Class. Quantum Grav. **18** (2001) 1205–1216 www.iop.org/Journals/cq PII: S0264-9381(01)18644-2

Co-accelerated particles in the C-metric

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Received 6 November 2000

Abstract

With appropriately chosen parameters, the C-metric represents two uniformly accelerated black holes moving in the opposite directions on the axis of the axial symmetry (the z-axis). The acceleration is caused by nodal singularities located on the z-axis.

In the present paper, geodesics in the C-metric are examined. In general, there exist three types of timelike or null geodesics in the C-metric: geodesics describing particles (a) falling under the black hole horizon; (b) crossing the acceleration horizon; and (c) orbiting around the z-axis and co-accelerating with the black holes.

Using an effective potential, it can be shown that there exist stable timelike geodesics of the third type if the product of the parameters of the C-metric, mA, is smaller than a certain critical value. Null geodesics of the third type are always unstable. Special timelike and null geodesics of the third type are also found in an analytical form.

PACS numbers: 0420, 0420J, 0425

1. Introduction and summary

The C-metric is a vacuum solution of the Einstein equations of the Petrov type D. Kinnersley and Walker [1] showed that it represents black holes uniformly accelerated by nodal singularities in opposite directions along the axis of the axial symmetry. In coordinates $\{x, y, p, q\}$ adapted to its algebraical structure, the C-metric reads as follows:

$$\mathrm{d}s^2 = \frac{1}{A^2(x+y)^2}\left(G^{-1}\,\mathrm{d}x^2 + F^{-1}\,\mathrm{d}y^2 + G\,\mathrm{d}p^2 - F\,\mathrm{d}q^2\right), \tag{1}$$

where the functions F, G are the cubic polynomials

$$F = -1 + y^2 - 2mAy^3, \tag{2}$$

$$G = 1 - x^2 - 2mAx^3, \tag{3}$$

with m and A being constant. As we are choosing the signature +2, we take $G > 0$.

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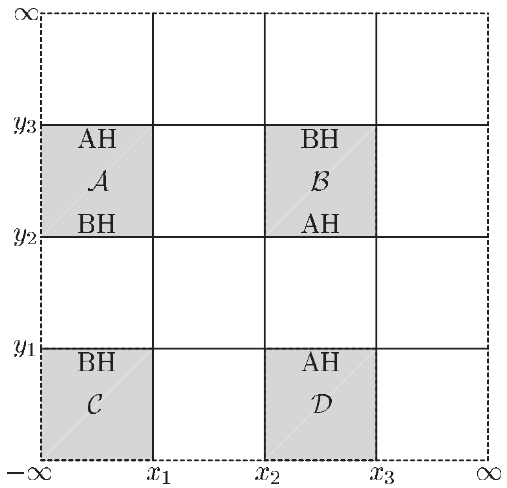


Figure 1. The character of the regions determined by roots of the polynomials F and G is illustrated schematically. Static regions are shaded, black hole and acceleration horizons are denoted by BH and AH, respectively. Broken lines represent curvature singularities.

Although this form of the C-metric (1) is simple, it is not suitable for a physical interpretation of the solution. Since the metric (1) has two Killing vectors $\partial/\partial p$, $\partial/\partial q$ it is possible to transform it in its static regions given by

$$G > 0, \qquad F > 0 \tag{4}$$

to the Weyl form [2] (see section 3). By further transformation [2] one can show that the C-metric is, in fact, a radiative boost–rotation symmetric spacetime [2, 3] (see section 4). The class of boost–rotation symmetric spacetimes is the only class of exact radiative solutions of the full nonlinear Einstein equations that are known in an analytical form, describe moving objects, and are asymptotically flat (see [3–5] for a general treatise). Several generalizations of the C-metric are known, let us mention the charged C-metric [6, 7] and the spinning C-metric [8, 9].

Farhoosh and Zimmermann [10] studied a special class of geodesics in the C-metric—test particles moving on the symmetry axis. We examine general geodesics starting in the most physical static region of the C-metric (region $\mathcal{B}$, see figure 1) with the help of an effective potential. It turns out that there exist three types of timelike (null) geodesics: (a) geodesics describing particles falling under the black hole horizon and then on the curvature singularity; (b) those describing particles crossing the acceleration horizon and reaching future timelike (null) infinity—they are not co-accelerated with the black holes; and (c) geodesics describing particles spinning around the axis of the axial symmetry (the z-axis), co-accelerating with the black holes along this axis and reaching future null infinity. We investigate stability of timelike and null geodesics of the third type using the effective potential in the coordinates $\{x, y, p, q\}$ (section 2) and in the Weyl coordinates (section 3) in which it is easy to see that the stability of timelike geodesics does not depend on the distribution of conical singularities located on the z-axis. We show that null geodesics of this type are always unstable and that there exist stable timelike geodesics of the considered type if the product of the parameters of the C-metric, mA, is smaller than a certain critical value (22). This result indicates that a black hole or a star, having satellites in the equatorial plane, which starts to accelerate in the direction perpendicular to this equatorial plane can retain some of its satellites (which is, in fact, not very surprising) only if the acceleration is sufficiently small.

We present special geodesics of the third type in an analytical form representing particles (or zero-rest-mass particles) orbiting around the axis of the axial symmetry in a *constant distance* and uniformly accelerating along the z-axis (dragged by the black holes). They are given by x = constant, y = constant in the coordinates $\{x, y, p, q\}$ (see section 2), by $\bar{\rho}$ = constant, $\bar{z}$ = constant in the Weyl coordinates (section 3) and finally we examine them in the coordinates adapted to the boost–rotation symmetry (ρ = constant, $z^2 - t^2$ = constant) where their physical interpretation can be easily understood (section 4).

In [11] it was shown that the Schwarzschild metric can be obtained from the C-metric in the Weyl coordinates by the limiting procedure $A \to 0$ (note that one can obtain the Weyl coordinates used in [11] by multiplying the Weyl coordinates in our paper by a factor of A). Using this procedure one can find that the unstable null geodesic in the C-metric representing photon-like particles orbiting around the z-axis corresponds to the unstable circular photon orbit in the Schwarzschild metric.

2. Geodesics in $\{x, y, p, q\}$ coordinates

First, we study geodesics in the coordinates $\{x, y, p, q\}$ in which the C-metric has the form (1). The polynomials F and G entering the metric have three different real roots iff the condition

$$27m^2A^2 < 1 \tag{5}$$

holds. Then the C-metric contains four different static regions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ (see figure 1) where the polynomials F, G satisfy (4). The metric (1) in each of these static regions can be transformed into the Weyl form with different metric functions (see [3]).

Curvature invariants diverge at $x \to \pm\infty$ or $y \to \pm\infty$ [3] where curvature singularities are located (see figure 1). Region $\mathcal{B}$ is the only static region without curvature singularities. In fact, it is the most physical static region describing a uniformly accelerated black hole (see [3] for an analysis of the other regions). Acceleration horizons and black hole horizons are at $y = y_i$ where y_i are the roots of the equation $F = 0$ and are denoted in figure 1 by AH and BH, respectively.

In this paper we consider only region $\mathcal{B}$ which exists iff condition (5) is satisfied.

If a metric has a Killing vector ξ^α then there exists a conserved quantity $\xi^\alpha U_\alpha$ for timelike geodesics with a tangent vector U^α and $\xi^\alpha k_\alpha$ for null geodesics with a tangent vector k^α. Since the C-metric (1) has two Killing vectors $\partial/\partial p$ and $\partial/\partial q$, corresponding covariant components of the 4-velocity, U_p, U_q, for particles and components k_p, k_q of the wave 4-vector for zero-rest-mass particles are conserved along geodesics and thus

$$\frac{\mathrm{d}p(\tau)}{\mathrm{d}\tau} = LA^2\frac{\left(x(\tau)+y(\tau)\right)^2}{G(x(\tau))}, \tag{6}$$

$$\frac{\mathrm{d}q(\tau)}{\mathrm{d}\tau} = EA^2\frac{\left(x(\tau)+y(\tau)\right)^2}{F(y(\tau))}, \tag{7}$$

where L and E are constants of motion, τ is a proper time for timelike geodesics and an affine parameter for null geodesics.

Let us examine special geodesics $x(\tau) = x_0$, $y(\tau) = y_0$, where x_0 and y_0 are constants. Then substituting (6) and (7) into the geodesic equations we obtain

$$\frac{\left(1 + mAx_0{}^3 + x_0y_0 + 3mAx_0{}^2y_0\right)L^2}{G(x_0)} - \frac{G(x_0)\,E^2}{F(y_0)} = 0, \tag{8}$$

$$\frac{F(y_0)\,L^2}{G(x_0)} - \frac{\left(-1 + mAy_0{}^3 - x_0y_0 + 3mAx_0y_0{}^2\right)E^2}{F(y_0)} = 0. \tag{9}$$

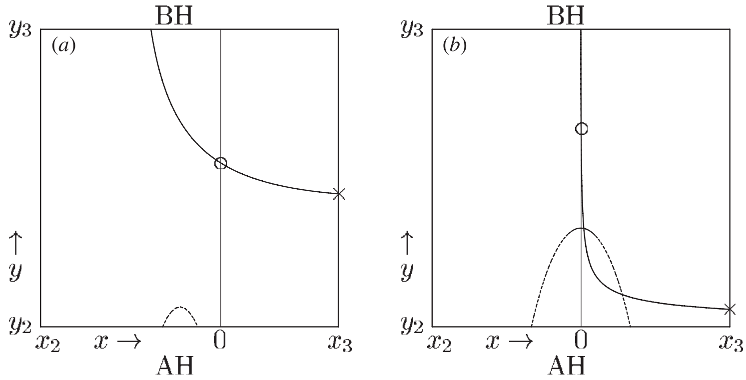


Figure 2. The full curve is given by (10) (the circle represents the null geodesic given by (15), each point on the curve between the circle and the cross represents a timelike geodesic (14) and the remaining points correspond to spacelike geodesics), the broken curve is given by $V_{,xx}V_{,yy} - V_{,xy}^2 = 0$, where L was substituted from (12) after performing the derivatives for: (*a*) $m = \frac{1}{2}$, $A = \frac{1}{3}$ (not satisfying (22)); (*b*) $m = 0.02$, $A = 0.05$ (satisfying (22))—between the intersections of the two plotted curves the geodesics (14) are stable and the corresponding potential V has its local minimum there.

A linear combination of these two equations leads to the condition

$$3m^2A^2x_0^2y_0^2 + mA\,(x_0y_0 + 3)\,(y_0 - x_0) - 1 = 0. \tag{10}$$

Points $[x_0, y_0]$ in region $\mathcal{B}$ satisfying this condition are plotted in figure 2.

The norm of the 4-velocity is

$$A^2\,(x_0 + y_0)^2 \left(\frac{L^2}{G(x_0)} - \frac{E^2}{F(y_0)} \right) = \epsilon, \tag{11}$$

where $\epsilon = -1, 0$ for timelike and null geodesics, respectively.

From (8) and (11) for a given timelike geodesic ($\epsilon = -1$), i.e. given x_0, y_0, constants L and E read

$$L^2 = \frac{G(x_0)^2}{A^2\,(x_0 + y_0)^3\,(1 + 3mAx_0)\,x_0}, \tag{12}$$

$$E^2 = \frac{F(y_0)^2}{A^2\,(x_0 + y_0)^3\,(1 - 3mAy_0)\,y_0}. \tag{13}$$

In region $\mathcal{B}$, L^2 and E^2 are positive only for $x_0 > 0$ where thus timelike geodesics exist (see figure 2). Thus all timelike geodesics in region $\mathcal{B}$ with x, y being constant are given by

$$\begin{aligned} x(\tau) &= x_0, \\ y(\tau) &= y_0, \\ p(\tau) &= LA^2\frac{(x_0 + y_0)^2}{G(x_0)}\tau, \\ q(\tau) &= EA^2\frac{(x_0 + y_0)^2}{F(y_0)}\tau, \end{aligned} \tag{14}$$

where $x_0 \in (0, x_3)$, y_0 is given by (10), and L, E are given by (12) and (13), respectively.

Spacelike geodesics have $x_0 < 0$ and we do not consider them further.

There exists a null geodesic ($\epsilon = 0$, the circle in figure 2):

$$
\begin{aligned}
x(\tau) &= 0, \\
y(\tau) &= \frac{1}{3mA}, \\
p(\tau) &= L\frac{1}{9m^2}\tau, \\
q(\tau) &= E\frac{3A^2}{1-27m^2A^2}\tau, \\
L^2 &= E^2\frac{27m^2A^2}{1-27m^2A^2}.
\end{aligned}
\tag{15}
$$

Note that L^2 is positive since the condition (5) is assumed to be satisfied.

Let us now examine the stability of *general* timelike geodesics. For this purpose we first construct an effective potential of a general freely falling particle whose 4-velocity has the norm

$$
-1 = \frac{-1}{A^2(x+y)^2}\left[-\frac{1}{G}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 - \frac{1}{F}\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2 - G\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}\right)^2 + F\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2\right]. \tag{16}
$$

Substituting from (6) and (7) into (16) we obtain the condition

$$
\frac{F}{E^2A^4(x+y)^4}\left[\frac{1}{G}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + \frac{1}{F}\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2\right] = \frac{E^2-V^2}{E^2}, \tag{17}
$$

where the effective potential V has the form

$$
V = \sqrt{F\left(\frac{1}{A^2(x+y)^2} + \frac{L^2}{G}\right)}. \tag{18}
$$

From (17) it follows that a freely falling particle with given E has access only to those regions where $E > V$. The potential V goes to infinity for $x \to x_2$ or $x \to x_3$ and consequently particles cannot reach the left- and right-hand edges of the square $\mathcal{B}$. Thus there remain only three possibilities: (a) particles leave the square $\mathcal{B}$ across the upper edge, i.e. they fall under the black hole horizon (on which $V = 0$); (b) particles leave the square $\mathcal{B}$ through the lower edge (on which also $V = 0$), i.e. they cross the acceleration horizon; and (c) particles remain in the square $\mathcal{B}$, i.e. they are co-accelerated with the black holes (see section 4).

Let us now study the stability of geodesics of the third type. There exist stable geodesics of this type if V has its local minimum in region $\mathcal{B}$, i.e. if there exists a point (x^*, y^*) in region $\mathcal{B}$ where

$$
V_{,x}(x^*, y^*) = V_{,y}(x^*, y^*) = 0, \tag{19}
$$

$$
V_{,xx}(x^*, y^*)V_{,yy}(x^*, y^*) - V^2_{,xy}(x^*, y^*) > 0, \tag{20}
$$

$$
V_{,xx}(x^*, y^*) > 0. \tag{21}
$$

For given m, A and L there is only one point (x^*, y^*) in region $\mathcal{B}$ satisfying (19). It lies on the curve (10) with L given by (12), i.e. it corresponds to the geodesic (14). Condition (21) is always satisfied, however, condition (20) is quite complicated and numerical calculations (see also figure 2 and the text below) show that the necessary condition for satisfying (20) is

$$
mA \lesssim 4.54 \times 10^{-3}. \tag{22}
$$

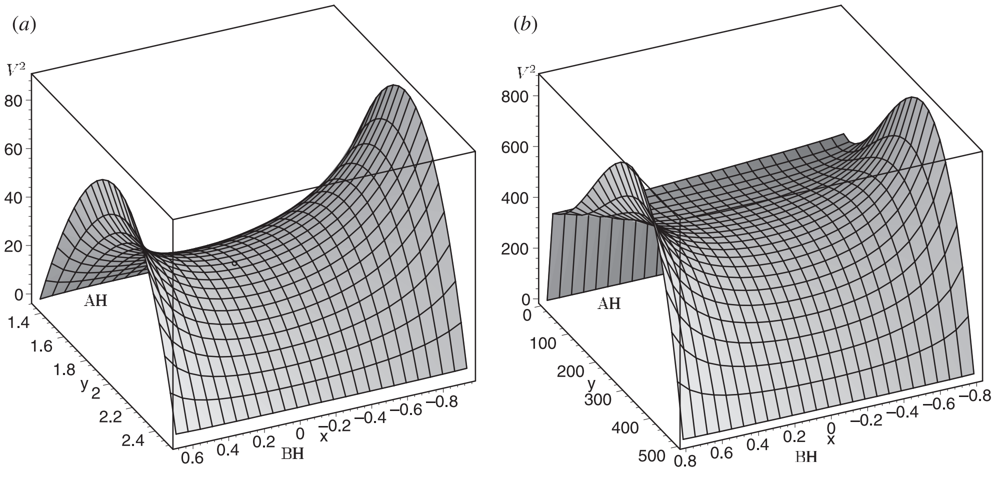


Figure 3. The function V as a function of x, y for $L = 0.084$ and (a) $m = \frac{1}{2}$, $A = \frac{1}{3}$ (not satisfying (22)); (b) $m = 0.02$, $A = 0.05$ (satisfying (22)), where a local minimum exists.

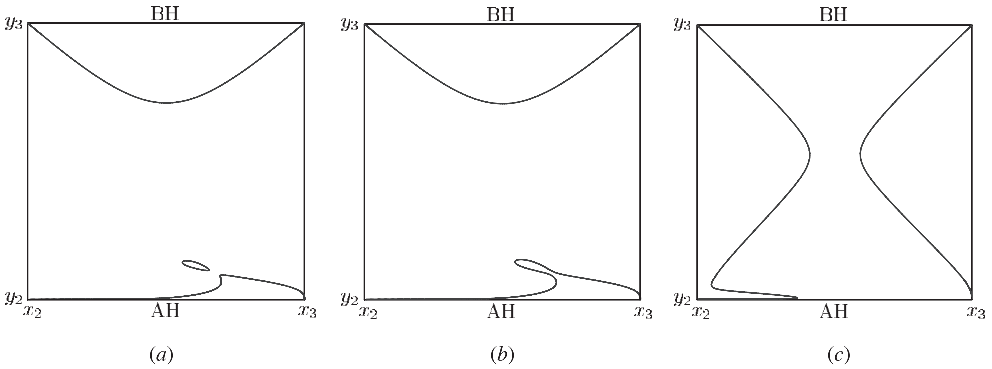
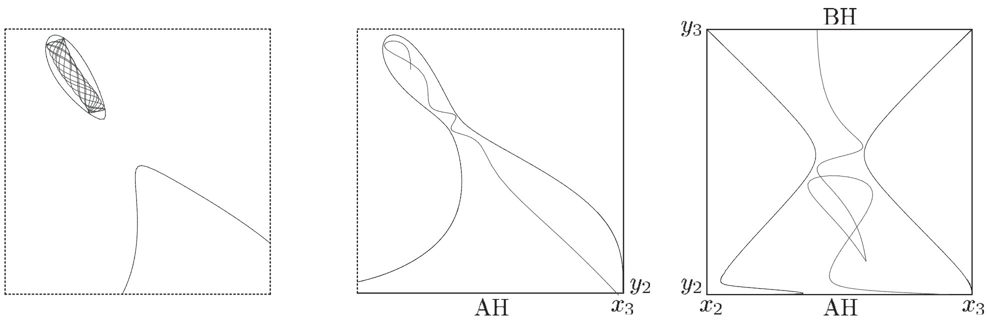


Figure 4. Curves V = constant for $m = 0.02$, $A = 0.05$, $L = 0.084$: (a) $V = \sqrt{372.9}$; (b) $V = \sqrt{373}$; (c) $V = \sqrt{429}$.

Figures 3(a) and (b) illustrate the behaviour of the potential V for parameters m, A which do not and do satisfy condition (22), respectively.

For parameters m, A not satisfying condition (22) there is no local minimum of the potential V and thus a small perturbation causes a freely falling particle moving along the geodesic (14) to fall either under the black hole horizon or under the acceleration horizon.

For parameters m, A satisfying condition (22) and for suitable L (see figure 2) there exists a region (the region bounded by a closed curve in figure 4(a)) from which freely falling particle with E lower than a certain critical value cannot escape (see figure 5(a)). In this region the considered geodesics are stable. It will be made clear in section 4 that these trapped particles are co-accelerated with the uniformly accelerated black hole. If the parameter E of these trapped particles is increased over a certain critical value they fall under the acceleration horizon (and thus they are not co-accelerated with the black hole, see figures 4(b), 4(c), 5(b) and 5(c)) or under the black hole horizon (see figures 4(c) and 5(c)).



(*a*) (*b*) (*c*)

Figure 5. Curves V = constant as in figure 4 and numerically obtained geodesics for $m = 0.02$, $A = 0.05$, $L = 0.084$. All geodesics start at the same point (x_0, y_0) which lies on the curve (10) but have different initial velocities, i.e. different constants of motion E: (*a*) $E = \sqrt{372.9}$—a geodesic of a co-accelerated particle; (*b*) $E = \sqrt{373}$—a geodesic of a particle crossing the acceleration horizon; (*c*) $E = \sqrt{429}$—geodesics of particles which fall under the acceleration or the black hole horizon depending on the direction of an initial velocity.

Similarly, we may derive an effective potential Λ for zero-rest-mass test particles substituting (6) and (7) into the relation for the norm of the wave 4-vector $k_\alpha k^\alpha = 0$:

$$\frac{F}{E^2 A^4 (x+y)^4}\left[\frac{1}{G}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + \frac{1}{F}\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\right)^2\right] = 1 - \frac{L^2}{E^2}\frac{1}{\Lambda^2}, \tag{23}$$

where

$$\Lambda = \sqrt{\frac{G}{F}}. \tag{24}$$

Thus photon-like particles can only reach regions where $L/E < \Lambda$. In the considered part $\mathcal{B}$ of the spacetime, the function Λ has vanishing first derivatives at the point $x = 0$, $y = 1/(3mA)$ (the circle in figure 2), however, there is not a local extreme and thus the geodesic (15) is unstable.

3. Geodesics in the Weyl coordinates

For interpreting the geodesics (14) and (15) we transform them into the Weyl coordinates in this section and into the coordinates adapted to the boost and rotation symmetries in the next section.

As was mentioned earlier the C-metric in each of its static regions can be transformed into the static Weyl form

$$\mathrm{d}s^2 = \mathrm{e}^{-2U}\left[\mathrm{e}^{2\nu}(\mathrm{d}\bar{\rho}^2 + \mathrm{d}\bar{z}^2) + \bar{\rho}^2\mathrm{d}\bar{\phi}^2\right] - \mathrm{e}^{2U}\mathrm{d}\bar{t}^2 \tag{25}$$

by transformation (see [2])

$$\begin{aligned}
\bar{z} &= \frac{1 + mAxy(x-y) + xy}{A^2(x+y)^2}, \\
\bar{\rho} &= \frac{\sqrt{FG}}{A^2(x+y)^2}, \\
\bar{\phi} &= p, \\
\bar{t} &= q.
\end{aligned} \tag{26}$$

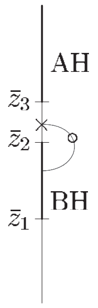


Figure 6. Region $\mathcal{B}$ in the Weyl coordinates: the axis $\bar{\rho} = 0$ with the black hole horizon (BH, $\bar{z}_1 < \bar{z} < \bar{z}_2$) and the acceleration horizon (AH, $\bar{z} > \bar{z}_3$) and the curve given by equation (35) for $m = \frac{1}{2}$, $A = \frac{1}{3}$ (the circle corresponds to the null geodesic (37), each point between the circle and the cross represents a timelike geodesic (36) and the other points on the curve correspond to spacelike geodesics).

Transforming the C-metric in region $\mathcal{B}$ into the Weyl coordinates we obtain

$$e^{2U} = \frac{[R_1 - (\bar{z} - \bar{z}_1)]\,[R_3 - (\bar{z} - \bar{z}_3)]}{R_2 - (\bar{z} - \bar{z}_2)}, \tag{27}$$

$$e^{2\nu} = \frac{1}{4}\frac{m^2}{A^6(\bar{z}_2 - \bar{z}_1)^2(\bar{z}_3 - \bar{z}_2)^2} \times \frac{\left[R_2R_3 + \bar{\rho}^2 + (\bar{z} - \bar{z}_2)(\bar{z} - \bar{z}_3)\right]\left[R_1R_2 + \bar{\rho}^2 + (\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2)\right]}{R_1R_2R_3\left[R_1R_3 + \bar{\rho}^2 + (\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_3)\right]}\, e^{2U}, \tag{28}$$

where functions R_1, R_2 and R_3 are defined by

$$R_i = \sqrt{(\bar{z} - \bar{z}_i)^2 + \bar{\rho}^2}, \tag{29}$$

and $\bar{z}_1 < \bar{z}_2 < \bar{z}_3$ are the roots of the equation

$$2A^4\bar{z}_i^3 - A^2\bar{z}_i^2 + m^2 = 0. \tag{30}$$

As was shown in [2], the C-metric in the Weyl coordinates corresponds to the field of a rod between $\bar{z}_1$ and $\bar{z}_2$, a semi-infinite line mass at $\bar{z} > \bar{z}_3$ and conical singularities for $\bar{z} < \bar{z}_1$ and $\bar{z}_2 < \bar{z} < \bar{z}_3$ keeping them apart. The rod between $\bar{z}_1$ and $\bar{z}_2$ represents the black hole horizon and the semi-infinite line mass at $\bar{z} > \bar{z}_3$ represents the acceleration horizon (see figure 6).

The regularity condition of the axis

$$e^{2\nu}(\bar{\rho} = 0, \bar{z}) = 1 \tag{31}$$

is not satisfied at points where nodal singularities appear. Since the metric (25) with metric functions

$$\begin{aligned} e^{-2U'} &= a\, e^{-2U}, \\ e^{2\nu'} &= b\, e^{2\nu}, \end{aligned} \tag{32}$$

where a and b are constants, also satisfies the vacuum Einstein equations, by choosing the constant b appropriately we may regularize (i.e. fulfil the condition (31)) either part of the axis $\bar{z} < \bar{z}_1$ or $\bar{z}_2 < \bar{z} < \bar{z}_3$ (see [3]).

Conditions (6) and (7) for geodesics in the Weyl coordinates (corresponding to the existence of two Killing vectors $\partial/\partial\bar{t}$, $\partial/\partial\bar{\phi}$) read as follows:

$$\frac{\mathrm{d}\bar{\phi}(\tau)}{\mathrm{d}\tau} = L\frac{\mathrm{e}^{2U}}{\bar{\rho}(\tau)^2}, \tag{33}$$

$$\frac{\mathrm{d}\bar{t}(\tau)}{\mathrm{d}\tau} = E\mathrm{e}^{-2U}. \tag{34}$$

Due to the transformation (26), the geodesics $x(\tau) = x_0 = \text{constant}$, $y(\tau) = y_0 = \text{constant}$, discussed in the previous section, now have the form $\bar{\rho}(\tau) = \bar{\rho}_0 = \text{constant}$, $\bar{z}(\tau) = \bar{z}_0 = \text{constant}$. Condition (10) in terms of the coordinates $\bar{\rho}$, $\bar{z}$ reads

$$R_1R_3 - R_3R_2 - R_1R_2 = 0, \tag{35}$$

where R_1, R_2, R_3 are given by (29) and the corresponding curve is plotted in figure 6.

Timelike geodesics (14) have in the Weyl coordinates the form

$$\begin{aligned}
\bar{\rho}(\tau) &= \bar{\rho}_0,\\
\bar{z}(\tau) &= \bar{z}_0,\\
\bar{\phi}(\tau) &= L\frac{\mathrm{e}^{2U(\bar{\rho}_0,\bar{z}_0)}}{\bar{\rho}_0^2}\tau,\\
\bar{t}(\tau) &= E\mathrm{e}^{-2U(\bar{\rho}_0,\bar{z}_0)}\tau,
\end{aligned} \tag{36}$$

where $\bar{z}_0$, $\bar{\rho}_0$ are constants satisfying (35), and E, L are constants of motion (33), (34).

Similarly, the null geodesic (15) in the Weyl coordinates reads (the circle in figure 6)

$$\begin{aligned}
\bar{z}(\tau) &= 9m^2,\\
\bar{\rho}(\tau) &= \sqrt{3(1-27m^2A^2)}\,\frac{m}{A},\\
\bar{\phi}(\tau) &= L\frac{1}{9m^2}\tau,\\
\bar{t}(\tau) &= E\frac{3A^2}{1-27m^2A^2}\tau.
\end{aligned} \tag{37}$$

Analogously as in section 2, (16)–(18), the motion of a freely falling particle with the constants of motion E, L is restricted to a region where $E > V$, the effective potential V being

$$V = \sqrt{\mathrm{e}^{2U}\left(1 + \frac{L^2\mathrm{e}^{2U}}{\bar{\rho}^2}\right)}. \tag{38}$$

Note, that since the potential V and the condition (35) do not depend on the function $\mathrm{e}^{2\nu}$, after changing the constant b, i.e. changing the distribution of nodal singularities, the geodesics (36) remain geodesics and, moreover, this change does not affect their stability.

In figure 7 (analogous to figure 4) curves with different values of V^2 are plotted. There again appears a region bounded by a closed curve from which trapped particles with a given parameter E cannot escape. Particles with E higher than a certain critical value fall either under the black hole horizon (the axis between w_1, w_2 where $w = |\bar{z}|^{1/4}\,\mathrm{sign}\,\bar{z}$) or under the acceleration horizon (the axis between w_3 and ∞).

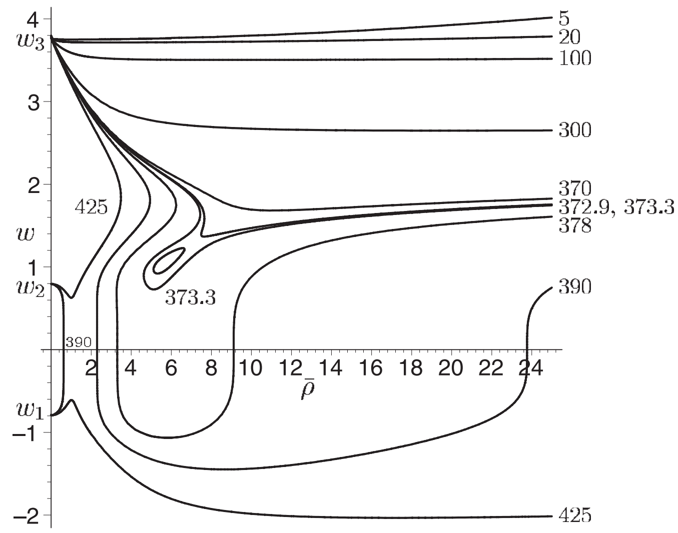


Figure 7. Curves V^2 = constant for $m = 0.02$, $A = 0.05$, $L = 0.084$ (to compactify the picture coordinate $w = |\bar z|^{1/4}\,\mathrm{sign}\,\bar z$ is used instead of $\bar z$; w_i correspond to $\bar z_i$). The potential V is infinite on the axis at $w \in (-\infty, w_1)$ and $w \in (w_2, w_3)$ and null at the black hole and acceleration horizons at $w \in (w_1, w_2)$, $w \in (w_3, \infty)$, respectively.

4. Geodesics in the canonical coordinates adapted to the boost–rotation symmetry

To find an interpretation of geodesics studied in the previous sections we transform the metric (25) by the transformation

$$\begin{aligned}
&\bar\rho^2 = \rho^2(z^2 - t^2),\\
&\bar\phi = \phi,\\
&\bar z - \bar z_3 = \tfrac{1}{2}(t^2 + \rho^2 - z^2),\\
&\bar t = \operatorname{arctanh}(t/z)
\end{aligned} \tag{39}$$

into the form

$$\begin{aligned}
\mathrm{d}s^2 = {}& -\mathrm{e}^{\lambda}\,\mathrm{d}\rho^2 - \rho^2\mathrm{e}^{-\mu}\,\mathrm{d}\phi^2\\
&- \frac{1}{z^2 - t^2}\left[(\mathrm{e}^{\lambda}z^2 - \mathrm{e}^{\mu}t^2)\,\mathrm{d}z^2 - 2zt(\mathrm{e}^{\lambda} - \mathrm{e}^{\mu})\,\mathrm{d}z\,\mathrm{d}t + (\mathrm{e}^{\lambda}t^2 - \mathrm{e}^{\mu}z^2)\,\mathrm{d}t^2\right]
\end{aligned} \tag{40}$$

which is adapted to the boost and rotation symmetries (see [4] and [3]). The inverse transformation to (39) has the form

$$\begin{aligned}
\rho &= \sqrt{\sqrt{\bar\rho^2 + (\bar z - \bar z_3)^2} + (\bar z - \bar z_3)},\\
z &= \pm\sqrt{\sqrt{\bar\rho^2 + (\bar z - \bar z_3)^2} - (\bar z - \bar z_3)}\,\cosh\bar t,\\
t &= \pm\sqrt{\sqrt{\bar\rho^2 + (\bar z - \bar z_3)^2} - (\bar z - \bar z_3)}\,\sinh\bar t,
\end{aligned} \tag{41}$$

where either upper or lower signs are valid.

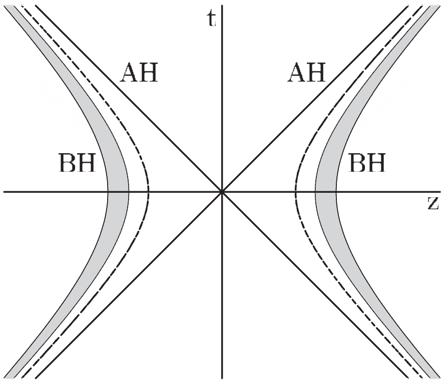


Figure 8. Uniformly accelerated black holes (the shaded region) and co-accelerated test particles (broken curves).

From (41) it follows that geodesics we are interested in, satisfying $\bar{\rho} = \bar{\rho}_0 =$ constant, $\bar{z} = \bar{z}_0 =$ constant in the Weyl coordinates, in the coordinates $\{t, \rho, z, \phi\}$ satisfy $\rho =$ constant and $z^2 - t^2 =$ constant (the worldline is a hyperbola in the (z, t)-plane which corresponds to a uniformly accelerated motion along the z-axis, see figure 8). Geodesics of this type (corresponding to (36) and (37)) now have the form

$$\begin{aligned}
\rho(\tau) &= K_1, \\
\phi(\tau) &= c_2\tau, \\
z(\tau) &= \pm K_2 \cosh c_1\tau, \\
t(\tau) &= K_2 \sinh c_1\tau,
\end{aligned} \tag{42}$$

where the constants K_1, K_2, c_1 and c_2 read

$$\begin{aligned}
K_1 &= \sqrt{\sqrt{\bar{\rho}_0^2 + (\bar{z}_0 - \bar{z}_3)^2} + (\bar{z}_0 - \bar{z}_3)}, \\
K_2 &= \sqrt{z^2 - t^2} = \sqrt{\sqrt{\bar{\rho}_0^2 + (\bar{z}_0 - \bar{z}_3)^2} - (\bar{z}_0 - \bar{z}_3)}, \\
c_1 &= E\mathrm{e}^{-2U(\bar{\rho}_0, \bar{z}_0)}, \\
c_2 &= L\frac{\mathrm{e}^{2U(\bar{\rho}_0, \bar{z}_0)}}{\bar{\rho}_0^2}.
\end{aligned}$$

These geodesics describe particles orbiting the z-axis and uniformly accelerating along the z-axis.

Acknowledgments

VP was supported by the grant GACR-202/00/P030 and AP by the grant GACR-202/00/P031. The work was also supported partially by the National Science Foundation under the grant NSF–Czech Rep. INT-9724783.

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