INSTITUTE OF PHYSICS PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **21** (2004) 2873–2897 PII: S0264-9381(04)72742-2

Bianchi identities in higher dimensions

V Pravda[1], **A Pravdová**[1], **A Coley**[2] **and R Milson**[2]

[1] Mathematical Institute, Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic
[2] Department of Mathematics and Statistics, Dalhousie U., Halifax, Nova Scotia B3H 3J5, Canada

E-mail: pravda@math.cas.cz, pravdova@math.cas.cz, aac@mathstat.dal.ca
and milson@mathstat.dal.ca

Received 28 November 2003
Published 18 May 2004
Online at stacks.iop.org/CQG/21/2873
DOI: 10.1088/0264-9381/21/12/007

Abstract
A higher dimensional frame formalism is developed in order to study implications of the Bianchi identities for the Weyl tensor in vacuum spacetimes of the algebraic types III and N in arbitrary dimension n. It follows that the principal null congruence is geodesic and expands isotropically in two dimensions and does not expand in $n-4$ spacelike dimensions or does not expand at all. It is shown that the existence of such principal geodesic null congruence in vacuum (together with an additional condition on twist) implies an algebraically special spacetime. We also use the Myers–Perry metric as an explicit example of a vacuum type D spacetime to show that principal geodesic null congruences in vacuum type D spacetimes do not share this property.

PACS number: 04.50.+h

1. Introduction

For dimensions $n < 4$ the Weyl tensor vanishes identically and for $n = 4$ it has very special properties. It is of interest to determine which of the properties of four-dimensional (4D) spacetimes can be straightforwardly generalized to higher dimensions and which need to be modified or do not hold at all.

Recently a classification of algebraic tensor types in Lorentzian manifolds of arbitrary dimension was developed [1]. For the Weyl tensor in 4D this classification reproduces the Petrov classification and for the Ricci tensor in 4D the Segre classification.

In 4D it follows from the Bianchi identities[3]

$$R_{ab\{cd;e\}} = 0 \tag{1}$$

that in algebraically special vacuum spacetimes the multiple principal null direction of the Weyl tensor is geodesic and shearfree. In this paper a higher dimensional frame formalism

[3] Note that in this paper we use two different operations denoted by {}. In the first case {} acts on three indices and stands for $R_{ab\{cd;e\}} = R_{abcd;e} + R_{abde;c} + R_{abec;d}$. In the other case {} acts on four indices and is given by (5).

is developed in order to study implications of the Bianchi identities, which are given in appendix B, for vacuum spacetimes of algebraic types N and III in higher dimensions. Although in most applications it is necessary to perform calculations in a given spacetime dimension n, in this paper we present results without specifying the dimension and hence these results are valid in any dimension.

In 4D, for algebraically special vacuum spacetimes some of the tetrad components of the Bianchi identities in the Newman–Penrose formalism [2] lead to simple algebraic equations (i.e. equations with no derivatives). In higher dimensions these algebraic equations are much more complex and the number of independent equations, as well as the number of unknowns, depends on the dimension of the spacetime.

We show that in vacuum type III and N spacetimes of arbitrary dimension the multiple principal null direction (PND) is geodesic. For type N spacetimes (section 3) the symmetric expansion matrix $\boldsymbol{S}$ has just one non-vanishing doubly degenerate eigenvalue. Consequently, the principal null geodesic congruence expands isotropically in two dimensions and does not expand in $n-2$ dimensions. Thus, shear does not vanish for $n>4$. The antisymmetric twist matrix $\boldsymbol{A}$ has only one independent component. All other components may be set to zero by appropriately choosing the frame. For type III, we prove similar results in the generic case (section 4) and in the non-twisting case (appendix C.1). The complete proof, including all possible degenerate cases, is presented for five-dimensional spacetimes (appendix C.2).

In section 5, we show that for a vacuum spacetime the properties of $\boldsymbol{S}$ and $\boldsymbol{A}$ matrices mentioned above imply that the spacetime is algebraically special. We also show that an arbitrary vacuum spacetime admitting a non-expanding and non-twisting geodesic null congruence (i.e., a higher dimensional generalization of the vacuum Kundt class) is algebraically special.

These statements cannot be regarded as a generalization of the Goldberg–Sachs theorem for higher dimensions (see section 5 for details) since in appendix D it is shown that the Myers–Perry metric, which is of the algebraic type D, has the expansion matrix $\boldsymbol{S}$ with one doubly degenerate eigenvalue and also another non-vanishing eigenvalue.

In section 6, we conclude with a discussion of potential applications.

2. Preliminaries

The Newman–Penrose formalism [2] in 4D is based on using a null tetrad $\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}$ and $\bar{\boldsymbol{m}}$, where $\boldsymbol{\ell}$ and $\boldsymbol{n}$ are real null vectors and $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ are complex null vectors instead of an orthonormal basis, thus taking advantage of the null cone structure of spacetimes. For n-dimensional calculations it seems to be more practical to choose a pair of null vectors $\boldsymbol{\ell}, \boldsymbol{n}$ and an orthonormal set of real spacelike vectors $\boldsymbol{m}^{(i)}$. We thus need two types of indices: indices $a, b, \ldots$ with values $0, \ldots, n-1$ and indices $i, j, \ldots$ going from 2 to $n-1$. We will observe Einstein's summation convention for both of these types of indices. However, for indices $i, j, \ldots$, there is no difference between covariant and contravariant components and thus we will not distinguish between subscripts and superscripts.

The frame

$$\boldsymbol{m}^{(0)} = \boldsymbol{n}, \qquad \boldsymbol{m}^{(1)} = \boldsymbol{\ell}, \qquad \boldsymbol{m}^{(i)} \tag{2}$$

thus satisfies

$$\ell^a \ell_a = n^a n_a = \ell^a m^{(i)}_a = n^a m^{(i)}_a = 0, \qquad \ell^a n_a = 1, \qquad m^{(i)a} m^{(j)}_a = \delta_{ij},$$

and the metric has the form

$$g_{ab} = 2\ell_{(a} n_{b)} + \delta_{ij} m^{(i)}_a m^{(j)}_b.$$

If one would like to completely generalize the Newman–Penrose formalism for higher dimensions it would be necessary to denote Ricci rotation-coefficients and all independent components of the Riemann, Weyl and Ricci tensors and then rewrite all frame components of the Bianchi identities (1) and the Ricci identities

$$V_{a;bc} = V_{a;cb} + R^{s}{}_{abc}V_s,$$

where V is an arbitrary vector, as well as expressions for commutators [8] of covariant derivatives in the directions of the frame vectors

$$D \equiv \ell^a \nabla_a, \qquad \triangle \equiv n^a \nabla_a, \qquad \delta_i \equiv m^{(i)a} \nabla_a. \tag{3}$$

However, in this paper we are only interested in studying consequences of the Bianchi identities, which are given in detail in appendix B, and thus we do not introduce the Ricci identities except for equation (103) given in section 5.

2.1. Decomposition of the Riemann and Weyl tensors

In order to construct a basis in the space of 4-rank tensors with symmetries

$$R_{abcd} = \tfrac{1}{2}(R_{[ab][cd]} + R_{[cd][ab]}), \tag{4}$$

we introduce the operation $\{\,\}$

$$w_{\{a}x_b y_c z_{d\}} \equiv \tfrac{1}{2}(w_{[a}x_{b]}y_{[c}z_{d]} + w_{[c}x_{d]}y_{[a}z_{b]}). \tag{5}$$

When decomposing the Riemann tensor in terms of the frame vectors we also have to take into account that

$$R_{a\{bcd\}} = 0. \tag{6}$$

Now let us decompose the Riemann tensor into its frame components and sort them by their boost weights (see [1]):

$$\begin{aligned} R_{abcd} = {} & \overbrace{4R_{0i0j}n_{\{a}m_b^{(i)}n_c m_{d\}}^{(j)}}^{2} \\ & + \overbrace{\left.\begin{aligned} & 8R_{010i}n_{\{a}\ell_b n_c m_{d\}}^{(i)} + 4R_{0ijk}n_{\{a}m_b^{(i)}m_c^{(j)}m_{d\}}^{(k)} \\ & + 4R_{0101}n_{\{a}\ell_b n_c \ell_{d\}} + 4R_{01ij}n_{\{a}\ell_b m_c^{(i)}m_{d\}}^{(j)} \\ & + 8R_{0i1j}n_{\{a}m_b^{(i)}\ell_c m_{d\}}^{(j)} + R_{ijkl}m_{\{a}^{(i)}m_b^{(j)}m_c^{(k)}m_{d\}}^{(l)} \end{aligned}\right\}}^{1}{}^{0} \\ & + \overbrace{8R_{101i}\ell_{\{a}n_b\ell_c m_{d\}}^{(i)} + 4R_{1ijk}\ell_{\{a}m_b^{(i)}m_c^{(j)}m_{d\}}^{(k)}}^{-1} \\ & + \overbrace{4R_{1i1j}\ell_{\{a}m_b^{(i)}\ell_c m_{d\}}^{(j)}}^{-2}. \end{aligned}$$

The Riemann frame components in this relation are subject to constraints following from (4) and (6),

$$\begin{aligned} & R_{0[i|0|j]} = 0, \\ & R_{0i(jk)} = R_{0\{ijk\}} = 0, \\ & R_{ijkl} = R_{\{ijkl\}}, \qquad R_{i\{jkl\}} = 0, \qquad R_{01ij} = 2R_{0[i|1|j]}, \\ & R_{1i(jk)} = R_{1\{ijk\}} = 0, \\ & R_{1[i|1|j]} = 0. \end{aligned} \tag{7}$$

Let us check that we have an appropriate number of independent frame components. It is well known that an n-dimensional Riemann tensor has

$$\frac{n^2(n^2-1)}{12} \tag{8}$$

independent components. Counting independent frame components of various boost weights, we obtain

$$\overbrace{2\left(\frac{m(m+1)}{2}\right)}^{2,-2}+\overbrace{2\left(\frac{(m+1)m(m-1)}{3}+m\right)}^{1,-1}+\overbrace{\frac{m^2(m^2-1)}{12}+m^2+1}^{0},$$

where $m=n-2$. This is in agreement with (8).

Similarly, it is possible to decompose the Weyl tensor but due to its tracelessness we have the following additional conditions:

$$\begin{aligned} &C_{0i0i}=C_{1i1i}=0,\\ &C_{010i}=C_{0jij}, \qquad C_{101i}=C_{1jij},\\ &2C_{0i1j}=C_{01ij}-C_{ikjk}, \qquad C_{0101}=-\tfrac{1}{2}C_{ijij}. \end{aligned} \tag{9}$$

It is well known that an n-dimensional Weyl tensor has

$$\frac{(n+2)(n+1)n(n-3)}{12} \tag{10}$$

independent components. By counting the independent scalars of various boost weights, we obtain

$$\overbrace{2\left(\frac{(m+2)(m-1)}{2}\right)}^{2,-2}+\overbrace{2\left(\frac{(m+1)m(m-1)}{3}\right)}^{1,-1}+\overbrace{\frac{m^2(m^2-1)}{12}+\frac{m(m-1)}{2}}^{0},$$

which is in agreement with (10).

The primary algebraic classification of the Weyl tensor in higher dimensions [1] is based on whether all Weyl frame components of a boost weight higher than a specified number can be transformed away by an appropriate choice of the null direction ℓ. If it is indeed possible, we call the corresponding direction a Weyl aligned null direction (WAND) of an appropriate order (0, 1, 2, 3). If the set of WANDs of a given order is discrete, we call the corresponding directions principal null directions (PND).

Type III and N spacetimes admitting a WAND of orders 2 and 3, respectively, were introduced in [7]. We will use the notation based on that given in [7] which is suitable for these algebraic classes but becomes rather cumbersome in more general cases. Let us in accordance with [1, 7] state that a spacetime is of the algebraic type III if there exists a frame (2) in which the Weyl tensor has the form

$$C_{abcd}=8\Psi_i\ell_{\{a}n_b\ell_c m^{(i)}_{d\}}+8\Psi_{ijk}m^{(i)}_{\{a}m^{(j)}_b\ell_c m^{(k)}_{d\}}+8\Psi_{ij}\ell_{\{a}m^{(i)}_b\ell_c m^{(j)}_{d\}}, \tag{11}$$

with $\Psi_{ijk}\neq 0$. The case with $\Psi_{ijk}=0$ (and consequently from (12) also $\Psi_i=0$) is of the algebraic type N. Note, however, that in this paper the operation $\{\,\}$ differs by a factor $1/8$ from that given in [7].

The components of the Weyl tensor Ψ_i, Ψ_{ijk} and Ψ_{ij} are given by

$$\Psi_i=C_{101i}, \qquad \Psi_{ijk}=\tfrac{1}{2}C_{1kij}, \qquad \Psi_{ij}=\tfrac{1}{2}C_{1i1j}.$$

Note that Ψ_{ij} is symmetric and traceless. Ψ_{ijk} is antisymmetric in the first two indices and from (7) and (9) it also follows that

$$\Psi_i = 2\Psi_{ijj}, \tag{12}$$

$$\Psi_{\{ijk\}} = 0. \tag{13}$$

2.2. Decomposition of covariant derivatives of the frame vectors

Let us denote components of covariant derivatives of the frame vectors $\boldsymbol{\ell}$, $\boldsymbol{n}$, $\boldsymbol{m}^{(i)}$ by L_{ab}, N_{ab} and $\overset{i}{M}_{ab}$, respectively,

$$\ell_{a;b} = L_{cd}m_a^{(c)}m_b^{(d)}, \qquad n_{a;b} = N_{cd}m_a^{(c)}m_b^{(d)}, \qquad m^{(i)}_{a;b} = \overset{i}{M}_{cd}m_a^{(c)}m_b^{(d)}.$$

Since the norm of all frame vectors is constant, it follows that

$$L_{0a} = N_{1a} = \overset{i}{M}_{ia} = 0.$$

Also from the fact that all scalar products of the frame vectors are constant, we get

$$N_{0a} + L_{1a} = 0, \qquad \overset{i}{M}_{0a} + L_{ia} = 0, \qquad \overset{i}{M}_{1a} + N_{ia} = 0, \qquad \overset{i}{M}_{ja} + \overset{j}{M}_{ia} = 0. \tag{14}$$

We thus arrive at

$$\ell_{a;b} = L_{11}\ell_a\ell_b + L_{10}\ell_a n_b + L_{1i}\ell_a m_b^{(i)} + L_{i1}m_a^{(i)}\ell_b + L_{i0}m_a^{(i)}n_b + L_{ij}m_a^{(i)}m_b^{(j)}, \tag{15}$$

$$n_{a;b} = -L_{11}n_a\ell_b - L_{10}n_a n_b - L_{1i}n_a m_b^{(i)} + N_{i1}m_a^{(i)}\ell_b + N_{i0}m_a^{(i)}n_b + N_{ij}m_a^{(i)}m_b^{(j)}, \tag{16}$$

$$\begin{aligned} m^{(i)}_{a;b} = {} & -N_{i1}\ell_a\ell_b - N_{i0}\ell_a n_b - L_{i1}n_a\ell_b - L_{i0}n_a n_b - N_{ij}\ell_a m_b^{(j)} \\ & + \overset{i}{M}_{j1}m_a^{(j)}\ell_b - L_{ij}n_a m_b^{(j)} + \overset{i}{M}_{j0}m_a^{(j)}n_b + \overset{i}{M}_{kl}m_a^{(k)}m_b^{(l)}. \end{aligned} \tag{17}$$

2.3. Null geodesic congruences

In sections 3 and 4, we will show that the multiple PND in type III and N vacuum spacetimes is geodesic. Let us thus study properties of null geodesic congruences in higher dimensions. Analogous but in some cases not fully equivalent definitions are given in the appendix of the paper [10].

The congruence corresponding to $\boldsymbol{\ell}$ is geodesic if

$$\ell_{a;b}\ell^b \propto \ell_a \quad \Leftrightarrow \quad L_{i0} = 0.$$

It is always possible to rescale $\boldsymbol{\ell}$ (and consequently also $\boldsymbol{n}$) in such a way that $\ell_{a;b}\ell^b = 0$ and thus also $L_{10} = 0$. From now on we will use this parametrization. Then the covariant derivative of the vector $\boldsymbol{\ell}$ is

$$\ell_{a;b} = L_{11}\ell_a\ell_b + L_{1i}\ell_a m_b^{(i)} + L_{i1}m_a^{(i)}\ell_b + L_{ij}m_a^{(i)}m_b^{(j)}. \tag{18}$$

Let us decompose $\boldsymbol{L}$ into its symmetric and antisymmetric parts, $\boldsymbol{S}$ and $\boldsymbol{A}$,

$$L_{ij} = S_{ij} + A_{ij}, \qquad S_{ij} = S_{ji}, \qquad A_{ij} = -A_{ji},$$

where

$$S_{ij} = \ell_{(a;b)}m^{(i)a}m^{(j)b}, \qquad A_{ij} = \ell_{[a;b]}m^{(i)a}m^{(j)b}.$$

We define the expansion θ and the shear matrix σ_{ij} as follows:

$$\theta \equiv \frac{1}{n-2}\ell^a{}_{;a} = \frac{1}{n-2}[\boldsymbol{S}], \tag{19}$$

$$\sigma_{ij} \equiv \left(\ell_{(a;b)} - \theta\delta_{kl}m_a^{(k)}m_b^{(l)}\right)m^{(i)a}m^{(j)b} = S_{ij} - \frac{[\boldsymbol{S}]}{n-2}\delta_{ij}. \tag{20}$$

We will also denote $\sigma^2 \equiv \sigma_{ij}\sigma_{ij}$. Let us, for simplicity, call $\boldsymbol{A}$ the *twist matrix* and $\boldsymbol{S}$ the *expansion matrix*, though $\boldsymbol{S}$ contains information about both, expansion and shear. For simplicity we also introduce quantities

$$S \equiv \tfrac{1}{2}[\boldsymbol{S}], \qquad A^2 \equiv \tfrac{1}{2}A_{ij}A_{ij}. \tag{21}$$

3. Type N vacuum spacetimes

For type N spacetimes the Weyl tensor (11) has the form

$$C_{abcd} = 8\Psi_{ij}\ell_{\{a}m_b^{(i)}\ell_c m_{d\}}^{(j)}, \tag{22}$$

where $\boldsymbol{\Psi}$ is symmetric and traceless. Bianchi equations, given in appendix B, now reduce to

$$D\Psi_{ij} = -2\Psi_{k(i}\overset{k}{M}_{j)0} - \Psi_{ik}L_{kj} - 2\Psi_{ij}L_{10}, \tag{23}$$

$$\delta_{[k}\Psi_{j]i} = -\Psi_{is}\overset{s}{M}_{[jk]} + \overset{s}{M}_{i[j}\Psi_{k]s} + \Psi_{i[j}L_{k]1} + 2L_{1[j}\Psi_{k]i}, \tag{24}$$

$$0 = L_{k[i}\Psi_{j]k}, \tag{25}$$

$$0 = \Psi_{i[j}L_{k]0}, \tag{26}$$

$$0 = L_{k[j}\Psi_{m]i} + L_{i[m}\Psi_{j]k}, \tag{27}$$

$$0 = \Psi_{i\{k}A_{jm\}}. \tag{28}$$

Let us first show that from equation (26) it follows that the multiple PND $\boldsymbol{\ell}$ in type N vacuum spacetimes is geodesic. For simplicity we will denote L_{i0} as L_i.

The contraction of i with k in (26) leads to

$$\Psi_{ij}L_i = 0.$$

Now the contraction of (26) with Ψ_{ik} gives

$$\Psi_{ik}\Psi_{ik}L_j = 0.$$

Note that in type N spacetimes $\Psi_{ik}\Psi_{ik} > 0$ and thus the previous equation implies $L_i = 0$ and $\boldsymbol{\ell}$ is indeed geodesic.

By substituting $\boldsymbol{L} = \boldsymbol{A} + \boldsymbol{S}$ into (27) and using (28), we obtain

$$S_{k[j}\Psi_{m]i} + S_{i[m}\Psi_{j]k} = 0. \tag{29}$$

Let us now study in detail consequences of equations (25), (28) and (29). It is possible to find a general solution of these equations by taking into account their various contractions (hereafter we assume $\boldsymbol{S} \neq 0$ and $\boldsymbol{A} \neq 0$; the non-twisting case can be obtained as a special case with $\boldsymbol{A} = 0$).

Contracting i with j in (28) gives

$$\Psi_{ik}A_{im} = \Psi_{im}A_{ik} \quad \Rightarrow \quad \boldsymbol{\Psi}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{\Psi} = 0 \tag{30}$$

and contracting k with j in (29) leads to

$$\Psi_{ij}S_{jm} + \Psi_{jm}S_{ij} = 2S\Psi_{im} \quad \Rightarrow \quad \boldsymbol{\Psi}\cdot\boldsymbol{S} + \boldsymbol{S}\cdot\boldsymbol{\Psi} = 2S\boldsymbol{\Psi}. \tag{31}$$

The previous two equations imply

$$\boldsymbol{\Psi}\cdot\boldsymbol{L} + \boldsymbol{L}\cdot\boldsymbol{\Psi} = 2S\boldsymbol{\Psi}.$$

By contracting i with m in (31) we get

$$\Psi_{ij}S_{ij} = 0 \quad \Rightarrow \quad \Psi_{ij}L_{ij} = 0. \tag{32}$$

From (25) and (30) it follows that

$$\Psi_{ki}S_{kj} = \Psi_{kj}S_{ki} \quad \Rightarrow \quad \boldsymbol{\Psi}\cdot\boldsymbol{S} - \boldsymbol{S}\cdot\boldsymbol{\Psi} = 0. \tag{33}$$

Let us denote a trace $[\boldsymbol{\Psi}\cdot\boldsymbol{\Psi}]$ as p. Now we are in a position to formulate the following lemma.

Lemma 1. *In vacuum type N spacetimes with $\boldsymbol{L} \neq 0$ the following relations are satisfied:*

$$\text{(a)} \quad \boldsymbol{\Psi}\cdot\boldsymbol{S} = S\boldsymbol{\Psi}, \tag{34}$$

$$\text{(b)} \quad \boldsymbol{S}\cdot\boldsymbol{S} = S\boldsymbol{S}, \tag{35}$$

$$\text{(c)} \quad \boldsymbol{A}\cdot\boldsymbol{S} = \boldsymbol{S}\cdot\boldsymbol{A} = S\boldsymbol{A}, \tag{36}$$

$$\text{(d)} \quad S\boldsymbol{A}\cdot\boldsymbol{A} = -A^2\boldsymbol{S}, \tag{37}$$

$$\text{(e)} \quad 2S\boldsymbol{\Psi}\cdot\boldsymbol{\Psi} = p\boldsymbol{S}, \tag{38}$$

$$\text{(f)} \quad S\boldsymbol{L}\cdot\boldsymbol{L}^T = S\boldsymbol{L}^T\cdot\boldsymbol{L} = (S^2 + A^2)\boldsymbol{S}. \tag{39}$$

Proof.

(a) This equation is a direct consequence of (31) and (33).

(b) By multiplying (29) by S_{ip} and using lemma 1(a) we get

$$\Psi_{kj}S_{im}S_{ip} + S\Psi_{mp}S_{kj} = \Psi_{km}S_{ij}S_{ip} + S\Psi_{jp}S_{km}.$$

An appropriate linear combination of this equation with (29) leads to

$$\Psi_{kj}(S_{im}S_{ip} - SS_{mp}) = \Psi_{km}(S_{ij}S_{ip} - SS_{pj}),$$

which, denoting $X_{mp} \equiv S_{im}S_{ip} - SS_{mp}$, takes the form

$$\Psi_{kj}X_{mp} = \Psi_{km}X_{jp}. \tag{40}$$

Note that the matrix $\boldsymbol{X}$ is symmetric. Now contracting k with j gives

$$\Psi_{jm}X_{jp} = 0. \tag{41}$$

Multiplying (40) with X_{mr} and using (41) leads to $X_{mp}X_{mr} = 0$, which, contracting p with r, gives $X_{mp}X_{mp} = 0$ and consequently $X_{mp} = 0$.

(c) Multiplying (28) with S_{kp} and using lemma 1(a) leads to

$$\psi_{ij}A_{km}S_{kp} + S\psi_{ip}A_{mj} + \psi_{im}A_{jk}S_{kp} = 0. \tag{42}$$

Now contracting i with m and using (30) gives

$$\psi_{ij}Y_{ip} = 0, \tag{43}$$

where $Y_{ip} \equiv A_{ki}S_{kp} + SA_{ip}$. Now substituting $A_{ki}S_{kp} = Y_{ip} - SA_{ip}$ into (42) and using (28) leads to

$$\psi_{ij}Y_{mp} = \psi_{im}Y_{jp}. \tag{44}$$

Multiplying (44) with Y_{mr} using (43) gives $Y_{mp}Y_{mr} = 0$ and consequently $Y_{mp} = 0$.

(d) Let us define another symmetric matrix $\boldsymbol{B}$ by $B_{ik} \equiv A_{ij}A_{jk}$. Note that $[\boldsymbol{B}] = -2A^2$. Multiplying (30) by A_{mr} and again applying (30) leads to

$$\psi_{ik}B_{ir} = \psi_{ir}B_{ik}. \tag{45}$$

Multiplying (28) by A_{mr} leads to

$$\psi_{ij}B_{kr} - \psi_{ik}B_{jr} + \psi_{im}A_{jk}A_{mr} = 0, \tag{46}$$

which after contracting k with r gives

$$\psi_{ik}B_{jk} = -A^2\Psi_{ij}. \tag{47}$$

Multiplying (46) with B_{kr} and using (45), (47) and (30) leads to $B_{kr}B_{kr} = 2A^4$ and similarly multiplying (46) by S_{kr} results in $B_{kr}S_{kr} = -2SA^2$. Let us now define a symmetric matrix $\boldsymbol{Q}$ by

$$Q_{ij} \equiv SB_{ij} + A^2S_{ij}.$$

Using previous formulae it turns out that $Q_{ij}Q_{ij} = 0$ and thus $Q_{ij} = 0$.

(e) This follows from multiplying (29) by Ψ_{kj}.

(f) This follows directly from lemma 1(b)–(d). □

Let us now use lemma 1 to prove the following lemmas for vacuum type N spacetimes with $\boldsymbol{L} \neq 0$.

Lemma 2. *The matrix $\boldsymbol{S}$ has at most two eigenvalues $\lambda = 0$ and $\lambda = S$.*

Proof. Let us denote the eigenvector of $\boldsymbol{S}$ by ξ. We thus have $S_{ij}\xi_j = \lambda\xi_i$. By multiplying lemma 1(b) by ξ we obtain

$$S_{ij}S_{jk}\xi_k = SS_{ik}\xi_k \quad \Rightarrow \quad \lambda^2 = S\lambda$$

and thus $\lambda = 0$ or $\lambda = S$. □

Lemma 3. *The following statements are equivalent:*

(a) Vector ξ is an eigenvector of $\boldsymbol{S}$ for the eigenvalue $\lambda = 0$.
(b) Vector ξ is an eigenvector of $\boldsymbol{A}$ for the eigenvalue $\lambda = 0$.
(c) Vector ξ is an eigenvector of $\boldsymbol{\Psi}$ for the eigenvalue $\lambda = 0$.

Proof.

(a) $\Rightarrow$ (b): Suppose that $S_{ij}\xi_j = 0$. Then by multiplying lemma 1(c) in the form $S_{ij}A_{jk} = SA_{ik}$ by ξ_i we obtain $A_{ik}\xi_i = 0$.
(b) $\Rightarrow$ (a): Suppose that $A_{ij}\xi_j = 0$. Then by multiplying lemma 1(d) in the form $SA_{ij}A_{jk} = -A^2S_{ik}$ by ξ_k we obtain $S_{ik}\xi_k = 0$.
(a) $\Rightarrow$ (c): Suppose that $S_{ij}\xi_j = 0$. Then by multiplying lemma 1(a) in the form $\Psi_{ij}S_{jk} = S\Psi_{ik}$ by ξ_k we obtain $\Psi_{ik}\xi_k = 0$.
(c) $\Rightarrow$ (a): Suppose that $\Psi_{ij}\xi_j = 0$. Then by multiplying lemma 1(e) in the form $2S\Psi_{ij}\Psi_{jk} = pS_{ik}$ by ξ_k we obtain $S_{ik}\xi_k = 0$. □

Lemma 4. *Every eigenvector of $\boldsymbol{\Psi}$ is also an eigenvector of $\boldsymbol{S}$.*

Proof. The case $\lambda = 0$ is solved in lemma 3. Let us now suppose that $\Psi_{ij}\xi_j = \lambda\xi_i, \lambda \neq 0$. Then by multiplying lemma 1(a) in the form $\Psi_{ij}S_{jk} = S\Psi_{ik}$ by ξ_i we obtain $S_{jk}\xi_j = S\xi_k$. □

Lemma 5. *The only possible eigenvalues of $\boldsymbol{\Psi}$ are $\lambda = 0$ and $\lambda = \pm\sqrt{p/2}$.*

Proof. Let us now suppose that $\Psi_{ij}\xi_j = \lambda\xi_i$, $\lambda \neq 0$. Then thanks to lemma 4 $S_{ij}\xi_j = S\xi_i$. By multiplying lemma 1(e) in the form $2S\Psi_{ij}\Psi_{jk} = pS_{ik}$ by ξ_k we obtain $\lambda^2 = p/2$. □

For every symmetric matrix $\boldsymbol{W}$ there exists an orthonormal basis of its eigenvectors ξ_i^a in which

$$W_{ij} = \sum_a \lambda_a \xi_i^a \xi_j^a.$$

Let us denote eigenvectors of $\boldsymbol{\Psi}$ corresponding to $\lambda = 0, \lambda = \sqrt{p/2}$ and $\lambda = -\sqrt{p/2}$ by $\boldsymbol{u}^\alpha$, $\boldsymbol{v}^A$ and $\boldsymbol{w}^{\tilde{A}}$, respectively. Here indices $\alpha, \beta, \ldots, A, B, \ldots,$ and $\tilde{A}, \tilde{B}, \ldots$ distinguish between vectors $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^{\alpha_{\max}}$, $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^{A_{\max}}$ and $\boldsymbol{w}^1, \ldots, \boldsymbol{w}^{\tilde{A}_{\max}}$. Now we have

$$\Psi_{ij} = \sqrt{\frac{p}{2}}\left(v_i^A v_j^A - w_i^{\tilde{A}} w_j^{\tilde{A}}\right). \tag{48}$$

Note that for indices $\alpha, \beta, \ldots, A, B, \ldots$ and $\tilde{A}, \tilde{B}, \ldots$ we also observe Einstein's summation convention. Thanks to lemmas 3 and 4

$$S_{ij} = S\left(v_i^A v_j^A + w_i^{\tilde{A}} w_j^{\tilde{A}}\right). \tag{49}$$

Let us now multiply (29) by $v_k^A v_j^B$. This leads to

$$\delta^{AB}\left(\sqrt{\frac{p}{2}}S_{il} + S\Psi_{il}\right) = S\sqrt{2p}\, v_l^A v_i^B$$

which for $A \neq B$ gives $v_l^A v_i^B = 0$ and thus just one value of index A is possible and consequently $A_{\max} = 1$. Let us thus denote $\boldsymbol{v}^1$ by $\boldsymbol{v}$. Similarly, we can find that $\tilde{A}_{\max} = 1$ and that there is thus just one vector $\boldsymbol{w} = \boldsymbol{w}^1$. Equations (48) and (49) now take the form

$$\Psi_{ij} = \sqrt{\frac{p}{2}}(v_i v_j - w_i w_j), \qquad S_{ij} = S(v_i v_j + w_i w_j). \tag{50}$$

Let us now introduce a new vector $\boldsymbol{V}$ by

$$V_i = A_{ij} v_j.$$

Thanks to lemma 1(c) and (d) we have

$$S_{ij}V_i = S_{ij}A_{ik}v_k = SV_j, \qquad V_i v_i = 0, \qquad V_i V_i = A^2$$

and thus $\boldsymbol{V}$ is just a multiple of $\boldsymbol{w}$ $\left(w_i = \pm\frac{1}{A}V_i\right)$ and we have freedom to set $w_i = \frac{1}{A}V_i$. Then we have

$$A_{ij}v_j = Aw_i, \qquad A_{ij}w_j = -Av_i.$$

Now by multiplying equation (28) by $v_i v_j$ we get

$$A_{kl} = A(w_k v_l - v_k w_l). \tag{51}$$

Note that $\boldsymbol{\Psi}$, $\boldsymbol{S}$ and $\boldsymbol{A}$ given by (50) and (51) satisfy equations (28), (29) and thus represent the general solution of these equations for $\boldsymbol{L} \neq 0$.

4. Type III vacuum spacetimes

For type III vacuum spacetimes the Weyl tensor is given by (11), where Ψ_{ijk} satisfies (12) and (13). The algebraic equations

$$\Psi_{ijk}L_k = L_{[i}\Psi_{j]}, \tag{52}$$

$$\Psi_{kl[j}L_{i]} = \Psi_{ij[k}L_{l]}, \tag{53}$$

$$L_{i[j}\Psi_{k]} + 2L_{s[j|}\Psi_{si|k]} = 0, \tag{54}$$

$$2A_{\{ij|}\Psi_{kl|m\}} + L_{l\{i}\Psi_{jm\}k} - L_{k\{i}\Psi_{jm\}l} = 0, \tag{55}$$

where we denote L_{i0} as L_i, follow from the Bianchi equations given in appendix B.

Let us show that from (52) and (53) it follows that the multiple PND ℓ in vacuum type III spacetimes is geodesic. By contracting l with j in (53) and using (52), (12) and (13) we obtain

$$2\Psi_{ijk}L_j = \Psi_k L_i. \tag{56}$$

One can see that for the class with $\Psi_k \neq 0$ it implies (as we can see by contracting this formula with L_i) that $L_i = 0$. Similarly, for the case $\Psi_k = 0$ these equations also imply that $L_i = 0$ (contract (53) with L_i and employ (52) and (56)). Thus, the multiple PND ℓ in vacuum type III spacetimes is geodesic.

Now the task is to solve (54) and (55). This turns out to be quite complicated. We will be able to find a general solution in an arbitrary dimension in the non-twisting case with $A_{ij} = 0$ (see appendix C.1). We also present a solution for the twisting case, but we can prove that it is a general solution only if we assume that Ψ_{ijk} is in a 'general form'. In appendix C.2, we show that this solution is indeed general in all degenerate cases in five dimensions.

Let us start with extracting some information from (54) and (55) by various contractions. By contracting (55) and using (54) we get

$$L\Psi_{ijk} + 2L_{[i|s}\Psi_{sk|j]} - 2S_{sk}\Psi_{ijs} + L_{[i|k}\Psi_{|j]} = 0, \tag{57}$$

where $L = 2S$ is the trace of $\boldsymbol{L}$. After adding (54) (where we replace indices i, j, k by k, i, j, respectively) to (57) we obtain

$$L\Psi_{ijk} + 4S_{[i|s}\Psi_{sk|j]} - 2S_{sk}\Psi_{ijs} + 2S_{[i|k}\Psi_{|j]} = 0, \tag{58}$$

which does not contain $\boldsymbol{A}$—the antisymmetric part of $\boldsymbol{L}$, and similarly

$$L\Psi_{ijk} - 4\Psi_{sk[i}A_{j]s} - 2\Psi_{[i}A_{j]k} - 2\Psi_{ijs}S_{sk} = 0. \tag{59}$$

Contraction of i with j in (54) leads to

$$L\Psi_k - 2L_{ik}\Psi_i + 2L_{si}\Psi_{sik} = 0 \tag{60}$$

and contraction of k with j in (58) (using (12) and (13)) to

$$L_{ij}\Psi_j = 2L_{js}\Psi_{ijs} \tag{61}$$

and (60), using (13), with (61) imply

$$L\Psi_i - L_{ij}\Psi_j - 2L_{ji}\Psi_j + 2L_{kj}\Psi_{ijk} = 0. \tag{62}$$

By substituting $\boldsymbol{L} = \boldsymbol{S} + \boldsymbol{A}$ into (60) and (62) we get

$$L\Psi_i - 2S_{ij}\Psi_j - 2A_{ji}\Psi_j - 4A_{jk}\Psi_{ijk} = 0, \tag{63}$$

$$L\Psi_i - 3S_{ij}\Psi_j - A_{ji}\Psi_j + 2(S_{jk} - A_{jk})\Psi_{ijk} = 0 \tag{64}$$

and their linear combinations give

$$L\Psi_i + 4A_{ij}\Psi_j - 4(S_{jk} + 2A_{jk})\Psi_{ijk} = 0, \tag{65}$$

$$L\Psi_i - 4S_{ij}\Psi_j + 4S_{jk}\Psi_{ijk} = 0. \tag{66}$$

By multiplying (54) by L_{ij} we arrive at

$$L_{ij}L_{ik}\Psi_j = \tfrac{1}{2}\ell\Psi_k, \tag{67}$$

where $\ell = L_{ij}L_{ij}$.

Inspired by the type N case, we again choose an orthonormal basis of eigenvectors of $\boldsymbol{S}$. We will denote vectors corresponding to non-zero eigenvalues of $\boldsymbol{S}$ as $\boldsymbol{v}^1, \boldsymbol{v}^2, \ldots, \boldsymbol{v}^{A_{\max}}$ and vectors corresponding to the eigenvalue 0 as $\boldsymbol{u}^1, \boldsymbol{u}^2, \ldots, \boldsymbol{u}^{\alpha_{\max}}$. We thus have

$$S_{ij} = \sum_{A=1}^{A_{\max}} \lambda_A v_i^A v_j^A. \tag{68}$$

Unfortunately, there are three indices A in this formula and we thus will not use Einstein's summation convention for indices $A, B, \ldots$ (though we will still use it for indices $\alpha, \beta, \ldots$).

Let us now decompose other quantities appearing in (54) and (55)

$$\begin{aligned}
\Psi_i &= \sum_A a_A v_i^A + b_\alpha u_i^\alpha,\\
A_{ij} &= \sum_{AB} \mathcal{A}_{AB}\big(v_i^A v_j^B - v_j^A v_i^B\big) + \sum_A \mathcal{B}_{A\beta}\big(v_i^A u_j^\beta - v_j^A u_i^\beta\big) + \mathcal{C}_{\alpha\beta}\big(u_i^\alpha u_j^\beta - u_j^\alpha u_i^\beta\big),\\
\Psi_{ijk} &= \sum_{ABC} \mathcal{M}_{ABC}\big(v_i^A v_j^B - v_j^A v_i^B\big)v_k^C + \sum_{AB} \mathcal{N}_{AB\gamma}\big(v_i^A v_j^B - v_j^A v_i^B\big)u_k^\gamma\\
&\quad + \sum_{AC} \mathcal{O}_{A\beta C}\big(v_i^A u_j^\beta - v_j^A u_i^\beta\big)v_k^C + \sum_A \mathcal{P}_{A\beta\gamma}\big(v_i^A u_j^\beta - v_j^A u_i^\beta\big)u_k^\gamma\\
&\quad + \sum_C \mathcal{R}_{\alpha\beta C}\big(u_i^\alpha u_j^\beta - u_j^\alpha u_i^\beta\big)v_k^C + \mathcal{S}_{\alpha\beta\gamma}\big(u_i^\alpha u_j^\beta - u_j^\alpha u_i^\beta\big)u_k^\gamma,
\end{aligned} \tag{69}$$

where $\mathcal{A}, \mathcal{C}, \mathcal{M}, \mathcal{N}, \mathcal{R}$ and $\mathcal{S}$ are antisymmetric in the first two indices. Now we need to rewrite some of the equations from this section in terms of these new quantities (more complicated equations are given in appendix A).

From (12) and (66), it follows that

$$a_A = 4\sum_B \mathcal{M}_{ABB} + 2\mathcal{P}_{A\beta\beta}, \tag{70}$$

$$b_\alpha = -2\sum_B \mathcal{O}_{B\alpha B} + 4\mathcal{S}_{\alpha\beta\beta}, \tag{71}$$

and

$$(L - 4\lambda_A)a_A = -8\sum_B \lambda_B \mathcal{M}_{ABB}, \tag{72}$$

$$Lb_\alpha = 4\sum_B \lambda_B \mathcal{O}_{B\alpha B}, \tag{73}$$

respectively. Equation (58) implies

$$L\mathcal{S}_{\alpha\beta\gamma} = 0, \tag{74}$$

$$(L - 2\lambda_C)\mathcal{R}_{\alpha\beta C} = 0, \tag{75}$$

$$L\mathcal{P}_{B\alpha\gamma} + 2\lambda_B \mathcal{P}_{B\gamma\alpha} = 0, \tag{76}$$

$$(L - 2\lambda_C)\mathcal{O}_{A\beta C} + 4\lambda_A \mathcal{N}_{AC\beta} + \lambda_A b_\beta \delta_{AC} = 0, \tag{77}$$

$$2\lambda_A \mathcal{O}_{A\gamma B} - 2\lambda_B \mathcal{O}_{B\gamma A} + 2L\mathcal{N}_{AB\gamma} = 0, \tag{78}$$

$$(L - 2\lambda_C)\mathcal{M}_{ABC} + 2\lambda_A \mathcal{M}_{ACB} + 2\lambda_B \mathcal{M}_{CBA} + \lambda_C \delta_{C[A} a_{B]} = 0, \tag{79}$$

and (13) leads to

$$\mathcal{P}_{A[\beta\gamma]} + \mathcal{R}_{\beta\gamma A} = 0, \tag{80}$$

$$\mathcal{O}_{[A|\beta|C]} + \mathcal{N}_{CA\beta} = 0, \tag{81}$$

$$\mathcal{M}_{\{ABC\}} = 0, \tag{82}$$

$$\mathcal{S}_{\{ABC\}} = 0. \tag{83}$$

In the twisting case, equations (54) and (55), without specifying dimension, are quite complex, and there are too many distinct cases to be solved. We thus present the solution provided $L \neq 0$ and provided that for every pair A, C $(A \neq C)$ there exists β for which $\mathcal{O}_{A\beta C} \neq 0$. It turns out that the only eigenvalues of $\boldsymbol{S}$ that are compatible with (54) and (55) correspond to $A_{\max} = 2, \lambda_1 = L/2, \lambda_2 = L/2$. We also checked other cases (but not all of them) and they also lead to the same conclusion. For completeness, we treat all other possible cases in five dimensions in appendix C.2.

A linear combination of (78) and (81), [(78) $-$ L (81)], gives

$$(2\lambda_A + L)\mathcal{O}_{A\gamma B} = (2\lambda_B + L)\mathcal{O}_{B\gamma A}. \tag{84}$$

For $A \neq B$ [(77) $-$ $2\lambda_A$ (81)] leads to

$$(L + 2\lambda_A - 2\lambda_B)\mathcal{O}_{A\gamma B} = 2\lambda_A \mathcal{O}_{B\gamma A}|_{A\neq B}, \tag{85}$$

and [$2\lambda_A$ (84) $-$ $(2\lambda_B + L)$ (85)] gives

$$\left(\lambda_A^2 + \lambda_B^2 - \lambda_A\lambda_B - \frac{L^2}{4}\right)\mathcal{O}_{A\gamma B} = 0|_{A\neq B}. \tag{86}$$

If $\mathcal{O}_{A\gamma B} \neq 0$, we obtain from (86)

$$\left(\lambda_A^2 + \lambda_B^2 - \lambda_A\lambda_B - \frac{L^2}{4}\right) = 0|_{A\neq B}. \tag{87}$$

If for each pair A, B there exists γ such that $\mathcal{O}_{A\gamma B} \neq 0$ then (87) is valid for all A, B, $A \neq B$. In the case $A_{\max} = 2$, we have $L = \lambda_1 + \lambda_2$ and equation (86) leads to $(\lambda_1 - \lambda_2)^2 = 0$ and consequently $\lambda_1 = \lambda_2 = \frac{L}{2}$.

In the case $A_{\max} > 2$, by subtracting

$$\left(\lambda_A^2 + \lambda_C^2 - \lambda_A\lambda_C - \frac{L^2}{4}\right) = 0|_{A\neq C} \tag{88}$$

from (87) we obtain

$$(\lambda_B - \lambda_C)(\lambda_B + \lambda_C - \lambda_A) = 0|_{A\neq B, A\neq C, B\neq C}$$

and thus

$$\lambda_B - \lambda_C = 0 \tag{89}$$

or

$$\lambda_B + \lambda_C - \lambda_A|_{A\neq B, A\neq C, B\neq C} = 0.$$

However, in the second case one can show that simply by summing this equation with interchanged indices

$$(\lambda_B + \lambda_C - \lambda_A = 0) + (\lambda_B + \lambda_A - \lambda_C = 0) \Rightarrow \lambda_A = 0$$

which cannot happen as $\lambda_A \neq 0$ by definition and thus (89) is satisfied and by substituting $\lambda_A = \lambda_B = \lambda$ in (87) we obtain $\lambda = \pm\frac{L}{2}$. Note however that $\sum_A \lambda_A = L$ and thus the case $\lambda_A = -L/2$ is excluded and $A_{\max} = 2$. Thus we can conclude with

Lemma 6. *For vacuum type III spacetimes with $L \neq 0$, providing that for every pair A, C ($A \neq C$) there exists β for which $\mathcal{O}_{A\beta C} \neq 0$, the expansion matrix $\boldsymbol{S}$ has just one non-vanishing eigenvalue $S = \frac{L}{2}$ which has multiplicity 2.*

Let us now study the case with two non-vanishing eigenvalues of $\boldsymbol{S}$, $\lambda_1 = \lambda_2 = L/2$, in general, i.e. without any assumptions about $\mathcal{O}_{A\beta C}$.

4.1. The case $A_{\max} = 2$, $\lambda_1 = \lambda_2 = L/2$ and $\Psi_i \neq 0$

From (74) and (76) it follows that

$$\mathcal{S}_{\alpha\beta\gamma} = 0, \qquad \mathcal{P}_{A\beta\gamma} = -\mathcal{P}_{A\gamma\beta}. \tag{90}$$

By assuming that the index $A \neq C$ in (77) we obtain

$$\mathcal{N}_{AC\beta} = 0.$$

Since $\mathcal{N}$ is antisymmetric in the first two indices, $\mathcal{N}_{AC\beta} = 0$ for all combinations of indices. By substituting $A = C$ in (77) we obtain

$$b_\beta = 0$$

and thus Ψ_i is an eigenvector of $\boldsymbol{S}$ corresponding to $L/2$. Equation (77) now implies

$$\mathcal{O}_{A\beta C} = \mathcal{O}_{C\beta A}$$

and (71) implies

$$\sum_A \mathcal{O}_{A\beta A} = 0. \tag{91}$$

Substituting $A \neq C$, $B \neq C$ into (79) leads to

$$\mathcal{M}_{ABC} = 0|_{A\neq C, B\neq C},$$

(70) leads to

$$a_A = 4\sum_B \mathcal{M}_{ABB} \quad \Rightarrow \quad a_1 = 4\mathcal{M}_{122}, a_2 = 4\mathcal{M}_{211} \tag{92}$$

and (80) gives

$$\mathcal{P}_{A\beta\gamma} + \mathcal{R}_{\beta\gamma A} = 0. \tag{93}$$

Substituting the index $B = A$ into (A.5), summing over the index A and using (A.1), (92), (91) and (90) we arrive at $\sum_A \mathcal{B}_{A\beta} a_A = 0$. Consequently, the vector Φ

$$\Phi_i = A_{ij}\Psi_j = \sum_B 2\mathcal{A}_{AB} a_B v_i^A,$$

which is orthogonal to Ψ_i, is also an eigenvector of $\boldsymbol{S}$. We thus have two orthogonal eigenvectors of $\boldsymbol{S}$, Ψ_i and Φ_i. Let us denote $\Psi_i\Psi_i$ by ψ^2 and $\Phi_i\Phi_i$ by ϕ^2. Now we choose an orthogonal basis with these two vectors which corresponds to

$$v_i^1, v_i^2 \longmapsto \frac{\Psi_i}{\psi}, \frac{\Phi_i}{\phi}.$$

In this basis we will denote components corresponding to Ψ_i and Φ_i by indices P and F, respectively. $\boldsymbol{S}$ now takes the form

$$S_{ij} = S\left(\frac{\Psi_i\Psi_j}{\psi^2} + \frac{\Phi_i\Phi_j}{\phi^2}\right). \tag{94}$$

This equation implies $S_{ij}S_{ij} = 2S^2$. Now using $\ell = 2(A^2 + S^2)$ it follows from the previous equations and (67) that

$$A_{ik}\Phi_i = A^2\Psi_k$$

and thus

$$\phi^2 = \Phi_i\Phi_i = \Phi_i A_{ik}\Psi_k = A^2\Psi_k\Psi_k = A^2\psi^2.$$

A_{ij} as an antisymmetric matrix takes the general form

$$\begin{aligned} A_{ij} = {} & \mathcal{A}_{PF}(\Psi_i\Phi_j - \Psi_j\Phi_i) + \mathcal{B}_{P\alpha}\big(\Psi_i u_j^\alpha - \Psi_j u_i^\alpha\big) \\ & + \mathcal{B}_{F\alpha}\big(\Phi_i u_j^\alpha - \Phi_j u_i^\alpha\big) + \mathcal{C}_{\alpha\beta}\big(u_i^\alpha u_j^\beta - u_j^\alpha u_i^\beta\big) \end{aligned}$$

but from $A_{ij}\Psi_j = \Phi_i$ and $A_{ik}\Phi_i = A^2\Psi_k$ we get

$$\mathcal{B}_{P\alpha} = 0, \qquad \mathcal{A}_{PF} = -\frac{1}{\psi^2}, \qquad \mathcal{B}_{F\alpha} = 0$$

and thus

$$A_{ij} = \frac{1}{\psi^2}(\Phi_i\Psi_j - \Phi_j\Psi_i) + \mathcal{C}_{\alpha\beta}\big(u_i^\alpha u_j^\beta - u_j^\alpha u_i^\beta\big).$$

Thanks to (90)–(93) we can rewrite Ψ_{ijk} in the form

$$\begin{aligned} \Psi_{ijk} = {} & \frac{1}{2\phi^2}(\Psi_i\Phi_j - \Psi_j\Phi_i)\Phi_k + \frac{\mathcal{O}_{P\alpha P}}{\psi^2}\big(\Psi_i u_j^\alpha - \Psi_j u_i^\alpha\big)\Psi_k - \frac{\mathcal{O}_{P\alpha P}}{\phi^2}\big(\Phi_i u_j^\alpha - \Phi_j u_i^\alpha\big)\Phi_k \\ & + \frac{\mathcal{O}_{P\alpha F}}{\psi\phi}\big(\Psi_i u_j^\alpha - \Psi_j u_i^\alpha\big)\Phi_k + \frac{\mathcal{O}_{P\alpha F}}{\psi\phi}\big(\Phi_i u_j^\alpha - \Phi_j u_i^\alpha\big)\Psi_k \\ & + \frac{\mathcal{P}_{P\beta\gamma}}{\psi}\big(\Psi_i u_j^\beta - \Psi_j u_i^\beta\big)u_k^\gamma + \frac{\mathcal{P}_{F\beta\gamma}}{\phi}\big(\Phi_i u_j^\beta - \Phi_j u_i^\beta\big)u_k^\gamma \\ & - \frac{\mathcal{P}_{P\beta\gamma}}{\psi}\big(u_i^\beta u_j^\gamma - u_j^\beta u_i^\gamma\big)\Psi_k - \frac{\mathcal{P}_{F\beta\gamma}}{\phi}\big(u_i^\beta u_j^\gamma - u_j^\beta u_i^\gamma\big)\Phi_k. \end{aligned} \tag{95}$$

By substituting (95) into (A.15) we obtain $\mathcal{A}_{PF}\mathcal{P}_{P\gamma\delta} = 0$, $\mathcal{A}_{PF}\mathcal{P}_{F\gamma\delta} = 0$ and since $\mathcal{A}_{PF} \neq 0$ we also get $\mathcal{P}_{P\gamma\delta} = 0 = \mathcal{P}_{F\gamma\delta}$. Then (A.14) or (A.13) gives $\mathcal{C}_{\alpha\beta}\mathcal{M}_{CDE} = 0$ which, since $\mathcal{M}_{CDE}$ does not vanish, implies $\mathcal{C}_{\alpha\beta} = 0$. In this case we thus have

$$A_{ij} = \frac{1}{\psi^2}(\Phi_i\Psi_j - \Phi_j\Psi_i), \tag{96}$$

and

$$\begin{aligned} \Psi_{ijk} = {} & \frac{1}{2\phi^2}(\Psi_i\Phi_j - \Psi_j\Phi_i)\Phi_k + \frac{\mathcal{O}_{P\alpha P}}{\psi^2}\big(\Psi_i u_j^\alpha - \Psi_j u_i^\alpha\big)\Psi_k - \frac{\mathcal{O}_{P\alpha P}}{\phi^2}\big(\Phi_i u_j^\alpha - \Phi_j u_i^\alpha\big)\Phi_k \\ & + \frac{\mathcal{O}_{P\alpha F}}{\psi\phi}\big(\Psi_i u_j^\alpha - \Psi_j u_i^\alpha\big)\Phi_k + \frac{\mathcal{O}_{P\alpha F}}{\psi\phi}\big(\Phi_i u_j^\alpha - \Phi_j u_i^\alpha\big)\Psi_k. \end{aligned} \tag{97}$$

We do not need to examine the rest of equations in appendix A since S_{ij}, A_{ij} and Ψ_{ijk} given by (94), (96) and (97), respectively, already satisfy both equations (54) and (55) and thus represent their solution.

Note that from (94), (96) and (97) we obtain for type III spacetimes relations equal or analogous to equations given in lemma 1:

$$\text{(a)}\quad \Psi_{ijs}S_{sk} = S\Psi_{ijk}, \tag{98}$$

$$\text{(b)}\quad S_{ik}S_{jk} = SS_{ij}, \tag{99}$$

$$\text{(c)}\quad A_{ik}S_{kj} = S_{ik}A_{kj} = SA_{ij}, \tag{100}$$

$$\text{(d)}\quad SA_{ik}A_{kj} = -A^2S_{ij}, \tag{101}$$

$$\text{(e)}\quad SL_{ik}L_{jk} = SL_{ki}L_{kj} = (S^2 + A^2)S_{ij}. \tag{102}$$

5. Comments on a possible generalization of the Goldberg–Sachs theorem for higher dimensions

In sections 3 and 4, it is shown that for type N and III vacuum spacetimes the expansion and twist matrices $\boldsymbol{S}$ and $\boldsymbol{A}$ have very specific properties given by (50), (51) and (94), (96) for type N and III, respectively. Note, however, that while in higher dimensions as well as in 4D the multiple PND is geodesic, it is not shearfree for $n > 4$. The question thus arises whether there exist some properties of matrices $\boldsymbol{S}$ and $\boldsymbol{A}$ that are satisfied (together with the condition that the spacetime possesses a geodesic null congruence) if and only if the vacuum spacetime is algebraically special. The answer is unclear at present. Moreover, the conditions for $\boldsymbol{S}$ and $\boldsymbol{A}$ that hold for types N and III are not satisfied for type D spacetimes (see appendix D). Let us here, as a first step towards such possible generalization, show that from the Ricci identities it follows that

Lemma 7.

(a) Suppose that an otherwise arbitrary vacuum spacetime admits a non-expanding and non-twisting geodesic null congruence (i.e. $\boldsymbol{S} = 0 = \boldsymbol{A}$). Then, the spacetime in question is algebraically special.

(b) Suppose that an otherwise arbitrary vacuum spacetime admits an expanding and twisting geodesic null congruence, and that its $\boldsymbol{S}$ and $\boldsymbol{A}$ matrices, in an appropriately chosen frame, have the form (50), (51). Then, the spacetime in question is algebraically special.

Proof. The contraction of the Ricci identities $\ell_{a;bc} - \ell_{a;cb} - R^s{}_{abc}\ell_s = 0$ with $\ell^a m^{(i)b} m^{(j)c}$ assuming that $\boldsymbol{\ell}$ is geodesic with an affine parametrization leads to

$$DL_{ij} + L_{is}L_{sj} + L_{is}\overset{s}{M}_{j0} + L_{sj}\overset{s}{M}_{i0} + R_{0i0j} = 0. \tag{103}$$

If $\boldsymbol{L} = \boldsymbol{S} + \boldsymbol{A} = 0$ then $R_{0i0j} = C_{0i0j} = 0$ for all i, j and the spacetime is thus algebraically special. This proves part (a) of lemma 7.

If we switch i and j in (103) and add and subtract the two equations we obtain

$$2DS_{ij} + L_{is}L_{sj} + L_{js}L_{si} + 2S_{is}\overset{s}{M}_{j0} + 2S_{sj}\overset{s}{M}_{i0} + 2R_{0i0j} = 0, \tag{104}$$

$$2DA_{ij} + L_{is}L_{sj} - L_{js}L_{si} + 2A_{is}\overset{s}{M}_{j0} + 2A_{sj}\overset{s}{M}_{i0} = 0. \tag{105}$$

We further assume that we have chosen vectors $\boldsymbol{m}^{(i)}$ in such a way that $\boldsymbol{S}$ is diagonal with the only non-zero entries being $S_{22} = S_{33}$ and that $\boldsymbol{A}$ has only two non-vanishing components $A_{23} = -A_{32}$. Then (104) takes the form

$$2DS_{ij} + \frac{2}{S}(S^2 - A^2)S_{ij} + 2S_{is}\overset{s}{M}_{j0} + 2S_{sj}\overset{s}{M}_{i0} + 2R_{0i0j} = 0. \tag{106}$$

If $i, j > 3$ then $R_{0i0j} = 0$. If $i \neq j < 4$ then (106) implies $2S_{22}\overset{2}{M}_{30} + 2S_{33}\overset{3}{M}_{20} + 2R_{0203} = 0$ which using (14) leads to R_{0203}. If $i = j < 4$ then again from (106) we get

$2DS+2(S^2-A^2)+2R_{0202}=2DS+2(S^2-A^2)+2R_{0303}=0$ which together with $R_{0i0i}=0$ gives $R_{0202}=R_{0303}=0$. For $j>3$ and $i\in(2,3)$ equations (106) and (105) turn to be

$$2S_{is}\overset{s}{M}_{j0}+2R_{0i0j}=0 \quad\Longrightarrow\quad S_{22}\overset{2}{M}_{j0}+R_{020j}=0, \quad S_{33}\overset{3}{M}_{j0}+R_{030j}=0, \tag{107}$$

$$2A_{is}\overset{s}{M}_{j0}=0 \quad\Longrightarrow\quad A_{23}\overset{3}{M}_{j0}=A_{32}\overset{2}{M}_{j0}=0 \quad\Longrightarrow\quad \overset{2}{M}_{j0}=\overset{3}{M}_{j0}=0 \tag{108}$$

which leads to $R_{020j}=R_{030j}=0$ and hence $R_{0i0j}=C_{0i0j}=0$ for all i, j and the spacetime is thus algebraically special. This proves part (b) of lemma 7. □

Note also that by contracting i with j in (106) we obtain for higher dimensions the same relation which is also valid for 4D

$$DS=A^2-S^2. \tag{109}$$

6. Discussion

In this paper, we present a higher dimensional frame formalism. The complete set of frame components of the Bianchi identities, which are in this context usually called the Bianchi equations, is given in appendix B. For algebraically general spacetimes these equations are quite complicated. However, for algebraically special cases they are much simpler (e.g., see section 3 for the type N case). In 4D it is possible to use the Bianchi and Ricci equations to construct many algebraically special solutions of Einstein's field equations. The hope is that it is possible to do a similar thing in higher dimensions, at least for the simplest algebraically special spacetimes. The vast majority of today's known higher dimensional exact solutions are simple generalizations of 4D solutions. The present approach may lead to new, genuinely higher dimensional exact solutions. Type N and D (see [1, 3] for the definition of type D in higher dimensions) solutions may be of particular physical interest.

In particular, we study the consequences of the algebraic Bianchi equations for type N and III spacetimes. It turns out that the principal null direction ℓ in these spacetimes is geodesic. For vacuum type N spacetimes with non-vanishing expansion or twist we also prove that the corresponding components of the Weyl tensor and expansion and twist matrices, in an appropriately chosen frame, can be expressed as

$$\Psi_{ij}=\sqrt{\frac{p}{2}}(v_iv_j-w_iw_j), \qquad S_{ij}=S(v_iv_j+w_iw_j), \qquad A_{kl}=A(w_kv_l-v_kw_l).$$

Note that we do not obtain any constraints for Ψ_{ij} in non-twisting and non-expanding type N spacetimes. We also establish similar results for vacuum type III spacetimes.

The Weyl tensor is said to be reducible if we can decompose it into two parts

$$C^a{}_{bcd}=C^{\tilde{a}}{}_{\tilde{b}\tilde{c}\tilde{d}}+C^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}}$$

where indices $a, b, \ldots$ have values from 0 to $n-1$, whereas indices $\tilde{a}, \tilde{b}, \ldots$ have values from 0 to $N-1$, and indices $\hat{a}, \hat{b}, \ldots$ from N to $n-1$ [3]. In this sense the Weyl tensor in vacuum type N spacetimes with non-vanishing expansion or twist is reducible, with a nontrivial four-dimensional part and a vanishing $(n-4)$-dimensional part.

In 4D the well-known Goldberg–Sachs theorem [4] states that 'a vacuum metric is algebraically special if and only if it contains a shearfree geodesic null congruence'. This theorem (and also its non-vacuum generalizations—see [5]) is very useful for constructing algebraically special exact solutions. At present it is unclear if and to what extent this theorem may be generalized for higher dimensions (see section 5 for details).

Recently, all 4D spacetimes with vanishing curvature invariants (i.e., vanishing invariants constructed from the Riemann tensor and its covariant derivatives of an arbitrary order—VSI spacetimes) were determined in [6]. In [7], the generalization to higher dimensions was discussed. All these spacetimes are of type III, N or O. The results presented in this paper will enable us to explicitly express curvature invariants involving derivatives of the Riemann tensor in higher dimensions. For example, for type N vacuum spacetimes we obtain, thanks to lemma 1,

$$I = C^{abcd;pq}C_{amcn;pq}C^{tmun;rs}C_{tbud;rs} = 3^2 2^{10} p^2 (S^2 + A^2)^4.$$

Consequently, we will be able to prove that higher dimensional VSI spacetimes are expansion and twist free, thereby proving the assertion made in [7] (see [8]).

Acknowledgments

We are grateful to the referee for constructive criticism. AP and VP would like to thank Dalhousie University for its hospitality while part of this work was carried out. RM and AC were partially supported by a research grant from NSERC. VP was supported by grant GACR-202/03/P017 and AP by grant KJB1019403. AP and VP also thank the NČLF for support.

Appendix A. Components of equations (54) and (55)

Equation (54) gives

$$\mathcal{C}_{\alpha[\beta}b_{\gamma]} + \sum_F \mathcal{B}_{F[\beta|}\mathcal{P}_{F\alpha|\gamma]} + 4\mathcal{C}_{\phi[\beta|}\mathcal{S}_{\phi\alpha|\gamma]} = 0, \tag{A.1}$$

$$2\mathcal{C}_{\alpha\beta}a_C + \mathcal{B}_{C\alpha}b_\beta - 2\lambda_C\mathcal{P}_{C\alpha\beta} + \sum_F 4\mathcal{A}_{CF}\mathcal{P}_{F\alpha\beta} + 2\sum_F \mathcal{B}_{F\beta}\mathcal{O}_{F\alpha C} + 8\mathcal{C}_{\phi\beta}\mathcal{R}_{\phi\alpha C} + 4\mathcal{B}_{C\phi}\mathcal{S}_{\phi\alpha\beta} = 0, \tag{A.2}$$

$$\mathcal{B}_{A[\beta}b_{\gamma]} + 4\sum_F \mathcal{B}_{F[\beta|}\mathcal{N}_{FA|\gamma]} - 4\mathcal{C}_{\phi[\beta|}\mathcal{P}_{A\phi|\gamma]} = 0, \tag{A.3}$$

$$\lambda_B\mathcal{O}_{B\alpha C} - \lambda_C\mathcal{O}_{C\alpha B} + \mathcal{B}_{[C|\alpha}a_{|B]} + 4\sum_F \mathcal{A}_{F[B|}\mathcal{O}_{F\alpha|C]} + 4\mathcal{B}_{[B|\phi}\mathcal{R}_{\alpha\phi|C]} = 0, \tag{A.4}$$

$$\lambda_A\delta_{AB}b_\gamma - \mathcal{B}_{A\gamma}a_B + 2\mathcal{A}_{AB}b_\gamma + 4\lambda_B\mathcal{N}_{BA\gamma} + 8\sum_F \mathcal{A}_{FB}\mathcal{N}_{FA\gamma} + 2\mathcal{B}_{B\phi}\mathcal{P}_{A\phi\gamma} - 4\sum_F \mathcal{B}_{F\gamma}\mathcal{M}_{FAB} + 4\mathcal{C}_{\phi\gamma}\mathcal{O}_{A\phi B} = 0, \tag{A.5}$$

$$\lambda_A\delta_{A[B}a_{C]} + 2\mathcal{A}_{A[B}a_{C]} + 2\lambda_B\mathcal{M}_{BAC} - 2\lambda_C\mathcal{M}_{CAB} + 8\sum_F \mathcal{A}_{F[B|}\mathcal{M}_{FA|C]} + 2\mathcal{B}_{[B|\phi}\mathcal{O}_{A\phi|C]} = 0. \tag{A.6}$$

Equation (55) leads to

$$2\mathcal{C}_{\{\alpha\beta|}\mathcal{S}_{\gamma\delta|\varepsilon\}} + \mathcal{C}_{\delta\{\alpha}\mathcal{S}_{\beta\varepsilon\}\gamma} - \mathcal{C}_{\gamma\{\alpha}\mathcal{S}_{\beta\varepsilon\}\delta} = 0, \tag{A.7}$$

$$2\mathcal{C}_{\beta\varepsilon}\mathcal{R}_{\gamma\delta A} + \mathcal{C}_{\delta[\varepsilon|}\mathcal{P}_{A|\beta]\gamma} + \mathcal{C}_{\gamma[\beta|}\mathcal{P}_{A|\varepsilon]\delta} + 2\mathcal{B}_{A[\beta}\mathcal{S}_{\gamma\delta|\varepsilon]} + \mathcal{B}_{A[\gamma}\mathcal{S}_{\beta\varepsilon|\delta]} = 0, \tag{A.8}$$

$$2\mathcal{C}_{\{\alpha\beta|}\mathcal{P}_{C\delta|\varepsilon\}} + 2\mathcal{C}_{\delta\{\alpha}\mathcal{R}_{\beta\varepsilon\}C} - \mathcal{B}_{C\{\alpha}\mathcal{S}_{\beta\varepsilon\}\delta} = 0, \tag{A.9}$$

$$-4\mathcal{B}_{[A|\varepsilon}\mathcal{R}_{\gamma\delta|B]} + \mathcal{B}_{[B|\delta}\mathcal{P}_{|A]\varepsilon\gamma} - \mathcal{B}_{[B|\gamma}\mathcal{P}_{|A]\varepsilon\delta} + 4\mathcal{C}_{[\delta|\varepsilon}\mathcal{N}_{AB|\gamma]} + 4\mathcal{A}_{AB}\mathcal{S}_{\gamma\delta\varepsilon} = 0, \tag{A.10}$$

$$2\mathcal{B}_{A[\beta|}\mathcal{P}_{C\delta|\varepsilon]} + \mathcal{B}_{C[\beta|}\mathcal{P}_{A|\varepsilon]\delta} + 2\mathcal{C}_{\beta\varepsilon}\mathcal{O}_{C\delta A} - 2\mathcal{C}_{\delta[\beta|}\mathcal{O}_{A|\varepsilon]C} - \mathcal{B}_{A\delta}\mathcal{R}_{\beta\varepsilon C} - (\lambda_A\delta_{AC} + 2\mathcal{A}_{CA})\mathcal{S}_{\beta\varepsilon\delta} = 0, \tag{A.11}$$

$$4\mathcal{C}_{\{\alpha\beta|}\mathcal{N}_{CD|\varepsilon\}} + \mathcal{B}_{D\{\alpha}\mathcal{R}_{\beta\varepsilon\}C} - \mathcal{B}_{C\{\alpha}\mathcal{R}_{\beta\varepsilon\}D} = 0, \tag{A.12}$$

$$4\mathcal{C}_{\alpha\beta}\mathcal{M}_{CDE} + 4\mathcal{B}_{E[\alpha|}\mathcal{N}_{CD|\beta]} + \mathcal{B}_{D[\beta|}\mathcal{O}_{E|\alpha]C} - \mathcal{B}_{C[\beta|}\mathcal{O}_{E|\alpha]D} + 2\lambda_E\delta_{E[D}\mathcal{R}_{\alpha\beta|C]} + 4\mathcal{A}_{[D|E}\mathcal{R}_{\alpha\beta|C]} = 0, \tag{A.13}$$

$$2\mathcal{C}_{\alpha\gamma}\mathcal{M}_{BED} + \mathcal{B}_{D\alpha}\mathcal{N}_{BE\gamma} + 2\mathcal{B}_{[B|\alpha}\mathcal{O}_{D\gamma|E]} + \mathcal{B}_{[B|\gamma}\mathcal{O}_{E\alpha|D]} - 2\mathcal{A}_{BE}\mathcal{P}_{D\gamma\alpha} + 2\mathcal{A}_{D[B}\mathcal{P}_{E]\alpha\gamma} + \lambda_D\delta_{D[B}\mathcal{P}_{E]\alpha\gamma} = 0, \tag{A.14}$$

$$\mathcal{B}_{[A|\gamma}\mathcal{N}_{B]E\delta} + \mathcal{B}_{[B|\delta}\mathcal{N}_{A]E\gamma} + \mathcal{B}_{E[\gamma}\mathcal{N}_{AB|\delta]} + 2\mathcal{A}_{\{AB|}\mathcal{R}_{\gamma\delta|E\}} = 0, \tag{A.15}$$

$$4\mathcal{B}_{[E|\alpha}\mathcal{M}_{CD|B]} + 2\mathcal{B}_{[D|\alpha}\mathcal{M}_{BE|C]} + 4\mathcal{A}_{BE}\mathcal{N}_{CD\alpha} + 2\mathcal{A}_{D[B}\mathcal{O}_{E]\alpha C} + 2\mathcal{A}_{C[E}\mathcal{O}_{B]\alpha D} + \lambda_B\delta_{B[D|}\mathcal{O}_{E\alpha|C]} + \lambda_E\delta_{E[C|}\mathcal{O}_{B\alpha|D]} = 0, \tag{A.16}$$

$$\mathcal{B}_{\{A|\gamma}\mathcal{M}_{BE\}D} - 2\mathcal{A}_{\{AB|}\mathcal{O}_{D\gamma|E\}} + 2\mathcal{A}_{D\{A}\mathcal{N}_{BE\}\gamma} + \lambda_D\delta_{D\{A}\mathcal{N}_{BE\}\gamma} = 0, \tag{A.17}$$

$$4\mathcal{A}_{\{AB|}\mathcal{M}_{CD|E\}} + 2\mathcal{A}_{D\{A}\mathcal{M}_{BE\}C} - 2\mathcal{A}_{C\{A}\mathcal{M}_{BE\}D} + \lambda_D\delta_{D\{A}\mathcal{M}_{BE\}C} - \lambda_C\delta_{C\{A}\mathcal{M}_{BE\}D} = 0. \tag{A.18}$$

Appendix B. Bianchi equations

$$DR_{101i} + \triangle R_{010i} - \delta_i R_{0101} = 2R_{010[i}L_{1]1} - R_{101i}L_{10} - R_{01is}L_{s1} - 2R_{101s}L_{si} + 2R_{[1|i1s}L_{s|0]} - R_{0101}N_{i0} - 2R_{010s}N_{si} + R_{01is}N_{s0} + 2R_{0s[0|i}N_{s|1]} - R_{010s}\overset{s}{M}_{i1} - R_{101s}\overset{s}{M}_{i0}, \tag{B.1}$$

$$-\triangle R_{01ij} + 2\delta_{[j|}R_{101|i]} = 2R_{101[j|}L_{1|i]} + 2R_{101[i|}L_{|j]1} + 2R_{1[i|1s}L_{s|j]} + R_{1sij}L_{s1} + 2R_{0101}N_{[ji]} + 2R_{010[j}N_{i]1} + 2R_{01[i|s}N_{s|j]} + 2R_{0s1[j|}N_{s|i]} - R_{0sij}N_{s1} + 2R_{01[i|s}\overset{s}{M}_{|j]1} + 2R_{101s}\overset{s}{M}_{[ji]}, \tag{B.2}$$

$$-DR_{01ij} + \delta_{[i|}R_{010|j]} = 2R_{0101}L_{[ij]} + 2R_{010[j|}L_{1|i]} + 2R_{01[i|s}L_{s|j]} + 2R_{0[i|1s}L_{s|j]} + 2R_{101[i}L_{j]0} + R_{1sij}L_{s0} + 2R_{010[j}N_{i]0} + 2R_{0[j|0s}N_{s|i]} - R_{0sij}N_{s0} + 2R_{01[i|s}\overset{s}{M}_{|j]0} + 2R_{010s}\overset{s}{M}_{[ij]}, \tag{B.3}$$

$$DR_{1i1j} - \triangle R_{0j1i} - \delta_j R_{101i} = 2R_{101i}L_{[1j]} - 2R_{1i1j}L_{10} + 2R_{1i[j|s}L_{s|1]} + 2R_{010[1|}N_{i|j]} - 2R_{101(i|}N_{|j)0} + 2R_{0[j|is}N_{s|1]} + R_{0s1i}N_{sj} - 2R_{1(ij)s}N_{s0} + 2R_{[0|j1s}\overset{s}{M}_{i|1]} + 2R_{[0|s1i}\overset{s}{M}_{j|1]} + R_{101s}\overset{s}{M}_{ij}, \tag{B.4}$$

$$DR_{0i1j} - \triangle R_{0i0j} + \delta_j R_{010i} = 2R_{010[j|}L_{i|1]} + 2R_{010i}L_{(1j)} - 2R_{0i0j}L_{11} + R_{101j}L_{i0} + 2R_{0i[j|s}L_{s|1]} + 2R_{0[j|is}L_{s|1]} - R_{1jis}L_{s0} + R_{010i}N_{j0} + 2R_{0i[0|s}N_{s|j]} - R_{010s}\overset{s}{M}_{ij} + 2R_{0s[0|j}\overset{s}{M}_{i|1]} + 2R_{0i[0|s}\overset{s}{M}_{j|1]}, \tag{B.5}$$

$$-DR_{1kij} + \triangle R_{0kij} - \delta_k R_{01ij} = -2R_{0[1|ij}L_{|k]1} + 2R_{0k1[i}L_{j]1} + 2R_{101[i}L_{j]k} - 2R_{1[i|1k}L_{|j]0} - R_{1kij}L_{10} + 2R_{[1|sij}L_{s|k]} - 2R_{010[i}N_{j]k} - R_{01ij}N_{k0} + 2R_{0[i|0k}N_{|j]1} + 2R_{0[j|1k}N_{|i]0} + 2R_{[k|sij}N_{s|0]} + 2R_{01[i|s}\overset{s}{M}_{|j]k} + 2R_{[0|kjs}\overset{s}{M}_{i|1]} + 2R_{[1|kis}\overset{s}{M}_{j|0]} + 2R_{[1|sij}\overset{s}{M}_{k|0]}, \tag{B.6}$$

$$\begin{aligned}\triangle R_{0ijk} + \delta_{[k|}R_{0i1|j]} &= 2R_{0i1[j|}L_{|k]1} + 2R_{0[i|jk}L_{|1]1} + 2R_{101[j|}L_{i|k]} + 2R_{1[k|is}L_{s|j]}\\ &\quad - R_{isjk}L_{s1} + 2R_{010i}N_{[jk]} + 2R_{0i0[j}N_{k]1} + 2R_{0i[k|s}N_{s|j]}\\ &\quad + 2R_{0i1s}\overset{s}{M}_{[kj]} + 2R_{0s1[k|}\overset{s}{M}_{i|j]} + 2R_{0i[k|s}\overset{s}{M}_{|j]1} - R_{0sjk}\overset{s}{M}_{i1},\end{aligned} \tag{B.7}$$

$$\begin{aligned}DR_{0ijk} + \delta_{[k|}R_{0i0|j]} &= -2R_{010i}L_{[jk]} + 2R_{010[k|}L_{i|j]} + 4R_{0i0[j|}L_{1|k]} + 2R_{0i1[j}L_{k]0}\\ &\quad + 2R_{0[i|jk}L_{|1]0} + 2R_{0i[k|s}L_{s|j]} + 2R_{0[k|is}L_{s|j]} - R_{isjk}L_{s0} + 2R_{0i0[j|}N_{|k]0}\\ &\quad + 2R_{0i0s}\overset{s}{M}_{[kj]} + 2R_{0[k|0s}\overset{s}{M}_{i|j]} + 2R_{0i[k|s}\overset{s}{M}_{|j]0} - R_{0sjk}\overset{s}{M}_{i0},\end{aligned} \tag{B.8}$$

$$\begin{aligned}DR_{1ijk} + \delta_{[k|}R_{0|j]1i} &= 2R_{101i}L_{[jk]} + 2R_{1i1[j}L_{k]0} - R_{1ijk}L_{10} + 2R_{1i[k|s}L_{s|j]}\\ &\quad + R_{010j}N_{ik} + 2R_{01[j|k}N_{i|0]} - 2R_{0[k|1i}N_{|j]0} + 2R_{0[k|is}N_{s|j]} - R_{isjk}N_{s0}\\ &\quad + 2R_{0[k|1s}\overset{s}{M}_{i|j]} - 2R_{0s1i}\overset{s}{M}_{[jk]} + 2R_{1i[k|s}\overset{s}{M}_{|j]0} - R_{1sjk}\overset{s}{M}_{i0},\end{aligned} \tag{B.9}$$

$$\begin{aligned}\triangle R_{1ijk} + \delta_{[k|}R_{1i1|j]} &= 2R_{1i1[j}L_{k]1} + 4R_{1i1[k|}L_{1|j]} - R_{1ijk}L_{11} + R_{01jk}N_{i1} + 2R_{0[j|1i}N_{|k]1}\\ &\quad + 2R_{101[k|}N_{i|j]} + 2R_{101i}N_{[kj]} + 2R_{1i[k|s}N_{s|j]} + 2R_{1[k|is}N_{s|j]} - R_{isjk}N_{s1}\\ &\quad - 2R_{1i1s}\overset{s}{M}_{[jk]} + 2R_{1[k|1s}\overset{s}{M}_{i|j]} + 2R_{1i[k|s}\overset{s}{M}_{|j]1} - R_{1sjk}\overset{s}{M}_{i1},\end{aligned} \tag{B.10}$$

$$\begin{aligned}-\delta_{\{i|}R_{01|jk\}} &= R_{101\{i}L_{jk\}} - R_{101\{i}L_{kj\}} + R_{1s\{ij|}L_{s|k\}} + R_{010\{i}N_{kj\}} - R_{010\{i}N_{jk\}}\\ &\quad - R_{0sij}N_{sk} + 2R_{0s[i|k}N_{s|j]} + R_{01\{i|s}\overset{s}{M}_{|jk\}} - R_{01\{i|s}\overset{s}{M}_{|kj\}},\end{aligned} \tag{B.11}$$

$$\begin{aligned}-DR_{ijkm} + \delta_{[k|}R_{0|m]ij} &= 2R_{01ij}L_{[km]} + 2R_{0k[1|j}L_{|i]m} + 2R_{0[m|1i}L_{j|k]} + 2R_{0m[i|j}L_{|1]k}\\ &\quad + 2R_{1[j|km}L_{|i]0} + 2R_{1[m|ij}L_{|k]0} + 2R_{ij[k|s}L_{s|m]} + 2R_{0i0[m|}N_{j|k]}\\ &\quad + 2R_{0j0[k|}N_{i|m]} + 2R_{0[j|km}N_{|i]0} + 2R_{0[m|ij}N_{|k]0} + 2R_{0[k|is}\overset{s}{M}_{j|m]}\\ &\quad + 2R_{0[m|js}\overset{s}{M}_{i|k]} + 2R_{0sij}\overset{s}{M}_{[km]} + 2R_{[i|skm}\overset{s}{M}_{|j]0} + 2R_{ij[k|s}\overset{s}{M}_{|m]0},\end{aligned} \tag{B.12}$$

$$\begin{aligned}-\triangle R_{ijkm} + \delta_{[k|}R_{1|m]ij} &= 2R_{1i[1|m}L_{j|k]} + 2R_{1[j|1k}L_{|i]m} + 2R_{1j[k|m}L_{i|1]} + 2R_{1kij}L_{[1m]}\\ &\quad + 2R_{1mij}L_{[k1]} - 2R_{01ij}N_{[km]} + 2R_{0[i|1m}N_{|j]k} + 2R_{0[j|1k}N_{|i]m}\\ &\quad + 2R_{0[j|km}N_{|i]1} + 2R_{0[m|ij}N_{|k]1} + 2R_{ij[k|s}N_{s|m]} + 2R_{1k[i|s}\overset{s}{M}_{|j]m}\\ &\quad + 2R_{1m[j|s}\overset{s}{M}_{|i]k} + 2R_{1sij}\overset{s}{M}_{[km]} + 2R_{ij[k|s}\overset{s}{M}_{|m]1} + 2R_{[i|skm}\overset{s}{M}_{|j]1},\end{aligned} \tag{B.13}$$

$$\begin{aligned}\delta_{\{j|}R_{1i|mk\}} &= R_{1i1\{j}L_{mk\}} - R_{1i1\{j}L_{km\}} + R_{1i\{jk|}L_{1|m\}} + R_{01\{jm|}N_{i|k\}} - R_{0\{j|1i}N_{|km\}}\\ &\quad + R_{0\{j|1i}N_{|mk\}} + R_{is\{jk|}N_{s|m\}} - R_{1i\{j|s}\overset{s}{M}_{|mk\}} + R_{1i\{j|s}\overset{s}{M}_{|km\}} + R_{1s\{jk|}\overset{s}{M}_{i|m\}},\end{aligned} \tag{B.14}$$

$$\begin{aligned}\delta_{\{j|}R_{0i|mk\}} &= R_{01\{jk|}L_{i|m\}} - R_{0i1\{j}L_{km\}} + R_{0i1\{j}L_{mk\}} + R_{0i\{jm|}L_{1|k\}} + R_{is\{jk|}L_{s|m\}}\\ &\quad + R_{0i0\{j}N_{mk\}} - R_{0i0\{j}N_{km\}} + R_{0i\{j|s}\overset{s}{M}_{|km\}} - R_{0i\{j|s}\overset{s}{M}_{|mk\}} + R_{0s\{jk|}\overset{s}{M}_{i|m\}},\end{aligned} \tag{B.15}$$

$$\begin{aligned}\delta_{\{k|}R_{ij|nm\}} &= R_{1j\{km|}L_{i|n\}} - R_{1i\{km|}L_{j|n\}} - R_{1\{k|ij}L_{|mn\}} + R_{1\{k|ij}L_{|nm\}}\\ &\quad + R_{0j\{km|}N_{i|n\}} - R_{0i\{km|}N_{j|n\}} + R_{0\{k|ij}N_{|nm\}} - R_{0\{k|ij}N_{|mn\}}\\ &\quad + R_{ij\{k|s}\overset{s}{M}_{|mn\}} - R_{ij\{k|s}\overset{s}{M}_{|nm\}} + R_{is\{km|}\overset{s}{M}_{j|n\}} - R_{js\{km|}\overset{s}{M}_{i|n\}}.\end{aligned} \tag{B.16}$$

Appendix C. Proof for vacuum a type III spacetimes

C.1. Non-twisting case in an arbitrary dimension

In this case, $\boldsymbol{A} = 0$ and thus $\boldsymbol{L} = \boldsymbol{S}$ is symmetric (we assume $\boldsymbol{S} \neq 0$ since in the non-twisting and non-expanding case both sides of equations (54) and (55) vanish).

From (60) we obtain

$$S_{ij}\Psi_j = \frac{L}{2}\Psi_i. \tag{C.1}$$

Note that providing $\Psi_i \neq 0$ from (C.1) and (67) we obtain $L^2 = 2\ell$ and thus the case $L = 0$ leads to $\boldsymbol{S} = 0$.

Equation (59) leads to

$$L\Psi_{ijk} = 2\Psi_{ijs}S_{sk}, \tag{C.2}$$

which using the decomposition (69) implies

$$\begin{aligned} &L\mathcal{N}_{AB\gamma} = 0, && L\mathcal{P}_{A\beta\gamma} = 0, && L\mathcal{S}_{\alpha\beta\gamma} = 0, \\ &(L - 2\lambda_C)\mathcal{M}_{ABC} = 0, && (L - 2\lambda_C)\mathcal{O}_{A\beta C} = 0, && (L - 2\lambda_C)\mathcal{R}_{\alpha\beta C} = 0. \end{aligned} \tag{C.3}$$

First study the case with $L = 0$. From equations (C.3) we get $\mathcal{M}_{ABC} = 0, \mathcal{O}_{A\beta C} = 0$, $\mathcal{R}_{\alpha\beta C} = 0$ and thus from (76) and (81) $\mathcal{P}_{A\beta\gamma} = 0, \mathcal{N}_{AB\gamma} = 0$ and further from (A.11) $\mathcal{S}_{\alpha\beta\gamma} = 0$ which imply $\Psi_{ijk} = 0$, i.e., type N or O spacetime and thus there are no non-trivial solutions of (54) and (55) of type III with $L = 0$.

Now we may proceed to the case $L \neq 0$. Then (C.3) and (80) imply

$$\mathcal{N}_{AB\gamma} = 0, \qquad \mathcal{P}_{A\beta C} = 0, \qquad \mathcal{S}_{\alpha\beta\gamma} = 0, \qquad \mathcal{R}_{\alpha\beta C} = 0. \tag{C.4}$$

From (C.1) we get

$$\lambda_A a_A = \frac{L}{2}a_A, \qquad \text{and} \qquad b_\alpha = 0, \tag{C.5}$$

which leads to

$$a_A = 0 \qquad \text{or} \qquad \lambda_A = \frac{L}{2}. \tag{C.6}$$

(i) If all $a_A \neq 0$ then all $\lambda_A = L/2$ (C.6) and thus $A_{\max} = 2, \lambda_1 = \lambda_2 = L/2$. The corresponding solution is given by (C.20).

(ii) If at least one $a_A \neq 0$ then corresponding $\lambda_A = L/2$ and from (79) with $B = C \neq A$ and (A.6) with $A = C \neq B$ and after interchanging A with B we get

$$a_A = \frac{2L - 4\lambda_B + 4\lambda_A}{\lambda_B}\mathcal{M}_{ABB}|_{B\neq A}, \tag{C.7}$$

$$a_A = 4\frac{\lambda_A}{\lambda_B}\mathcal{M}_{ABB}|_{B\neq A}. \tag{C.8}$$

Comparing the right-hand sides we obtain $2L - 4\lambda_B + 4\lambda_A = 4\lambda_A$ and thus $\lambda_B = L/2$ for all $B \neq A$. However, the remaining $\lambda_A = L/2$ and in this case again $A_{\max} = 2, \lambda_1 = \lambda_2 = L/2$.

(iii) If all $a_A = 0$ then (C.8) implies

$$\mathcal{M}_{ABB} = 0 \tag{C.9}$$

for all A, B. We treat separately the case with all $\mathcal{M}_{ABC} = 0$ and the case with some $\mathcal{M}_{ABC} \neq 0$.

(A) If all $\mathcal{M}_{ABC} = 0$ then there exists at least one non-vanishing $\mathcal{O}_{A\beta C}$ in order to have a type III Weyl tensor.

For $A_{\max} \geqslant 3$, from equation (A.16) for $B = C, C \neq E, C \neq D, D \neq E$ we get $\lambda_C \mathcal{O}_{E\alpha D} = 0$ which implies that all $\mathcal{O}_{E\alpha D} = 0|_{E\neq D}$. Equation (77) with $C = A$, i.e., $(L - 2\lambda_A)\mathcal{O}_{A\beta A} = 0$, implies

$$\mathcal{O}_{A\beta A} = 0 \qquad \text{or} \qquad \lambda_A = \frac{L}{2}. \tag{C.10}$$

From equation (A.16) (for $B = C$, $D = E$, $C \neq E$)

$$\lambda_C \mathcal{O}_{D\alpha D} = -\lambda_D \mathcal{O}_{C\alpha C}|_{D\neq C}. \tag{C.11}$$

If there exists $\lambda_C \neq L/2$ then from (C.10) $\mathcal{O}_{C\alpha C} = 0$ and thus from (C.11) all $\mathcal{O}_{D\alpha D} = 0$ and $\Psi_{ijk} = 0$, the spacetime is of type N or O. The other possibility that all $\lambda_A = L/2$ is not compatible with $A_{\max} \geqslant 3$ and so $A_{\max} \leqslant 2$.

For $A_{\max} = 2$, (71) with (C.11) imply

$$\lambda_2 = \lambda_1 = \frac{L}{2} \qquad \text{and} \qquad \mathcal{O}_{2\beta 2} = -\mathcal{O}_{1\beta 1} \tag{C.12}$$

or $\mathcal{O}_{2\beta 2} = \mathcal{O}_{1\beta 1} = 0$. In the second case, equation (81) implies

$$\mathcal{O}_{2\beta 1} = \mathcal{O}_{1\beta 2} \tag{C.13}$$

and from (78) (or (A.4)) we get $\lambda_2 = \lambda_1 = L/2$.

For $A_{\max} = 1$, from (C.10) we immediately get $\mathcal{O}_{1\beta 1} = 0$, type N or O spacetimes. To summarize, the case $\mathcal{M}_{ABC} = 0$ ($\forall$ A, B, C) always leads to $A_{\max} = 2$, $\lambda_1 = \lambda_2 = L/2$ or to type N or O spacetimes.

(B) Let us now assume, for $A_{\max} \geqslant 4$, that there exists $\mathcal{M}_{ABC} \neq 0|_{B\neq C}$. Then (A.18) for $C \neq D$, $A \neq D$, $A \neq C$, $B = D$, $C \neq E$ and $D \neq E$ implies $\lambda_D \mathcal{M}_{AEC} = 0$ and thus all $\mathcal{M}_{ABC} = 0|_{B\neq C}$ which is in contradiction with our assumptions and consequently $A_{\max} \leqslant 3$.

Equation (79) with $C \neq A$ and $B \neq C$ and (A.6) gives

$$(L - 2\lambda_C + 2\lambda_A)\mathcal{M}_{CAB} = (L - 2\lambda_C + 2\lambda_B)\mathcal{M}_{CBA}|_{C\neq A, C\neq B}, \tag{C.14}$$

$$\lambda_A \mathcal{M}_{CAB} = \lambda_B \mathcal{M}_{CBA}. \tag{C.15}$$

These two equations lead to

$$(L - 2\lambda_C)\mathcal{M}_{CAB} = (L - 2\lambda_C)\mathcal{M}_{CBA} \tag{C.16}$$

and so either

$$\lambda_C = \frac{L}{2} \wedge \lambda_A \mathcal{M}_{CAB} = \lambda_B \mathcal{M}_{CBA} \tag{C.17}$$

or

$$\lambda_C \neq \frac{L}{2} \wedge \mathcal{M}_{CAB} = \mathcal{M}_{CBA} \wedge \lambda_A \mathcal{M}_{CAB} = \lambda_B \mathcal{M}_{CBA}, \tag{C.18}$$

i.e.,

$$\mathcal{M}_{CAB} = \mathcal{M}_{CBA} = 0 \qquad \text{or} \qquad \mathcal{M}_{CAB} = \mathcal{M}_{CBA} \neq 0 \wedge \lambda_A = \lambda_B. \tag{C.19}$$

Let us assume that $A_{\max} = 3$. Let $\lambda_1 \neq L/2$. Then the case (a) $\mathcal{M}_{123} = \mathcal{M}_{132} = 0$ (C.18) implies $\mathcal{M}_{231} = 0$ (82) which is the case $\mathcal{M}_{ABC} = 0$ studied above (A). The condition (b) $\mathcal{M}_{123} = \mathcal{M}_{132} \neq 0$ (C.18) also implies $\mathcal{M}_{231} = 0$ (82) and for $\lambda_2 \neq L/2$ we get $\mathcal{M}_{213} = \mathcal{M}_{231} = 0$ (C.18) and thus also $\mathcal{M}_{123} = 0$ and this is again the case $\mathcal{M}_{ABC} = 0$. However, if $\lambda_2 = L/2$ then $\lambda_1 \mathcal{M}_{213} = \lambda_3 \mathcal{M}_{231} = 0$ (C.17) and since $\mathcal{M}_{231} = 0$ we again obtain the case $\mathcal{M}_{ABC} = 0$.

If $\lambda_1 = L/2$ then $\lambda_2 = L/2$ immediately implies $\lambda_3 = 0$ and $\lambda_2 \neq L/2$ corresponds to the case analysed in the previous paragraph.

$A_{\max} < 3$ again leads to all $\mathcal{M}_{ABC} = 0$ (C.9).

To summarize, the general solution of (54) and (55) in the non-twisting case may be written, substituting (C.4), (C.5), (C.8), (C.12) and (C.13) into (69), in the form

$$\Psi_i = 4\mathcal{M}_{122}v_i^1 + 4\mathcal{M}_{211}v_i^2, \qquad S_{ij} = \frac{L}{2}(v_i^1 v_j^1 + v_i^2 v_j^2),$$
$$\Psi_{ijk} = 2\mathcal{M}_{122}\big(v_i^1 v_j^2 - v_j^1 v_i^2\big)v_k^2 + 2\mathcal{M}_{211}\big(v_i^2 v_j^1 - v_j^2 v_i^1\big)v_k^1 + \mathcal{O}_{1\alpha 1}\big(v_i^1 u_j^\alpha - v_j^1 u_i^\alpha\big)v_k^1$$
$$- \mathcal{O}_{1\alpha 1}\big(v_i^2 u_j^\alpha - v_j^2 u_i^\alpha\big)v_k^2 + \mathcal{O}_{1\alpha 2}\big(v_i^1 u_j^\alpha - v_j^1 u_i^\alpha\big)v_k^2 + \mathcal{O}_{1\alpha 2}\big(v_i^2 u_j^\alpha - v_j^2 u_i^\alpha\big)v_k^1. \tag{C.20}$$

C.2. Twisting type III spacetimes in five dimensions

In section 4, we studied possible non-zero eigenvalues λ_A, $A = 1, \ldots, A_{\max}$, of $\boldsymbol{S}$ which are compatible with equations (54) and (55), providing that for every pair A, C ($A \neq C$) there exists β for which $\mathcal{O}_{A\beta C} \neq 0$. It turned out that it follows from (54) and (55) that $A_{\max} = 2$ and $\lambda_1 = \lambda_2 = L/2$. Here we study all other possible cases in 5D and show that the only case with non-vanishing eigenvalues of $\boldsymbol{S}$ is $A_{\max} = 2$ and $\lambda_1 = \lambda_2 = L/2$. The corresponding form of Ψ_{ijk}, Ψ_i, $\boldsymbol{S}$ and $\boldsymbol{A}$ is given in section 4.

In 5D, $A_{\max}$ can have values 0, 1, 2 or 3. We will treat these cases separately (the cases $A_{\max} = 0$ and $A_{\max} = 1$ can be easily solved in arbitrary dimensions).

(i) The case $A_{\max} = 0$ (globally) corresponds to $\boldsymbol{S} = 0$ and one of the Ricci equations gives $D\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}$, which implies (take trace of both sides) that $\boldsymbol{A} = 0$. Thus, in an arbitrary dimension, as in 4D, there are no non-expanding, twisting spacetimes.

(ii) If $A_{\max} = 1$, i.e., $\lambda_1 = L \neq 0$, then $\mathcal{M}_{ABC} = 0$, $\mathcal{N}_{AB\gamma} = 0$ and $\mathcal{A}_{AB} = 0$. From (74) and (75), it follows $\mathcal{S}_{\alpha\beta\gamma} = 0$ and $\mathcal{R}_{\alpha\beta C} = 0$. Equations (80) and (76) give $\mathcal{P}_{A\beta\gamma} = 0$. From (77) and (71), it follows $b_\beta = 0$ and $\mathcal{O}_{A\beta A} = 0$. Thus, the case $A_{\max} = 1$ leads to a type N or O spacetime.

(iii) In the case $A_{\max} = 2, \alpha_{\max} = 1$ and thus, thanks to antisymmetry, $\mathcal{R}_{\alpha\beta C} = 0$ and $\mathcal{S}_{\alpha\beta\gamma} = 0$. Then from (80) $\mathcal{P}_{A\beta\gamma} = \mathcal{P}_{A\gamma\beta}$ and from (76) $\mathcal{P}_{B\alpha\gamma} = 0$ or $\lambda_B = -L/2$.
Let us proceed further assuming that at least one non-vanishing $\mathcal{O}$ exists:

From (77) and (81), it follows that $\mathcal{O}_{112} = 0$ is equivalent to $\mathcal{O}_{211} = 0$. Similarly, from (77) and (71), $\mathcal{O}_{111} = 0$ is equivalent to $\mathcal{O}_{212} = 0$.

The case $\mathcal{O}_{112}, \mathcal{O}_{211} \neq 0$ was analysed in section 4 and leads to $\lambda_1 = \lambda_2 = L/2$. If $\mathcal{O}_{112} = 0 = \mathcal{O}_{211}$ and $\mathcal{O}_{111}, \mathcal{O}_{212} \neq 0$ then (81) gives $\mathcal{N}_{AB\gamma} = 0$ and from (77) and (71),

$$b_\beta = \frac{\lambda_1 - \lambda_2}{\lambda_1}\mathcal{O}_{1\beta 1} = \frac{\lambda_2 - \lambda_1}{\lambda_2}\mathcal{O}_{2\beta 2} = -2(\mathcal{O}_{1\beta 1} + \mathcal{O}_{2\beta 2}), \tag{C.21}$$

follows $\lambda_1 = \lambda_2 = L/2$. Next we have to analyse possible cases with $\mathcal{O}_{A\beta C} = 0\,\forall\, A, \beta, C$.

Let us now assume that all $\mathcal{O}$'s are zero and at least one $\mathcal{M}$ is non-vanishing. From (79) for $B = C$ and $B \neq A$ and from (72) we get

$$\lambda_2 a_1 = 2(3\lambda_1 - \lambda_2)\mathcal{M}_{122}, \qquad (-3\lambda_1 + \lambda_2)a_1 = -8\lambda_2\mathcal{M}_{122}, \tag{C.22}$$
$$\lambda_1 a_2 = 2(3\lambda_2 - \lambda_1)\mathcal{M}_{211}, \qquad (-3\lambda_2 + \lambda_1)a_2 = -8\lambda_1\mathcal{M}_{211}. \tag{C.23}$$

If $a_1 \neq 0$ (and thus also $\mathcal{M}_{122} \neq 0$) then the ratio of equations (C.22) gives $(3\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2) = 0$ which has solutions

$$\lambda_2 = \lambda_1 \qquad \text{or} \qquad \lambda_2 = -3\lambda_1. \tag{C.24}$$

Similarly, if $a_2 \neq 0$, $\mathcal{M}_{211} \neq 0$ then the ratio of equations (C.23) gives $(3\lambda_2 + \lambda_1)(\lambda_2 - \lambda_1) = 0$ with solutions $\lambda_2 = \lambda_1$ and $\lambda_1 = -3\lambda_2$. Thus for $a_1 \neq 0$ and $a_2 \neq 0$, the only solution is $\lambda_2 = \lambda_1 = L/2$.

If only one $a_A \neq 0$, without loss of generality $a_1 \neq 0$ and $a_2 = 0$, then $\mathcal{M}_{211} = 0$ and from (70) $\mathcal{P}_{211} = 0$ and from (A.14) it follows $\mathcal{P}_{111} = 0$. For the second solution in (C.24), $\lambda_2 = -3\lambda_1$, from (C.22) and (70) we obtain $a_1 = -4\mathcal{M}_{122}$ and $a_1 = 4\mathcal{M}_{122}$, respectively, and thus this case does not occur.

If $a_1 = 0 = a_2$ then (C.22), (C.23) imply that all $\mathcal{M}$'s are zero and from (70) also all $\mathcal{P}$'s vanish, which leads to type N or O spacetime.

(iv) For $A_{\max} = 3$, the only components of Ψ_{ijk}, Ψ_i and A_{ij} are $\mathcal{M}_{ABC}, a_A$ and $\mathcal{A}_{AB}$, respectively.

From (79) with $C = B$ we obtain the equation

$$a_A = \frac{2L + 4\lambda_A - 4\lambda_B}{\lambda_B} \mathcal{M}_{ABB}|_{B \neq A} \tag{C.25}$$

which together with (70) gives

$$a_1 = \frac{2L + 4\lambda_1 - 4\lambda_2}{\lambda_2} \mathcal{M}_{122} = \frac{2L + 4\lambda_1 - 4\lambda_3}{\lambda_3} \mathcal{M}_{133} = 4(\mathcal{M}_{122} + \mathcal{M}_{133}), \tag{C.26}$$

$$a_2 = \frac{2L + 4\lambda_2 - 4\lambda_1}{\lambda_1} \mathcal{M}_{211} = \frac{2L + 4\lambda_2 - 4\lambda_3}{\lambda_3} \mathcal{M}_{233} = 4(\mathcal{M}_{211} + \mathcal{M}_{233}), \tag{C.27}$$

$$a_3 = \frac{2L + 4\lambda_3 - 4\lambda_1}{\lambda_1} \mathcal{M}_{311} = \frac{2L + 4\lambda_3 - 4\lambda_2}{\lambda_2} \mathcal{M}_{322} = 4(\mathcal{M}_{311} + \mathcal{M}_{322}). \tag{C.28}$$

(A) If all $a_A \neq 0$ then the only solution is $\lambda_1 = \lambda_2 = L/2$ and $\lambda_3 = 0$, which corresponds to $A_{\max} = 2$ discussed above.

(B) Let us now study the case with one a_A vanishing (we can assume $a_3 = 0$ without loss of generality). Then from (C.28) either $\mathcal{M}_{311} + \mathcal{M}_{322} = 0$ and $\lambda_1 = \lambda_2 = L/2 + \lambda_3$, which implies $A_{\max} = 2$, $\lambda_3 = 0$, $\lambda_1 = \lambda_2 = L/2$, or $\mathcal{M}_{311} = \mathcal{M}_{322} = 0$ and then from (C.26) and (C.27) we get

$$3\lambda_1^2 - \lambda_2^2 - \lambda_3^2 + 2(-\lambda_1\lambda_2 - \lambda_1\lambda_3 + \lambda_2\lambda_3) = 0, \tag{C.29}$$

$$-\lambda_1^2 + 3\lambda_2^2 - \lambda_3^2 + 2(-\lambda_1\lambda_2 + \lambda_1\lambda_3 - \lambda_2\lambda_3) = 0. \tag{C.30}$$

Their difference gives

$$4(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - \lambda_3) = 0 \tag{C.31}$$

and thus either $\lambda_1 = \lambda_2$ and from (C.29) $\lambda_3 = 0$ or $\lambda_3 = \lambda_1 + \lambda_2$ and from (C.29) $\lambda_1\lambda_2 = 0$. This case thus again leads to $A_{\max} = 2$.

(C) If only one a_A is non-vanishing (we again choose $a_1 \neq 0$ and $a_2 = a_3 = 0$ without loss of generality) then (C.27), (C.28) imply either $\lambda_1 = L/2 + \lambda_3 = \lambda_2$ and $\lambda_1 = L/2 + \lambda_2 = \lambda_3$, which is not possible, or $\lambda_1 = L/2 + \lambda_3 = \lambda_2$, which gives $\lambda_1 = \lambda_2 = L/2$ and $\lambda_3 = 0$, or $\lambda_1 = L/2 + \lambda_2 = \lambda_3$, which gives $\lambda_1 = \lambda_3 = L/2$ and $\lambda_2 = 0$, or $\mathcal{M}_{211} = \mathcal{M}_{233} = \mathcal{M}_{311} = \mathcal{M}_{322} = 0$. All these cases, except the last one, are inconsistent with our assumption $A_{\max} = 3$. We thus need to check the last case $\mathcal{M}_{211} = \mathcal{M}_{233} = \mathcal{M}_{311} = \mathcal{M}_{322} = 0$, which has two possible branches corresponding to $\mathcal{M}_{123} = 0$ and $\mathcal{M}_{123} \neq 0$.

(a) $\mathcal{M}_{123} = 0$: from (79) (in the form (C.14))

$$(-\lambda_1 + 3\lambda_2 + \lambda_3)\mathcal{M}_{123} = (-\lambda_1 + \lambda_2 + 3\lambda_3)\mathcal{M}_{132}, \tag{C.32}$$

$$(3\lambda_1 - \lambda_2 + \lambda_3)\mathcal{M}_{213} = (\lambda_1 - \lambda_2 + 3\lambda_3)\mathcal{M}_{231}, \tag{C.33}$$

$$(3\lambda_1 + \lambda_2 - \lambda_3)\mathcal{M}_{312} = (\lambda_1 + 3\lambda_2 - 3\lambda_3)\mathcal{M}_{321}, \tag{C.34}$$

we get $\lambda_3 = 0$ and $\lambda_1 = L/2 = \lambda_2$ for $\mathcal{M}_{231} = \mathcal{M}_{132} \neq 0$, or $\mathcal{M}_{231} = \mathcal{M}_{132} = 0$. Then from (A.6) we get $\mathcal{A}_{AB} = 0$ and $a_1 = \frac{4\lambda_1}{\lambda_2}\mathcal{M}_{122} = \frac{4\lambda_1}{\lambda_3}\mathcal{M}_{133}$ which together with (C.26) gives $\lambda_2 = \lambda_3 = L/2$, $\lambda_1 = 0$.

(b) $\mathcal{M}_{123} \neq 0$: from (C.32)–(C.34) and (82) we get

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = 0 \tag{C.35}$$

with the solution $\lambda_3 = \lambda_1 + \lambda_2 \pm 2\sqrt{\lambda_1\lambda_2}$. By substituting this result into (C.29) we get $\lambda_1(\lambda_2 \pm \sqrt{\lambda_1\lambda_2}) = 0$ and thus either $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = L/2$ or $\lambda_1 = \lambda_2 = L/2$ and $\lambda_3 = 0$ ($\lambda_3 = 4\lambda_1$ does not satisfy (C.29)).

(D) If all $a_A = 0$ then from (C.26)–(C.28) either one of $\lambda_A = 0$ and the other two are equal to $L/2$ or all $\mathcal{M}_{ABB} = 0$. Then if

(a) $\mathcal{M}_{123} = 0$ from (C.32)–(C.34) and (82) we get either one of $\lambda_A = 0$ and the other two are again equal to $L/2$ or all $\mathcal{M}_{ABC} = 0$, which corresponds to a type N or O spacetime.

(b) If $\mathcal{M}_{123} \neq 0$ then from (C.32)–(C.34) and (82) we get (C.35). Further from (A.6) we get all $\mathcal{A}_{AB} = 0$ and then

$$\lambda_2\mathcal{M}_{123} = \lambda_3\mathcal{M}_{132}, \tag{C.36}$$

$$\lambda_1\mathcal{M}_{123} = -\lambda_3\mathcal{M}_{231}, \tag{C.37}$$

$$\lambda_1\mathcal{M}_{132} = \lambda_2\mathcal{M}_{231}, \tag{C.38}$$

which yields

$$0 = (\lambda_2 - \lambda_3)(\lambda_1 - \lambda_2 - \lambda_3), \tag{C.39}$$

$$0 = -\lambda_1^2 + \lambda_1\lambda_2 + 3\lambda_1\lambda_3 + 2\lambda_2\lambda_3 - 2\lambda_3^2, \tag{C.40}$$

$$0 = -\lambda_1^2 + 3\lambda_1\lambda_2 + \lambda_1\lambda_3 + 2\lambda_2\lambda_3 - 2\lambda_2^2, \tag{C.41}$$

with the solution either $\lambda_1 = \lambda_2 + \lambda_3$ (then (C.40) implies λ_2 or λ_3 is zero) or $\lambda_2 = \lambda_3$ and (C.40) implies $\lambda_1 = 0$ or $\lambda_1 = 4\lambda_2$; however, from (A.18) we get $(\lambda_2 - \lambda_1)\mathcal{M}_{123} = 0$ which is a contradiction.

Appendix D. An example of a type D vacuum spacetime

The five-dimensional rotating black-hole metric in Boyer–Lindquist coordinates has the form [9, 10] (we use the notation of [10])

$$\mathrm{d}s^2 = \frac{\rho^2}{4\Delta}\,\mathrm{d}x^2 + \rho^2\,\mathrm{d}\theta^2 - \mathrm{d}t^2 + (x + a^2)\sin^2\theta\,\mathrm{d}\phi^2 + (x + b^2)\cos^2\theta\,\mathrm{d}\psi^2 + \frac{{r_0}^2}{\rho^2}(\mathrm{d}t + a\,\sin^2\theta\,\mathrm{d}\phi + b\,\cos^2\theta\,\mathrm{d}\psi)^2,$$

where

$$\rho^2 = x + a^2\cos^2\theta + b^2\sin^2\theta, \qquad \Delta = (x + a^2)(x + b^2) - {r_0}^2 x.$$

After normalizing two null vectors L_+ and L_- given in (4.23) in [10], one can appropriately choose the rest of the frame vectors $\boldsymbol{m}^1$, $\boldsymbol{m}^2$ and $\boldsymbol{m}^3$. It turns out that all components of the Weyl tensor with the boost weights 2, 1, -1, -2 vanish and hence the spacetime is of the algebraic type D [1, 3]. One can also explicitly calculate the matrix $\boldsymbol{S}$. While the form of $\boldsymbol{S}$ depends on the choice of $\boldsymbol{m}^1$, $\boldsymbol{m}^2$ and $\boldsymbol{m}^3$, the characteristic polynomial of $\boldsymbol{S}$, $P_\lambda(\boldsymbol{S})$, does not. For $P_\lambda(\boldsymbol{S})$ we obtain

$$P_\lambda(\boldsymbol{S}) = \left(\lambda - \frac{1}{\sqrt{x}}\right)\left(\lambda - \frac{\sqrt{x}}{\rho^2}\right)^2 \tag{D.1}$$

and the diagonal form of $\boldsymbol{S}$ is

$$\boldsymbol{S}_{\text{diag}} = \begin{pmatrix} \frac{1}{\sqrt{x}} & 0 & 0 \\ 0 & \frac{\sqrt{x}}{\rho^2} & 0 \\ 0 & 0 & \frac{\sqrt{x}}{\rho^2} \end{pmatrix}. \tag{D.2}$$

From (20) it follows that shear is

$$\sigma = \sqrt{\frac{2}{3x}} \left(1 - \frac{x}{\rho^2}\right), \tag{D.3}$$

which is in accordance with equation (4.25) in [10]. Note that for $a = b = 0$ all eigenvalues of $\boldsymbol{S}$ are equal and shear is thus zero. Whereas for $a^2 + b^2 > 0$ the matrix $\boldsymbol{S}$ has two equal and one distinct eigenvalues and shear does not vanish. Note also that similarly as for types III and N, this algebraically special vacuum solution has non-vanishing shear (see also footnote 4 in paper [10]). Moreover, in this spacetime $\boldsymbol{S}$ does not have the properties proved in sections 3 and 4 for vacuum type N and type III spacetimes.

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