.7) it is evident that all invariants formed from the products of the derivatives (2.28) with *different* n's must necessarily vanish. Recalling the relations (2.14) and (2.20) between the Weyl tensor $C_{\alpha\beta\gamma\delta}$, the Weyl spinor Ψ_{ABCD} and the NP scalar Ψ_4, we can now formulate our two basic propositions.

Proposition 2. In type-N vacuum spacetimes with Λ, admitting a non-expanding and non-twisting null geodesic congruence, all invariants constructed from the Weyl tensor and its covariant derivatives of arbitrary order vanish.

Next, recall that in vacuum spacetimes with Λ Einstein's equations imply $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$, so that

$$C_{\alpha\beta\gamma\delta;\varepsilon} = R_{\alpha\beta\gamma\delta;\varepsilon}. \tag{3.9}$$

Then, regarding the spinor form (2.15) of the Riemann tensor we see that all invariants constructed from the Riemann tensor and its covariant derivatives of arbitrary order can be expressed in terms of the spinors

$$\Psi_4 o^A o^B o^C o^D + \tfrac{1}{3} \Lambda (\varepsilon^{AC} \varepsilon^{BD} + \varepsilon^{AD} \varepsilon^{BC}), \tag{3.10}$$

$$\nabla^{C_1 \dot{X}_1} (\Psi_4 o^A o^B o^C o^D), \tag{3.11}$$

$$\vdots$$

$$\nabla^{C_n \dot{X}_n} \cdots \nabla^{C_1 \dot{X}_1} (\Psi_4 o^A o^B o^C o^D). \tag{3.12}$$

Applying the considerations above and recalling that $\varepsilon^{AB} = o^A \iota^B - o^B \iota^A$, we can easily confirm that all terms in the invariants containing $\Psi_4 o^A o^B o^C o^D$ or their derivatives of arbitrary order, vanish. The only non-vanishing quantities can be formed from the constant term $\tfrac{1}{3} \Lambda (\varepsilon^{AC} \varepsilon^{BD} + \varepsilon^{AD} \varepsilon^{BC})$ and are dependent on Λ only. We thus finally arrive at:

Proposition 3. In type-N vacuum spacetimes with Λ, admitting a non-expanding and non-twisting null geodesic congruence, all invariants constructed from the products of the Riemann tensor and its covariant derivatives of arbitrary order vanish, provided they contain a derivative of the Riemann tensor. The invariants constructed from the Riemann tensor itself are all constants depending on Λ.

If $\Lambda = 0$, the Riemann tensor is equal to the Weyl tensor and all the invariants vanish by proposition 2.

4. Expanding and twisting solutions

This section is divided into two parts. In the first, we analyse type-N spacetimes with Λ with expanding but non-twisting null congruences. We shall show that invariants of zero and first order vanish (or are constants determined by Λ) as in the non-expanding and non-twisting case. However, we will succeed in finding a non-vanishing invariant of second order depending on the expansion and on Ψ_4. In the second part, we shall show that the invariants of the zero and first order again vanish, and we shall demonstrate how the non-vanishing invariant is modified when there is a non-zero twist.

Before we turn to the details, we wish to point out the main reason why, with the expansion present, a non-vanishing invariant may exist: it is due to the Newman–Penrose equation for the expansion, $D\rho = \rho^2$. Now $D^n \rho \neq 0$ for any n, and one cannot even formulate lemma 3, for example.

4.1. Non-twisting case

All metrics of vacuum type-N spacetimes with Λ were given by García-Díaz and Plebański [3]. In their coordinates, suitable for our purposes, the metric reads

$$\mathrm{d}s^2 = 2v^2 \, \mathrm{d}\xi \, \mathrm{d}\bar{\xi} + 2v\bar{A} \, \mathrm{d}\xi \, \mathrm{d}u + 2vA \, \mathrm{d}\bar{\xi} \, \mathrm{d}u + 2\psi \, \mathrm{d}u \, \mathrm{d}v + 2(A\bar{A} + \psi B) \, \mathrm{d}u^2, \qquad (4.1)$$

where

$$\begin{aligned}
A &= \epsilon\xi - vf \,, \\
B &= -\epsilon + \tfrac{1}{2}v(f_{,\xi} + \bar{f}_{,\bar{\xi}}) + \tfrac{1}{6}\Lambda v^2 \psi \,, \\
\psi &= 1 + \epsilon\xi\bar{\xi} \,, \\
\epsilon &= -1, 0, +1.
\end{aligned} \qquad (4.2)$$

It is useful to choose the null tetrad corresponding to the forms

$$\begin{aligned}
\omega^1 &= v \, \mathrm{d}\bar{\xi} + \bar{A} \, \mathrm{d}u, \\
\omega^2 &= v \, \mathrm{d}\xi + A \, \mathrm{d}u, \\
\omega^3 &= \psi \, \mathrm{d}u, \\
\omega^4 &= -\mathrm{d}v - B \, \mathrm{d}u.
\end{aligned} \qquad (4.3)$$

Using this tetrad, one obtains the NP coefficients as follows:

$$\sigma = \kappa = \varepsilon = \pi = \lambda = \alpha = \beta = 0,$$

$$\gamma = \frac{f_{,\xi}}{2\psi} + \frac{\Lambda v}{6}, \qquad \tau = -\frac{\epsilon\xi}{\psi v}, \qquad \rho = \frac{1}{v},$$

$$\nu = -\frac{f_{,\xi\xi}}{2\psi}, \qquad \mu = -\frac{\Lambda v}{6}, \qquad \Psi_4 = -\frac{f_{,\xi\xi\xi}}{2\psi v}.$$

(4.4)

In the following calculations, the specific forms of most of the NP coefficients are not important—we need to know only the value of $\rho = 1/v$, and which of the coefficients vanish. Then we can write down all the NP equations by specializing the NP equations in the appendix to the present case. The Bianchi identities (2.21), (2.22) now simplify to the form

$$D\Psi_4 = \rho\Psi_4, \qquad \delta\Psi_4 = \tau\Psi_4.$$

(4.5)

Using these and the NP equations, we can write the first spinor derivative of the Weyl spinor as

$$\nabla^{E\dot{F}}(\Psi_4 o^A o^B o^C o^D) = -(\Delta + 4\gamma)\Psi_4 \, S_{[0]}^{ABCDE} \, \bar{o}^{\dot{F}}$$

$$+ \bar{\delta}\Psi_4 \, S_{[0]}^{ABCDE} \, \bar{\iota}^{\dot{F}} + \Psi_4(\tau\bar{o}^{\dot{F}} - \rho\bar{\iota}^{\dot{F}}) \, S_{[1]}^{ABCDE},$$

(4.6)

where we have introduced quantities $S_{[j]}^{ABCDE}$ which denote the symmetrized products of the basis spinors o^A and ι^A, and the subscript $[j]$ gives the number of ι^A's, entering $S^{\cdots}$ (for example, $S_{[1]}^{AB} = o^A\iota^B + o^B\iota^A$). As usual, $\bar{S}^{\cdots}$ denotes the complex conjugate of $S^{\cdots}$, with $[j]$ being the number of $\bar{\iota}^{\dot{A}}$'s. From equation (4.6) it is then clear that all the first-order invariants of the Weyl tensor vanish since there are not enough ι's to be combined with o's, as was discussed in detail in the previous section.

Now consider the second derivatives. Since calculations become rather lengthy we used the computer algebra package *Maple V*. Again using the NP equations and Bianchi identities, we arrive at the following results:

$$\nabla^{G\dot{H}}\nabla^{E\dot{F}}(\Psi_4 o^A o^B o^C o^D) = \mathcal{A} \, S_{[0]}^{ABCDEG} \, \bar{S}_{[0]}^{\dot{F}\dot{H}} + \mathcal{B} \, S_{[1]}^{ABCDEG} \, \bar{S}_{[0]}^{\dot{F}\dot{H}}$$

$$+ \mathcal{C} \, S_{[0]}^{ABCDEG} \, \bar{S}_{[1]}^{\dot{F}\dot{H}} + \mathcal{D} \, S_{[2]}^{ABCDEG} \, \bar{S}_{[0]}^{\dot{F}\dot{H}} + \mathcal{E} \, S_{[1]}^{ABCDEG} \, \bar{S}_{[1]}^{\dot{F}\dot{H}}$$

$$+ \mathcal{F} \, S_{[2]}^{ABCDEG} \, \bar{S}_{[1]}^{\dot{F}\dot{H}} + \mathcal{G} \, S_{[1]}^{ABCDEG} \, \bar{S}_{[2]}^{\dot{F}\dot{H}} + \mathcal{H} \, S_{[2]}^{ABCDEG} \, \bar{S}_{[2]}^{\dot{F}\dot{H}},$$

(4.7)

where the functions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ read

$$\mathcal{A} = \left\{(\Delta + 9\gamma + \bar{\gamma})\Delta + 4(5\gamma + \bar{\gamma} + \Delta)\gamma - 5\tau\nu - \bar{\nu}\bar{\delta}\right\}\Psi_4,$$

$$\mathcal{B} = \delta\left\{(\Delta - 4\gamma)\Psi_4\right\},$$

$$\mathcal{C} = -\bar{\delta}\left\{(\Delta + 4\gamma)\Psi_4\right\},$$

$$\mathcal{D} = 2\Psi_4\tau^2, \tag{4.8}$$

$$\mathcal{E} = (D\Delta + 4\rho\gamma)\Psi_4,$$

$$\mathcal{F} = -2\Psi_4\rho\tau,$$

$$\mathcal{G} = -2\rho\bar{\delta}\Psi_4,$$

$$\mathcal{H} = 2\Psi_4\rho^2.$$

The above expression (4.7) indicates that a non-vanishing invariant may exist. Looking at the last term in (4.7), which is proportional to the function $\mathcal{H}$, we see that it contains, in contrast to all other terms, the same number of o's and ι's. Consequently, a combination of such terms may give a non-vanishing result. One can easily confirm, however, that simple squares or even cubes of such terms do not work. After a number of unsuccessful attempts to construct a non-vanishing expression we arrived at the invariant

$$I = R^{\alpha\beta\gamma\delta;\varepsilon\phi} R_{\alpha\mu\gamma\nu;\varepsilon\phi} R^{\lambda\mu\rho\nu;\sigma\tau} R_{\lambda\beta\rho\delta;\sigma\tau}, \tag{4.9}$$

or, regarding (3.9), equivalently

$$I = C^{\alpha\beta\gamma\delta;\varepsilon\phi} C_{\alpha\mu\gamma\nu;\varepsilon\phi} C^{\lambda\mu\rho\nu;\sigma\tau} C_{\lambda\beta\rho\delta;\sigma\tau}. \tag{4.10}$$

In terms of the spinors employed to arrive at the invariant, we obtain

$$I = 4\mathcal{H}^2\bar{\mathcal{H}}^2 \underset{[2]}{S_{BGDH}} \underset{[2]}{S^{BGDH}} \underset{[2]}{\bar{S}_{\dot{B}\dot{G}\dot{D}\dot{H}}} \underset{[2]}{\bar{S}^{\dot{B}\dot{G}\dot{D}\dot{H}}}, \tag{4.11}$$

where we have used the relation $\underset{[2]}{S^{ABCDEF}} \iota_E\iota_F = \underset{[2]}{S^{ABCD}}$. Since a straightforward calculation gives $\underset{[2]}{S^{BGDH}} \underset{[2]}{S_{BGDH}} = 6$, we find

$$I = 144\mathcal{H}^2\bar{\mathcal{H}}^2, \tag{4.12}$$

or, regarding (4.8) and (4.4), we finally obtain

$$I = 9(2\rho)^8\Psi_4{}^2\bar{\Psi}_4^2, \tag{4.13}$$

and

$$I = 144\frac{f_{\xi\xi\xi}^2 \bar{f}_{\bar{\xi}\bar{\xi}\bar{\xi}}^2}{\psi^4 v^{12}}. \tag{4.14}$$

4.2. Twisting case

Twisting type-N vacuum spacetimes with Λ are not known, except for the Hauser solution. With non-vanishing twist, ρ becomes complex, $\rho = \theta + i\omega$, and in contrast to non-twisting spacetimes, the NP coefficients α, β, λ in general do not vanish. Hence, the NP equations become much more complicated. However, inspecting the relation generalizing (4.6), we again find that the zeroth- and first-order invariants vanish. The calculations of the second derivatives (4.7) are only feasible using a computer algebra package (here again *Maple V*). The functions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ in (4.8) become much more lengthy, however the most relevant function, $\mathcal{H}$, remains the same. Therefore, using (4.12) we find that the invariant (4.9), respectively (4.10), becomes

$$I = 9(2\rho)^4(2\bar{\rho})^4\Psi_4{}^2\bar{\Psi}_4^2. \tag{4.15}$$

Before moving on to possible applications of the invariants, let us note that their form in the NP formalism (not in terms of the Weyl or Riemann tensor) could have been anticipated

by considering the behaviour of the NP quantities under the transformations (2.24) and (2.25). Taking into account only undifferentiated NP quantities and assuming a type-N spacetime with $\rho \neq 0$, one finds that the quantities which are invariant under (2.24) and (2.25) must be functionally dependent on $\rho^2 \bar{\rho}^2 \Psi_4 \bar{\Psi}_4$. All such invariants can thus be written in terms of the invariant (4.9). Of course, expression (4.9) is invariant under *any* tetrad and coordinate transformation.

5. Applications

The invariant (4.9) can be used to study the occurrence of singularities in various expanding type-N spacetimes.

5.1. Expanding, non-twisting spacetimes

For example, in the García-Díaz–Plebański spacetimes with the metric (4.1), we find that the invariant (4.9), which now becomes equal to the expression (4.14), is diverging if for $f_{\xi\xi\xi}^2 \bar{f}_{\bar{\xi}\bar{\xi}\bar{\xi}}^2 \neq 0$ there is: (i) $v = 0$; and (ii) $\psi = 1 + \epsilon\xi\bar{\xi} = 0$, i.e. if $\epsilon = -1$ and $\xi\bar{\xi} = 1$. It also diverges when v and ψ are finite but $f_{\xi\xi\xi}^2 \bar{f}_{\bar{\xi}\bar{\xi}\bar{\xi}}^2$ diverges. If f is a polynomial quadratic in ξ, the spacetime can be shown to be flat or de Sitter (if $\Lambda \neq 0$). If, however,

$$f(\xi, u) = c_0(u) + c_1(u)\xi + \cdots + c_n(u)\xi^n,$$

where $n \geqslant 6$, then I is singular at $\xi = \infty$, $\bar{\xi} = \infty$ for each $\epsilon = -1, 0, 1$.

5.2. Type-N twisting spacetimes

Recently two papers appeared discussing the physical meaning of the vacuum Einstein equations of Petrov type N with expanding and twisting null congruence. Stephani [18] has argued, within the framework of the linearized theory, that these solutions contain singular lines ('pipes') in space which at any time extend arbitrarily far away from a possible insular source. A few months ago, Finley *et al* [5], using an iterative approach, tried to resolve 'Stephani's paradox' by constructing the solution up to third order in their approximation scheme. They conclude that 'up to the third order, there do exist acceptable, regular solutions'. What does our invariant (4.15) say about the nature of these approximate solutions?

The standard form of the type-N twisting vacuum solutions (see e.g. [10]) reads (using here the usual signature $+2$)

$$ds^2 = 2\omega^1\omega^2 - 2\omega^3\omega^4,$$

$$\omega^1 = -\frac{d\zeta}{P\bar{\rho}} = \bar{\omega}^2,$$

$$\omega^3 = du + L\,d\zeta + \bar{L}\,d\bar{\zeta}, \tag{5.1}$$

$$\omega^4 = dr + W\,d\zeta + \overline{W}\,d\bar{\zeta},$$

where the real function $P(\zeta, \bar{\zeta}, u)$ and complex function $L(\zeta, \bar{\zeta}, u)$ appearing in the metric

satisfy the relations

$$\rho = \frac{-1}{r + i\Sigma}, \qquad 2i\Sigma = P^2(\bar\partial L - \partial\bar L),$$

$$W = \frac{L_{,u}}{\rho} + i\partial\Sigma, \qquad \partial \equiv \partial_\zeta - L\partial_u.$$

(5.2)

The function ρ is indeed the NP coefficient used before. Introducing a real function V by putting $P = V_{,u}$, the field equations can be written in the form

$$(\partial\partial\bar\partial\bar\partial V)_{,u} = P^{-1}(\partial\partial V)_{,u}(\bar\partial\bar\partial V)_{,u},$$

$$\mathrm{Im}(\partial\partial\bar\partial\bar\partial V) = 0, \qquad \partial[P^{-1}(\bar\partial\bar\partial V)_{,u}] = 0.$$

(5.3)

The coordinate system and field equations are invariant under gauge transformations [10, 18]

$$\zeta' = f(\zeta), \qquad u' = F(\zeta, \bar\zeta, u), \qquad r' = rF_{,u}^{-1}.$$

(5.4)

The NP component of the Weyl tensor is given by

$$\Psi_4 = P^2\rho\partial_u[P^{-1}(\bar\partial\bar\partial V)_{,u}].$$

(5.5)

Under the gauge transformation (5.4), the quantities appearing above transform as follows:

$$\rho' = F_{,u}\rho, \qquad \Sigma = F_{,u}\Sigma',$$

$$\partial' = f'^{-1}\partial, \qquad P' = F_{,u}^{-1}|f'|P,$$

$$L' = f'^{-1}(Lf_{,u} - F_{,\zeta}),$$

$$\Psi_4' = f'^2(\bar f')^{-2}F_{,u}^{-2}\Psi_4.$$

(5.6)

Using expressions (5.2) and (5.5) for ρ and Ψ_4, we can easily calculate the invariant (4.15) to obtain

$$I = 9(2\rho)^4(2\bar\rho)^4\Psi_4{}^2\bar\Psi_4^2 = 2304\frac{P^8}{(r^2 + \Sigma^2)^6}\big(\partial_u[P^{-1}(\bar\partial\bar\partial V)_{,u}]\big)^2\big(\partial_u[P^{-1}(\partial\partial V)_{,u}]\big)^2. \quad (5.7)$$

It can easily be checked, by using (5.6), that (5.7) does not change under the gauge transformation (5.4). Of course, in its original forms (4.9), (4.10), I is invariant under *any* coordinate transformation.

Now, Stephani [18] found the general solution of the linearized field equations in the form (using Stephani's notation)

$$P = 1 + \tfrac{1}{2}\zeta\bar\zeta,$$

$$L = B(\zeta, \bar\zeta) + \frac{C(u, \bar\zeta)}{(1 + \zeta\bar\zeta/2)^2} + \frac{\bar\zeta^2 D(u, \zeta)}{2(1 + \zeta\bar\zeta/2)^2} - \frac{\bar\zeta D_{,\zeta}}{(1 + \zeta\bar\zeta/2)} + D_{,\zeta\zeta},$$

(5.8)

where the functions $C(u, \bar\zeta)$ and $D(u, \zeta)$ are arbitrary and $B(\zeta, \bar\zeta)$ has to satisfy $\mathrm{Im}(B_{,\bar\zeta}) = 0$. In order that spacetime is not flat, i.e. $\Psi_4 \neq 0$, the condition

$$D_{,uu\zeta\zeta} \neq 0$$

(5.9)

has to be satisfied.

Using Stephani's solution (5.8) in (5.7), we find the invariant to be

$$I^L = 2304\frac{P^8}{r^{12}}D_{,\zeta\zeta\zeta uu}^2\bar D_{,\bar\zeta\bar\zeta\bar\zeta uu}^2.$$

(5.10)

Defining $\tilde{C}(u, \bar{\zeta}) \equiv C(u, \bar{\zeta})\bar{\zeta}^{-2}$ and $\tilde{D}(u, \zeta) \equiv D(u, \zeta)\zeta^{-2}$, Stephani calculates the expression

$$P^2 L_{,u\bar{\zeta}} = \frac{2\bar{\zeta}\tilde{C}_{,u}(u, \bar{\zeta})}{1 + \zeta\bar{\zeta}/2} + \bar{\zeta}^2\tilde{C}_{,u\bar{\zeta}} - \frac{2\zeta\tilde{D}_{,u}(u, \zeta)}{1 + \zeta\bar{\zeta}/2} - \zeta^2\tilde{D}_{,u\zeta}, \qquad (5.11)$$

which is invariant under the gauge transformation (5.4). Since the functions $\tilde{C}$ and $\tilde{D}$ are analytic in ζ and $\bar{\zeta}$, respectively, they will have singularities in the plane $(\zeta, \bar{\zeta})$, i.e. on the sphere (recall the standard convention $\zeta = \sqrt{2} \tan\frac{1}{2}\theta e^{i\varphi}$). The expression (5.11) will be regular only if $\tilde{C}_{,u} \sim \bar{\zeta}^{-1}$ and $\tilde{D}_{,u} \sim \zeta^{-1}$. However, in this case the spacetime is flat.

Clearly, the function $D(u, \zeta)$ has singularities in the plane $(\zeta, \bar{\zeta})$ and the invariant (5.10) will diverge unless $D_{,uu} \sim a\zeta^2 + b\zeta + c$. (Notice that if $D \sim \zeta^3$ then the invariant diverges due to P^8.) However, this implies flat spacetime again.

As mentioned above, Finley *et al* [5] constructed the twisting and diverging type-N solution up to third order of an iteration procedure. Their final expressions for the functions entering our invariant (5.7) read

$$\Sigma \approx \frac{1 - \frac{1}{2}\zeta\bar{\zeta}}{1 + \frac{1}{2}\zeta\bar{\zeta}} \operatorname{Im}(f_2) - 2\operatorname{Re}(a^{(1)}) \operatorname{Im}\left(\frac{\mathrm{d}f^{(2)}}{\mathrm{d}u}\right),$$

$$\Psi_4 \approx \frac{\bar{a}^{(1)}}{r}\left(\frac{1 + \frac{1}{2}\zeta\bar{\zeta}}{\bar{\zeta}}\right)^2 \frac{\mathrm{d}^3\bar{f}^2}{\mathrm{d}u^3}, \qquad (5.12)$$

where $a^{(1)}$ is a complex constant, $f_2(u) \equiv f^{(2)}(u) + f^{(3)}(u)$, $f^{(2)}(u)$, $f^{(3)}(u)$ being arbitrary functions of u. The curvature is non-vanishing so far as $f^{(2)}_{,uuu} \neq 0$. Both quantities, Σ/r and K/r^2 (for K see equation (5.9) in [5]), considered in [5], which are invariant under the gauge transformations (5.4), are indeed regular; the NP coefficient ρ (cf (5.2)) entering our curvature invariant (4.15) is also regular. However, $(\Psi_4\bar{\Psi}_4)^2$ is *not* regular on the $(\zeta, \bar{\zeta})$-sphere unless $\bar{a}^{(1)} f^{(2)}_{,uuu} = 0$, which corresponds to flat spacetime. Indeed, $(\Psi_4\bar{\Psi}_4)^2 \sim (1 + \frac{1}{2}\zeta\bar{\zeta})^8/(\zeta\bar{\zeta})^4$ diverges at $\zeta\bar{\zeta} \to \infty$ (which in the standard convention, with $\zeta = \sqrt{2} e^{i\varphi} \tan\frac{1}{2}\theta$, corresponds to $\theta = \pi$).

Therefore, we find that Stephani's conclusion based on the linearized theory remains true for the third-order solution analysed by Finley *et al*. Although this raises more doubts about the interpretation of type-N twisting solutions as representing radiation fields outside bounded sources, solutions of the full Einstein equations may perhaps bring us surprises.

Acknowledgments

Appendix. The Newman–Penrose equations in type-N spacetimes

The NP coefficients are defined in table A1 (see e.g. [19]), in which ∇ denotes, respectively, D, Δ, δ, $\bar{\delta}$; the first line gives the definition in terms of the basis spinors and the second in terms of the null tetrad.

Table A1. NP coefficients.

∇	$o^A \nabla o_A$ $m^a \nabla l_a$	$o^A \nabla \iota_A = \iota^A \nabla o_A$ $\frac{1}{2}(n^a \nabla l_a - \bar{m}^a \nabla m_a)$	$\iota^A \nabla \iota_A$ $-\bar{m}^a \nabla n_a$
D	κ	ε	π
Δ	τ	γ	ν
δ	σ	β	μ
$\bar{\delta}$	ρ	α	λ

In the vacuum type-N spacetimes, the null tetrad can be chosen so that

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0,$$

$$\Phi_{A\dot{A}B\dot{B}} = 0, \qquad \sigma = \kappa = 0,$$

the NP equations are

$$D\rho = \rho^2 + (\varepsilon + \bar{\varepsilon})\rho,$$

$$D\tau = (\tau + \bar{\pi})\rho + (\varepsilon - \bar{\varepsilon})\tau,$$

$$D\alpha - \bar{\delta}\varepsilon = (\rho + \bar{\varepsilon} - 2\varepsilon)\alpha - \bar{\beta}\varepsilon + (\varepsilon + \rho)\pi,$$

$$D\beta - \delta\varepsilon = (\bar{\rho} - \bar{\varepsilon})\beta - (\bar{\alpha} - \bar{\pi})\varepsilon,$$

$$D\gamma - \Delta\varepsilon = (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta - (\varepsilon + \bar{\varepsilon})\gamma - (\gamma + \bar{\gamma})\varepsilon + \tau\pi - \tfrac{1}{24}R,$$

$$D\lambda - \bar{\delta}\pi = \rho\lambda + \pi^2 + (\alpha - \bar{\beta})\pi - (3\varepsilon - \bar{\varepsilon})\lambda,$$

$$D\mu - \delta\pi = \bar{\rho}\mu + \pi\bar{\pi} - (\varepsilon + \bar{\varepsilon})\mu - \pi(\bar{\alpha} - \beta) + \tfrac{1}{12}R,$$

$$D\nu - \Delta\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + (\gamma - \bar{\gamma})\pi - (3\varepsilon + \bar{\varepsilon})\nu,$$

$$\Delta\lambda - \bar{\delta}\nu = -(\mu + \bar{\mu})\lambda - (3\gamma - \bar{\gamma})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \Psi_4, \qquad (A.1)$$

$$\delta\rho = \rho(\bar{\alpha} + \beta) + (\rho - \bar{\rho})\tau,$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \varepsilon(\mu - \bar{\mu}) + \tfrac{1}{24}R,$$

$$\delta\lambda - \bar{\delta}\mu = (\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta),$$

$$\delta\nu - \Delta\mu = (\mu^2 + \lambda\bar{\lambda}) + (\gamma + \bar{\gamma})\mu - \bar{\nu}\pi + (\tau - 3\beta - \bar{\alpha})\nu,$$

$$\delta\gamma - \Delta\beta = (\tau - \bar{\alpha} - \beta)\gamma + \mu\tau - \varepsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda},$$

$$\delta\tau = \bar{\lambda}\rho + (\tau + \beta - \bar{\alpha})\tau,$$

$$\Delta\rho - \bar{\delta}\tau = -\rho\bar{\mu} + (\bar{\beta} - \alpha - \bar{\tau})\tau + (\gamma + \bar{\gamma})\rho - \tfrac{1}{12}R,$$

$$\Delta\alpha - \bar{\delta}\gamma = (\rho + \varepsilon)\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma,$$

and the commutators are

$$(\Delta D - D\Delta) = (\gamma + \bar{\gamma})D + (\varepsilon + \bar{\varepsilon})\Delta - (\tau + \bar{\pi})\bar{\delta} - (\bar{\tau} + \pi)\delta,$$

$$(\delta D - D\delta) = (\bar{\alpha} + \beta - \bar{\pi})D - (\bar{\rho} + \varepsilon - \bar{\varepsilon})\delta,$$

$$(\delta\Delta - \Delta\delta) = -\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + \bar{\lambda}\bar{\delta} + (\mu - \gamma + \bar{\gamma})\delta,$$

$$(\bar{\delta}\delta - \delta\bar{\delta}) = (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta - (\bar{\alpha} - \beta)\bar{\delta} - (\bar{\beta} - \alpha)\delta.$$

(A.2)

References

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