Vanishing scalar invariant spacetimes in higher dimensions

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Abstract
We study manifolds with Lorentzian signature and prove that all scalar curvature invariants of all orders vanish in a higher dimensional Lorentzian spacetime if and only if there exists an aligned non-expanding, non-twisting, geodesic null direction along which the Riemann tensor has negative boost order.

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1. Introduction

Recently [1] it was proven that in four-dimensional (4D) pseudo-Riemannian or Lorentzian spacetimes all of the scalar invariants constructed from the Riemann tensor and its covariant derivatives of arbitrary order are zero if and only if the spacetime is of Petrov-type III, N or O, all eigenvalues of the Ricci tensor are zero (the Ricci tensor is consequently of Plebański–Petrov-type (PP-type) N or O, or alternatively, of Segre-type {(31)}, {(211)} or {(1111)}) and the common multiple null eigenvector of the Weyl and Ricci tensors is geodesic, shear-free, non-expanding and non-twisting; we shall refer to these spacetimes as vanishing scalar invariant (VSI) spacetimes. An equivalent characterization of VSI spacetimes in 4D is that there exists an aligned shear-free, non-expanding non-twisting, geodesic null direction ℓ^a along which the Riemann tensor has negative boost order. For Petrov-type O, the Weyl tensor vanishes and so it suffices that the null vector field ℓ^a associated with the Ricci tensor is again geodesic, shear-free, non-expanding and non-twisting. All of these spacetimes belong to Kundt's class, and hence the metric of these spacetimes can be expressed in an appropriate form in adapted coordinates [2, 3]. VSI spacetimes can be classified according to their Petrov-type, Segre type and the vanishing or non-vanishing of the quantity τ. This leads to 16 non-trivial distinct classes of VSI spacetimes, one of which is the vacuum pp-wave (Petrov-type N, vacuum, $\tau = 0$) spacetime, in addition to the trivial flat Minkowski spacetime. All of the corresponding metrics are displayed in [1]. The generalized pp-wave solutions are of Petrov-type N, PP-type O (with $\tau = 0$), and admit a covariantly constant null vector field [4].

We shall study VSI spacetimes in arbitrary N dimensions (not necessarily even, but $N = 10$ is of particular importance from string theory) and, in principle, for arbitrary signature. However, we shall focus our attention on Lorentzian manifolds with signature $N - 2$. We note that for Riemannian manifolds with signature N, flat space is the only VSI manifold. Manifolds with signature $N - 4$ with $N \geqslant 5$ are also of physical interest [5, 6]. In [7] we investigated N-dimensional Lorentzian spacetimes in which all of the scalar invariants constructed from the Riemann tensor and its covariant derivatives are zero. These spacetimes are higher dimensional generalizations of N-dimensional pp-wave spacetimes, which have been of interest recently in the context of string theory in curved backgrounds in higher dimensions. We presented a canonical form for the Riemann and Weyl tensors in a preferred null frame in arbitrary dimensions if all of the scalar curvature invariants vanish (thereby generalizing the theorem of [1] to higher dimensions). We shall prove the assertions in this paper, and we shall briefly discuss the algebraic structure of the resulting spacetimes. In particular, we shall prove:

Theorem 1. *All curvature invariants of all orders vanish in an N-dimensional Lorentzian spacetime if and only if there exists an aligned non-expanding ($S_{ij} = 0$), non-twisting ($A_{ij} = 0$), geodesic null direction ℓ^a along which the Riemann tensor has negative boost order.*

An analytical form of the conditions in theorem 1 are as follows:
$$R_{abcd} = 8A_i \ell_{\{a} n_b \ell_c m^i{}_{d\}} + 8B_{ijk} m^i{}_{\{a} m^j{}_b \ell_c m^k{}_{d\}} + 8C_{ij} \ell_{\{a} m^i{}_b \ell_c m^j{}_{d\}} \tag{1}$$
(i.e., the Riemann tensor is of algebraic type III or N [9]), and
$$\ell_{a;b} = L_{11} \ell_a \ell_b + L_{1i} \ell_a m^i{}_b + L_{i1} m^i{}_a \ell_b; \tag{2}$$
that is, the expansion matrix $S_{ij} = 0$, the twist matrix $A_{ij} = 0$ (which are the analogues of ρ, σ in 4D; see section 1.2 for the definitions), as well as $L_{i0} = 0 = L_{10}$ (corresponding to an affinely parametrized geodesic congruence ℓ_a; i.e., analogues of κ, $\epsilon + \bar{\epsilon}$, respectively—see equation (58)).

In this paper we shall first summarize the N-dimensional null frame formalism, the algebraical classification of a tensor based on boost order [7–9] and the Bianchi identities and their consequnces for vacuum-type III and N spacetimes [10]. The sufficiency and necessity of theorem 1 are then proven in sections 2 and 3, 4, respectively. The paper concludes with a discussion. Many of the details of the analysis are found in the appendices.

1.0.1. Notation

We shall consider a null frame $\boldsymbol{\ell} = \boldsymbol{m}_0, \boldsymbol{n} = \boldsymbol{m}_1, \boldsymbol{m}_2, \ldots, \boldsymbol{m}_{N-1}$ ($\boldsymbol{\ell}, \boldsymbol{n}$ null with $\ell^a \ell_a = n^a n_a = 0, \ell^a n_a = 1, \boldsymbol{m}_i$ real and spacelike $m_i{}^a m_{ja} = \delta_{ij}, i = 2, \ldots, N - 1$, all other products vanish) in an N-dimensional Lorentz-signature space(time), so that
$$g_{ab} = 2\ell_{(a} n_{b)} + \delta_{jk} m^j{}_a m^k{}_b. \tag{3}$$
Covariance is relative to the group of linear Lorentz transformations. Throughout, Roman indices a, b, c, A, B, C range from 0 to $N - 1$. Lowercase indices indicate an arbitrary basis, while the uppercase ones indicate a null frame. Spacelike indices i, j, k also indicate a null frame, but vary from 2 to $N - 1$ only. We will raise and lower the spacelike indices using δ_{ij}; e.g., $T_i = \delta_{ij} T^j$. We will observe Einstein's summation convention for both of these types of indices; however, for indices $i, j, \ldots$ there is no difference between covariant and contravariant components and thus we will not distinguish between subscripts and superscripts.

We also introduce the notation (compare with [7, 9])
$$w_{\{a} x_b y_c z_{d\}} \equiv \tfrac{1}{2}(w_{[a} x_{b]} y_{[c} z_{d]} + w_{[c} x_{d]} y_{[a} z_{b]}) \equiv \tfrac{1}{8}\{[w_p x_q][y_r z_s]\}. \tag{4}$$

Table 1. Boost weights of the Weyl scalars.

2	1	0	−1	−2
C_{0i0j}	C_{010i}, C_{0ijk}	$C_{0101}, C_{01ij}, C_{0i1j}, C_{ijkl}$	C_{011i}, C_{1ijk}	C_{1i1j}

1.1. Background

A *null rotation* about n is a Lorentz transformation of the form

$$\hat{n} = n, \qquad \hat{m}_i = m_i + z_i n, \qquad \hat{\ell} = \ell - z_i m^i - \tfrac{1}{2}\delta^{ij} z_i z_j n. \tag{5}$$

A null rotation about ℓ has an analogous form. A boost is a transformation of the form

$$\hat{n} = \lambda^{-1} n, \qquad \hat{m}_i = m_i, \qquad \hat{\ell} = \lambda \ell, \qquad \lambda \neq 0. \tag{6}$$

A spin is a transformation of the form

$$\hat{n} = n, \qquad \hat{m}_i = X_i^j m_j, \qquad \hat{\ell} = \ell, \tag{7}$$

where X_i^j is an orthogonal matrix.

Let $T_{a_1,\ldots,a_p}$ be a rank p tensor. For a fixed list of indices $A_1, \ldots, A_p$, we call the corresponding $T_{A_1,\ldots,A_p}$ a null-frame scalar. These scalars transform under a boost (6) according to

$$\hat{T}_{A_1,\ldots,A_p} = \lambda^b T_{A_1,\ldots,A_p}, \qquad b = b_{A_1} + \cdots + b_{A_p}, \tag{8}$$

where

$$b_0 = 1, \qquad b_i = 0, \qquad b_1 = -1. \tag{9}$$

We call the above b the *boost weight* of the scalar. We define the *boost order* of the tensor $\mathbf{T}$, as a whole, to be the boost weight of its leading term [8].

We can then decompose the Riemann tensor and sort its components by boost weight:

$$R_{abcd} = \overbrace{4R_{0i0j}n_{\{a}m^i{}_b n_c m^j{}_{d\}}}^{2} + \overbrace{8R_{010i}n_{\{a}\ell_b n_c m^i{}_{d\}} + 4R_{0ijk}n_{\{a}m^i{}_b m^j{}_c m^k{}_{d\}}}^{1} \tag{10}$$

$$+ \left\{ \begin{matrix} +4R_{0101}n_{\{a}\ell_b n_c \ell_{d\}} + 4R_{01ij}n_{\{a}\ell_b m^i{}_c m^j{}_{d\}} \\ +8R_{0i1j}n_{\{a}m^i{}_b \ell_c m^j{}_{d\}} + R_{ijkl}m^i{}_{\{a}m^j{}_b m^k{}_c m^l{}_{d\}} \end{matrix} \right\}^{0}$$

$$+ \overbrace{8R_{101i}\ell_{\{a}n_b \ell_c m^i{}_{d\}} + 4R_{1ijk}\ell_{\{a}m^i{}_b m^j{}_c m^k{}_{d\}}}^{-1} + \overbrace{4R_{1i1j}\ell_{\{a}m^i{}_b \ell_c m^j{}_{d\}}}^{-2}.$$

The Weyl tensor C_{abcd} has a boost-weight decomposition analogous to (10). Table 1 shows the boost weights for the scalars of the Weyl curvature tensor C_{abcd}. Thus, generically C_{abcd} has boost order 2. If all C_{0i0j} vanish, but some C_{010i}, or C_{0ijk} do not, then the boost order is 1, etc. The Weyl scalars also satisfy a number of additional relations, which follow from curvature tensor symmetries and from the trace-free condition:

$$C_{0i0}{}^i = 0,$$

$$C_{010j} = C_{0ij}{}^i, \qquad C_{0(ijk)} = 0,$$

$$C_{0101} = C_{0i1}{}^i, \qquad C_{0i1j} = -\tfrac{1}{2}C_{ikj}{}^k + \tfrac{1}{2}C_{01ij}, \qquad C_{i(jkl)} = 0, \tag{11}$$

$$C_{011j} = -C_{1ij}{}^i, \qquad C_{1(ijk)} = 0,$$

$$C_{1i1}{}^i = 0.$$

A priori, the assignment of a boost order to a tensor seems to depend on the choice of a null frame [8]. However, a null rotation about ℓ fixes the leading terms of a tensor, while boosts and spins subject the leading terms to an invertible transformation. It follows that the boost order of a tensor is a function of the null direction k. We shall therefore denote boost order by $\mathcal{B}(k)$ (with the choice of a tensor C determined by context). We will say that a null vector k is *aligned* with the Weyl tensor C whenever $\mathcal{B}(k) \leqslant 1$ [8, 9]. We will call the integer $1 - \mathcal{B}(k) \in \{0, 1, 2, 3\}$ the *order of alignment*.

Definition 2. *We will say that the* principal type *of a Lorentzian manifold is I, II, III, N according to whether there exists an aligned k of alignment order 0, 1, 2, 3, respectively (i.e., $\mathcal{B}(k) = 1, 0, -1, -2$, respectively). If no aligned k exists we will say that the manifold is of type G. If the Weyl tensor vanishes, we will say that the manifold is of type O.*

It follows that there exists a frame in which the components of the Weyl tensor satisfies:

$$
\begin{aligned}
&\text{Type I:} && C_{0i0j} = 0, \\
&\text{Type II:} && C_{0i0j} = C_{0ijk} = 0, \\
&\text{Type III:} && C_{0i0j} = C_{0ijk} = C_{ijkl} = C_{01ij} = 0, \\
&\text{Type N:} && C_{0i0j} = C_{0ijk} = C_{ijkl} = C_{01ij} = C_{1ijk} = 0.
\end{aligned}
\tag{12}
$$

The general types have various algebraically special subtypes [9], which include (the following only lists the additional conditions for the algebraic specializations): Type Ia $C_{010i} = 0$, subclasses of type II (with $C_{0101} = 0$, the traceless Ricci part of $C_{ijkl} = 0$, the Weyl part of $C_{ijkl} = 0$ and $C_{01ij} = 0$) and type IIIa $C_{011i} = 0$. Note that the conditions for the type II subcases can be combined to yield composite types. The *full type* of the Weyl tensor includes identifying its principal type and subclass and multiplicities, etc [8]. The special type D is defined by the fact that in canonical form all terms are of boost weight zero. For type III tensors, the principle null direction (PND) of order 2 is unique. There are no PNDs of order 1, and at most 1 PND of order 0. (For $N = 4$ there is always exactly 1 PND of order 0; for $N > 4$ this PND need not exist.) For type N tensors, the order 3 PND is the only PND of any order.

1.2. Bianchi identities

Covariant derivatives of the frame vectors can be expressed as [10]

$$
\ell_{a;b} = L_{11}\ell_a\ell_b + L_{10}\ell_a n_b + L_{1i}\ell_a m^i{}_b + L_{i1}m^i{}_a\ell_b + L_{i0}m^i{}_a n_b + L_{ij}m^i{}_a m^j{}_b,
\tag{13}
$$

$$
n_{a;b} = -L_{11}n_a\ell_b - L_{10}n_a n_b - L_{1i}n_a m^i{}_b + N_{i1}m^i{}_a\ell_b + N_{i0}m^i{}_a n_b + N_{ij}m^i{}_a m^j{}_b,
\tag{14}
$$

$$
\begin{aligned}
m^i{}_{a;b} = {}&-N_{i1}\ell_a\ell_b - N_{i0}\ell_a n_b - L_{i1}n_a\ell_b - L_{i0}n_a n_b - N_{ij}\ell_a m^j{}_b \\
&+ \overset{i}{M}{}_{j1}m^j{}_a\ell_b - L_{ij}n_a m^j{}_b + \overset{i}{M}{}_{j0}m^j{}_a n_b + \overset{i}{M}{}_{kl}m^k{}_a m^l{}_b
\end{aligned}
\tag{15}
$$

(where L_{11} is the analogue of the spin coefficient $\gamma + \bar{\gamma}$ in 4D, etc).

Let us decompose L into its symmetric and antisymmetric parts, S and A,

$$
L_{ij} = S_{ij} + A_{ij}, \qquad S_{ij} = S_{ji}, \qquad A_{ij} = -A_{ji}.
\tag{16}
$$

If ℓ corresponds to a null geodesic congruence with an affine parametrization, we can express the expansion θ and the shear matrix σ_{ij} as

$$
\theta \equiv \frac{1}{n-2}\ell^a{}_{;a} = \frac{1}{n-2}[S],
\tag{17}
$$

$$\sigma_{ij} \equiv \left(\ell_{(a;b)} - \theta\delta_{kl}m^k{}_a m^l{}_b\right) m_i{}^a m_j{}^b = S_{ij} - \frac{[S]}{n-2}\delta_{ij}. \tag{18}$$

For simplicity, let us call A the *twist matrix* and S the *expansion matrix*, although S contains information about both expansion and shear. We also introduce the quantities

$$S \equiv \tfrac{1}{2}[S], \qquad A^2 \equiv \tfrac{1}{2}A_{ij}A_{ij}. \tag{19}$$

We next introduce directional derivatives D, $\triangle$ and δ_i by

$$D \equiv \ell^a \nabla_a, \qquad \triangle \equiv n^a \nabla_a, \qquad \delta_i \equiv m_i{}^a \nabla_a, \qquad \nabla_a = n_a D + \ell_a \triangle + m^i{}_a \delta_i. \tag{20}$$

Commutators then have the form

$$\triangle D - D\triangle = L_{11}D + L_{10}\triangle + L_{i1}\delta_i - N_{i0}\delta_i, \tag{21}$$

$$\delta_i D - D\delta_i = (L_{1i} + N_{i0})D + L_{i0}\triangle + (L_{ji} - \overset{i}{M}_{j0})\delta_j, \tag{22}$$

$$\delta_i \triangle - \triangle\delta_i = N_{i1}D + (L_{i1} - L_{1i})\triangle + (N_{ji} - \overset{i}{M}_{j1})\delta_j, \tag{23}$$

$$\delta_i \delta_j - \delta_j \delta_i = (N_{ij} - N_{ji})D + (L_{ij} - L_{ji})\triangle + (\overset{j}{M}_{ki} - \overset{i}{M}_{kj})\delta_k. \tag{24}$$

For type III and type N vacuum spacetimes we will use the notation of [10] with

$$\Psi_i = C_{101i}, \qquad \Psi_{ijk} = \tfrac{1}{2}C_{1kij}, \qquad \Psi_{ij} = \tfrac{1}{2}C_{1i1j}.$$

The Weyl tensor can thus be expressed as

$$C_{abcd} = 8\Psi_i \ell_{\{a}n_b\ell_c m^i{}_{d\}} + 8\Psi_{ijk}m^i{}_{\{a}m^j{}_b \ell_c m^k{}_{d\}} + 8\Psi_{ij}\ell_{\{a}m^i{}_b \ell_c m^j{}_{d\}}. \tag{25}$$

Case $\Psi_{ijk} \neq 0$ is of type III, while $\Psi_{ijk} = 0$ (and consequently also $\Psi_i = 0$) corresponds to type N. Note that Ψ_{ij} is symmetric and traceless. Ψ_{ijk} is antisymmetric in the first two indices and in vacuum also satisfies[3]

$$\Psi_i = 2\Psi_{ijj}, \tag{26}$$

$$\Psi_{\{ijk\}} = 0. \tag{27}$$

Further constraints on Ψ_{ij}, Ψ_{ijk}, S, A and ℓ can be obtained by employing the Bianchi and Ricci identities

$$R_{ab\{cd;e\}} = 0, \tag{28}$$

$$V_{a;bc} = V_{a;cb} + R^s{}_{abc}V_s, \tag{29}$$

where V is an arbitrary vector.

The implications of the Bianchi identities (28) for vacuum-type III and N spacetimes were studied in [10]. It was shown that for these spacetimes ℓ is geodesic. Furthermore, if they have non-vanishing expansion S and twist A and u^α, v and w are orthonormal elements of the vector space spanned by vectors $m^{(i)}$ (see sections III and IV in [10]) then

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[3] Note that we use two different operations denoted by { }. In the first case { } acts on three indices and stands for $R_{ab\{cd;e\}} = R_{abcd;e} + R_{abde;c} + R_{abec;d}$. In the other case { } acts on four indices and is given by (4).

(1) in vacuum type N spacetimes with $S \neq 0$ it is always possible to choose vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ for which

$$\Psi_{ij} = \sqrt{\frac{p}{2}} (v_i v_j - w_i w_j), \tag{30}$$

$$S_{ij} = S(v_i v_j + w_i w_j), \tag{31}$$

$$A_{kl} = A(w_k v_l - v_k w_l), \tag{32}$$

where

$$p \equiv \Psi_{ij} \Psi_{ij}; \tag{33}$$

(2) in vacuum type III spacetimes with $S \neq 0$, $A \neq 0$, $\Psi_i \neq 0$ and with a 'general form' of Ψ_{ijk} (see [10] for details) it is possible to introduce a vector

$$\Phi_i \equiv A_{ij} \Psi_j \tag{34}$$

and then express $\boldsymbol{S}$, $\boldsymbol{A}$ and Ψ_{ijk} as

$$S_{ij} = S \left(\frac{\Psi_i \Psi_j}{\psi^2} + \frac{\Phi_i \Phi_j}{\phi^2} \right), \tag{35}$$

$$A_{ij} = \frac{1}{\psi^2} (\Phi_i \Psi_j - \Phi_j \Psi_i), \tag{36}$$

$$\Psi_{ijk} = \frac{1}{2\phi^2} (\Psi_i \Phi_j - \Psi_j \Phi_i) \Phi_k + \frac{\mathcal{O}_{1\alpha1}}{\psi^2} \left(\Psi_i u_j^\alpha - \Psi_j u_i^\alpha \right) \Psi_k - \frac{\mathcal{O}_{1\alpha1}}{\phi^2} \left(\Phi_i u_j^\alpha - \Phi_j u_i^\alpha \right) \Phi_k$$
$$+ \frac{\mathcal{O}_{1\alpha2}}{\psi\phi} \left(\Psi_i u_j^\alpha - \Psi_j u_i^\alpha \right) \Phi_k + \frac{\mathcal{O}_{1\alpha2}}{\psi\phi} \left(\Phi_i u_j^\alpha - \Phi_j u_i^\alpha \right) \Psi_k, \tag{37}$$

where $\phi^2 \equiv \Phi_i \Phi_i$, $\psi^2 = \Psi_i \Psi_i$. We have treated all possible degenerate situations for the non-twisting case, $A_{ij} = 0$, in arbitrary dimensions (see appendix C.1 in [10]) and for the twisting case in five dimensions (see appendix C.2 in [10]), and they all lead to special cases of the solution (35)–(37) which are given explicitly in [10]. We assert that the solution (35)–(37) is a general solution even for the twisting case in arbitrary dimensions. We have proven this in all of the general cases, and although we have not rigorously proven this for all of the degenerate cases there is evidence that it is indeed true.

2. The sufficiency proof

In this section, we start with the assumptions of theorem 1 and then show that all curvature invariants of all orders vanish. The corresponding form of the Riemann tensor (1), which implies appropriate types for the Weyl and Ricci tensors, was given in [7]. The congruence corresponding to ℓ is geodesic with vanishing shear, twist and expansion; i.e.,

$$L_{i0} = L_{ij} = 0. \tag{38}$$

Let us, for simplicity, choose an affine parametrization and a parallely propagated frame. Thus,

$$L_{10} = 0, \qquad N_{i0} = 0, \qquad \overset{i}{M}_{j0} = 0. \tag{39}$$

Due to the Bianchi identities

$$R_{ab\{cd;e\}} = 0, \tag{40}$$

we can express how the operator D acts on A_i, B_{ijk} and C_{ij}. Corresponding results may be obtained by evaluating the Bianchi identities with various combinations of null-frame indices, and by using the form (1) of the Riemann tensor. For example, for indices $101i0$ we obtain

$$DA_i = 0. \tag{41}$$

The other two equations of interest are obtained by using, respectively, indices $ij1k0$ and $1i1j0$:

$$DB_{ijk} = 0, \tag{42}$$

$$DC_{ij} = -B_{kji}L_{k1} + A_iL_{[1j]} + \tfrac{1}{2}\overset{k}{M}_{ij}A_k + \tfrac{1}{2}\delta_j A_i. \tag{43}$$

Evaluating the Ricci identities

$$\ell_{a;bc} - \ell_{a;cb} = R^d_{abc}\ell_d \tag{44}$$

and using indices $10i$, $i01$ and 110, respectively, yields

$$DL_{1i} = 0, \tag{45}$$

$$DL_{i1} = 0, \tag{46}$$

$$DL_{11} = -L_{1i}L_{i1}. \tag{47}$$

From equations (45)–(47), it follows that

$$D^2L_{11} = 0. \tag{48}$$

Similarly, evaluating

$$n_{a;bc} - n_{a;cb} = R^d_{abc}n_d \tag{49}$$

with indices $ij0$ and $i10$, respectively, we get

$$DN_{ij} = 0, \tag{50}$$

$$DN_{i1} = -N_{ij}L_{j1} + A_i. \tag{51}$$

From equations (41), (46) and (50) we obtain

$$D^2N_{i1} = 0. \tag{52}$$

Evaluating

$$m^i{}_{a;bc} - m^i{}_{a;cb} = R^d_{abc}m^i{}_d \tag{53}$$

with indices $j0k$ and $j10$, respectively, we obtain

$$D\overset{i}{M}_{jk} = 0, \tag{54}$$

$$D\overset{i}{M}_{j1} = -\overset{i}{M}_{jk}L_{k1}. \tag{55}$$

From equations (46) and (54) we get

$$D^2\overset{i}{M}_{j1} = 0. \tag{56}$$

Note that from the previous equations it also follows that

$$D^2C_{ij} = 0. \tag{57}$$

Table 2. Properties of quantities which are relevant for our proof.

Quantity	Boost weight	'D-equation'
L_{11}	-1	$D^2 L_{11} = 0$
L_{1i}	0	$DL_{1i} = 0$
L_{i1}	0	$DL_{i1} = 0$
N_{i1}	-2	$D^2 N_{i1} = 0$
N_{ij}	-1	$DN_{ij} = 0$
$\overset{i}{M}_{jk}$	0	$D\overset{i}{M}_{jk} = 0$
$\overset{i}{M}_{j1}$	-1	$D^2 \overset{i}{M}_{j1} = 0$
A_i	-1	$DA_i = 0$
B_{ijk}	-1	$DB_{ijk} = 0$
C_{ij}	-2	$D^2 C_{ij} = 0$

Let us now express covariant derivatives of the frame vectors and commutators for a geodesic, affinely parametrized, expansion and twist-free ℓ and the rest of the frame parallely propagated along ℓ

$$\ell_{a;b} = L_{11}\ell_a\ell_b + L_{1i}\ell_a m^i{}_b + L_{i1}m^i{}_a\ell_b, \tag{58}$$

$$n_{a;b} = -L_{11}n_a\ell_b - L_{1i}n_a m^i{}_b + N_{i1}m^i{}_a\ell_b + N_{ij}m^i{}_a m^j{}_b, \tag{59}$$

$$m^i{}_{a;b} = -N_{i1}\ell_a\ell_b - L_{i1}n_a\ell_b - N_{ij}\ell_a m^j{}_b + \overset{i}{M}_{j1}m^j{}_a\ell_b + \overset{i}{M}_{kl}m^k{}_a m^l{}_b; \tag{60}$$

$$\triangle D - D\triangle = L_{11}D + L_{i1}\delta_i, \tag{61}$$

$$\delta_i D - D\delta_i = L_{1i}D, \tag{62}$$

$$\delta_i \triangle - \triangle\delta_i = N_{i1}D + (L_{i1} - L_{1i})\triangle + (N_{ji} - \overset{i}{M}_{j1})\delta_j, \tag{63}$$

$$\delta_i\delta_j - \delta_j\delta_i = (N_{ij} - N_{ji})D + (\overset{j}{M}_{ki} - \overset{i}{M}_{kj})\delta_k. \tag{64}$$

Now we can proceed with the proof in a similar way to that in four dimensions [1]. Thus, we will only outline the key points. In all of the proofs we recall that ℓ is geodesic and affinely parametrized, the expansion and twist matrices vanish and the frame is parallely propagated along ℓ.

Definition 3. *We shall say that a weighted scalar η with boost weight b is balanced if $D^{-b}\eta = 0$ for $b < 0$ and $\eta = 0$ for $b \geqslant 0$.*

In analogy with the proof in 4D (employing table 2 and the commutators (61), (62)) we obtain

Lemma 4. *If η is a balanced scalar then $L_{11}\eta$, $L_{1i}\eta$, $L_{i1}\eta$, $N_{i1}\eta$, $N_{ij}\eta$, $\overset{i}{M}_{j1}\eta$, $\overset{i}{M}_{kl}\eta$, $D\eta$, $\delta_i\eta$, $\triangle\eta$ are balanced as well.*

Definition 5. *A balanced tensor is a tensor whose components are all balanced scalars.*

From (20) and lemma 4 (together with the assumptions in this section), it then follows that

Lemma 6. *A covariant derivative of an arbitrary order of a balanced tensor is again a balanced tensor.*

The Riemann tensor in the form (1) is balanced from equations (41), (42) and (57). Consequently, all of its covariant derivatives are also balanced and thus all curvature invariants in this case vanish. This completes the sufficiency part of the proof.

3. The necessity proof for zeroth-order invariants

In this section, we prove that the Riemann tensor for VSI spacetimes necessarily has negative boost order.

A scalar formed from the contractions of the Riemann curvature tensor is an invariant quantity in the sense that the contraction yields the same answer with respect to every choice of basis. There is an infinite number of different ways to perform such contractions. Hence, a given curvature tensor has associated with it an infinite set of invariants, some of which are generically non-vanishing. In this section, we classify the algebraically special spacetimes characterized by the condition that all zeroth-order (i.e., algebraic) invariants formed from R_{abcd} vanish. Henceforth, we shall refer to such spacetimes as belonging to the VSI_0 class. We will show that such spacetimes are necessarily of principal type III, N or O, with an aligned Ricci tensor of types PP-N, PP-O (or vacuum).

A scalar invariant has boost weight zero. It follows immediately that if the boost order of the curvature tensor is negative along some aligned null direction ℓ^a, then all scalar invariants must vanish. The converse is much harder to prove. In 4D it is well known [2, 11] that the vanishing of the fundamental second- and third-degree Weyl invariants, $I = J = 0$, implies that the Petrov type (principal type) is III, N or O. The condition that the Ricci tensor either vanishes or is of type PP-N or PP-O and is aligned, follows easily.

In higher dimensions, an entirely different approach is needed. We define a curvature-like tensor to be a rank 4 tensor with the following index symmetries:

$$R_{abcd} = -R_{bacd} = R_{cdab}.$$

The class of curvature-like tensors is more general than the class of Riemann curvature tensors, because we do not impose the algebraic Bianchi condition

$$R_{abcd} + R_{acdb} + R_{adbc} = 0. \tag{65}$$

Indeed, curvature-like tensors may be best characterized as symmetric, rank 2 tensors with bivector indices. On occasion we will, therefore, write curvature-like tensors as $R_{\alpha\beta} = R_{\beta\alpha}$, where α, β are bivector indices as defined in appendix C.

With every curvature-like R_{abcd} we associate the rank 2, symmetric covariant

$$R_{ab} = R_{acb}{}^{c},$$

which we will call the Ricci covariant. In addition, we could raise a bivector index and consider the transformation of bivector space $R_\alpha{}^\beta$ (see appendix C). By taking powers and then lowering an index we obtain additional covariants, e.g., the following curvature-like tensors $R_{\alpha\beta}^{(k)}$,

$$R_{\alpha\beta}^{(2)} = R_\alpha{}^\gamma R_{\gamma\beta}, \qquad R_{\alpha\beta}^{(3)} = R_\alpha{}^\gamma R_\gamma{}^\delta R_{\delta\beta}, \qquad \text{etc.}$$

Our main result is the following.

Theorem 7. *Let R_{abcd} be a non-zero, curvature-like tensor with vanishing zeroth-order invariants. Then, $R_{\alpha\beta}^{(3)} = 0$, and there exists an aligned null direction ℓ^a along which the boost order is negative. Generically, that is for $R_{\alpha\beta}^{(2)} \neq 0$, the principal type is III, with*

$$R_{abcd} = 8R_{011i}n_{\{a}\ell_b\,\ell_c m^i{}_{d\}} + 4R_{1ijk}\ell_{\{a}m^i{}_b m^j{}_c m^k{}_{d\}} + 4R_{1i1j}\ell_{\{a}m^i{}_b\,\ell_c m^j{}_{d\}}. \tag{66}$$

If $R_{\alpha\beta}^{(2)} = 0$, then the principal type is N, with

$$R_{abcd} = 4R_{1i1j}\ell_{\{a}m^i{}_b\,\ell_c m^j{}_{d\}}. \tag{67}$$

The proof will be broken up into a number of lemmas that treat special cases (the proofs of the lemmas are given in the appendices).

Lemma 8. *A curvature-like tensor of pure boost weight zero possesses a non-vanishing invariant.*

Lemma 9. *Let R_{abcd} be a curvature-like tensor with vanishing zeroth-order invariants. If the Ricci covariant $R_{ab} \neq 0$, then there exists an aligned null direction ℓ^a along which the boost order of R_{abcd} is negative.*

Lemma 10. *Let S_{abcd} be a symmetric, rank 4 tensor with vanishing zeroth-order invariants. Then there exists an aligned null direction ℓ^a; i.e. $S_{0000} = 0$.*

Lemma 11. *Let R_{abcd} be a curvature-like tensor with vanishing zeroth-order invariants. If $R_{\alpha\beta}^{(2)} = 0$ and $R_{ab} = 0$, then there exists an aligned null direction ℓ^a along which the boost order is negative.*

Lemma 12. *Let R_{abcd} be a curvature-like tensor with negative boost order. If $R_{\alpha\beta}^{(2)} = 0$ and $R_{ab} = 0$, then the principal type is N; i.e., (67) holds.*

Proof of theorem 7. If $R_{ab} \neq 0$, then the theorem follows by lemma 9. Henceforth, we suppose that $R_{ab} = 0$. If $R_{\alpha\beta}^{(2)} = 0$, then the theorem follows by lemmas 11 and 12. Henceforth, we suppose that $R_{\alpha\beta}^{(2)} \neq 0$.

By proposition 20 of the appendix, $R_{\alpha\beta}$ is nilpotent. Hence, for some sufficiently large k we have

$$P_{\alpha\beta} = R_{\alpha\beta}^{(k)} \neq 0, \qquad P_{\alpha\beta}^{(2)} = 0.$$

We now consider two cases, depending on whether the covariant $P_{ab} = P_{acb}{}^c$ is non-zero or whether it vanishes. In the first case $P_{ab} \neq 0$, and we proceed as we did in the proof to lemma 9 by constructing a rank 2 covariant

$$Q_{ab} = \ell_a\ell_b,$$

and then showing that the boost order is negative. If $P_{ab} = 0$, then we apply lemmas 11 and 12 to P_{abcd} to find an aligned ℓ^a such that

$$P_{abcd} = 4P_{1i1j}\ell_{\{a}m^i{}_b\,\ell_c m^j{}_{d\}}.$$

Consider the following rank 4 covariant

$$Q_{abcd} = P_{aebf}P_c{}^e{}_d{}^f = \lambda\ell_a\ell_b\ell_c\ell_d, \qquad \lambda = \sum_{ij}(P_{1i1j})^2 > 0.$$

Proceeding as in the proof of lemma 9, we can show that

$$R_{0i01} = R_{0ijk} = 0.$$

Since components of negative boost weight cannot contribute to an invariant, lemma 8 implies that the weight-zero components of R_{abcd} vanish. The theorem is thus proven. $\square$

4. Necessity proof

In this section, we prove the necessity part of theorem 1 assuming that the Riemann tensor has negative boost weight; i.e., the Weyl tensor is of type III, N or O and the Ricci tensor is of type N or O as was shown in theorem 7.

We will explicitly express two differential Weyl invariants for type N and III vacuum spacetimes. These differential invariants were originally used for similar proofs in 4D [13, 1, 14] and, as in 4D, they are zero only if both the expansion and twist vanish. Though the resulting expressions are simple, the calculations are quite extensive even with the use of MAPLE.

We cannot repeat similar calculations for non-vacuum spacetimes since, at present, the consequences of the Bianchi identities for non-vacuum spacetimes have not been fully studied and thus we cannot employ possible non-vacuum analogues of (30)–(37). For non-vacuum spacetimes we thus prove existence of non-vanishing differential Ricci invariants if ℓ is not geodesic or if it is expanding or twisting. The explicit form of these invariants may be reconstructed from the proof if necessary.

4.1. Type N vacuum spacetimes

Curvature invariants of the zeroth- and first-order (i.e., invariants containing the Riemann tensor and its first covariant derivative) vanish. Curvature invariants of the second order lead to invariants of matrices Ψ_{ij} (25) and L (16). Let us explicitly calculate the second-order invariant

$$I = C^{abcd;rs} C_{amcn;rs} C^{tmun;vw} C_{tbud;vw} \tag{68}$$

given in [13]. This expression can be rewritten as a polynomial invariant constructed from the components of the matrices Ψ and L which contain more than a thousand terms. A typical term is, for example,

$$2^7 \Psi_{ij} \Psi_{il} \Psi_{mr} \Psi_{kt} L_{jk} L_{lm} L_{np} L_{nq} L_{rp} L_{rq} L_{rs} L_{ts},$$

which, due to lemma 1 in [10], can be simplified to $2^7 p^2 (S^2 + A^2)^4$. We note that it is efficient to use part (f) of this lemma as often as possible, and only afterwards to decompose L into S and A and use the remaining equations in lemma 1. Extensive algebraical calculations in MAPLE lead to

$$I = 3^2 2^{10} p^2 (S^2 + A^2)^4. \tag{69}$$

This invariant clearly vanishes only if both quantities S and A are equal to zero, and we have thus completed the necessity part of the proof for type N vacuum spacetimes.

Let us check this formula in 4D. The relations between the frame vectors m^2 and m^3 and the standard null-tetrad vectors m and $\bar{m}$ are

$$m^a = \frac{1}{\sqrt{2}} \left(m_2{}^a - i m_3{}^a \right), \qquad \bar{m}^a = \frac{1}{\sqrt{2}} \left(m_2{}^a + i m_3{}^a \right). \tag{70}$$

In 4D, Ψ has two independent real components, Ψ_{22} and Ψ_{23}, which are related to the complex $\psi_4 = C_{abcd} n^a \bar{m}^b n^c \bar{m}^d$ by

$$\psi_4 = 2(\Psi_{22} + i\Psi_{23}), \qquad \bar{\psi}_4 = 2(\Psi_{22} - i\Psi_{23}), \qquad \psi_4 \bar{\psi}_4 = 4\left(\Psi_{22}{}^2 + \Psi_{23}{}^2\right). \tag{71}$$

Now we can recover the formula for the invariant I in 4D [13]:

$$I_{4D} = 3^2 2^8 (\theta^2 + \omega^2)^4 \psi_4^2 \bar{\psi}_4^2, \tag{72}$$

by using $S = \theta$, $A = \omega$, $p = \frac{1}{2} \psi_4 \bar{\psi}_4$.

4.2. Type III vacuum spacetimes

The zeroth-order curvature invariants again vanish; however, we can calculate the first-order non-vanishing invariant

$$I_{\text{III}} = C^{abcd;e} C_{amcn;e} C^{lmrn;s} C_{lbrd;s} \tag{73}$$

given in [14]. Due to equations (35)–(37), we can express this invariant, after extensive calculations in MAPLE, as

$$I_{\text{III}} = 64(S^2 + A^2)^2 [9\psi^2 + 27\psi(\mathcal{O}_{PP} + \mathcal{O}_{PF}) + 28(\mathcal{O}_{PP} + \mathcal{O}_{PF})^2], \tag{74}$$

where $\mathcal{O}_{PP} = \mathcal{O}_{P\alpha P}\mathcal{O}_{P\alpha P}$, $\mathcal{O}_{PF} = \mathcal{O}_{P\alpha F}\mathcal{O}_{P\alpha F}$.

We note that $\mathcal{O}_{PP}$, $\mathcal{O}_{PF}$ and ψ are non-negative and for type III spacetimes at least one of them is positive and thus the VSI condition for type III vacuum spacetimes implies $S = A = 0$. We also note that the solution (35)–(37) is expressed for simplicity in a frame which is adapted to the twisting case with non-vanishing Ψ_i. However, this solution may also be expressed in another frame in which we obtain the non-twisting case by simply putting $A = 0$ and the case with $\Psi_i = 0$ by putting $\Psi_i = 0$. Thus, we can obtain the resulting expressions for the invariant I_{III} in these special cases by substituting $A = 0$ or $\psi = 0$ in equation (74). We remark that the completeness of the proof for the vacuum type III case relies on the solution (35)–(37) being a general solution. Although there is good analytical evidence to support this for six dimensions and there is some support in higher dimensions, we have not rigorously proven this in all the degenerate cases (of measure zero) for the twisting case in dimension six and higher.

4.3. Ricci invariants

In this section, we show that for non-vacuum PP-N and PP-O spacetimes there exist non-vanishing first- and second-order (in derivatives) Ricci invariants if ℓ is not geodesic or if it admits expansion or twist. We start by proving several lemmas and then we apply them to the PP-N and PP-O cases separately.

Lemma 13. *If there exists a null frame in which a tensor $\boldsymbol{T}$ of rank $\boldsymbol{2}$ is of pure boost order zero, then $\boldsymbol{T}$ possesses a non-vanishing invariant.*

Proof. We prove this lemma by contradiction by assuming that a second rank tensor of pure boost order zero,

$$T_{ab} = T_{01}n_a\ell_b + T_{10}\ell_a n_b + T_{ij}m^i{}_a m^j{}_b, \tag{75}$$

has vanishing algebraical scalar invariants. Consequently,

$$T_{ac}^{(2)} \equiv T_{ab}T_c{}^b = T_{01}T_{10}(n_a\ell_c + \ell_a n_c) + T_{is}T_{js}m^i{}_a m^j_c \tag{76}$$

and

$$T_{ac}^{(2)} T^{(2)ac} = 2(T_{01})^2(T_{10})^2 + T_{ij}^{(2)} T_{ij}^{(2)}, \tag{77}$$

where

$$T_{ij}^{(2)} = T_{si}T_{sj}. \tag{78}$$

Thus, if $\boldsymbol{T}$ has vanishing algebraical invariants then

$$T_{ij}^{(2)} = 0 \Rightarrow T_{ij} = 0 \qquad \text{and} \qquad T_{01}T_{10} = 0. \tag{79}$$

Now, considering that

$$T_{ab}T^{ba} = (T_{01})^2 + (T_{10})^2 + T_{ij}T_{ji} \tag{80}$$

we conclude that

$$T_{01} = T_{10} = 0 \tag{81}$$

and thus a non-vanishing T cannot have vanishing algebraical invariants. □

Lemma 14. *If there exists a null frame in which a tensor T of rank **2** has boost order zero, then T possesses a non-vanishing invariant.*

Proof. Every tensor T with boost order zero can be divided into two parts

$$T = T^{(0)} + T^{(-)} \tag{82}$$

where $T^{(0)}$ is of pure boost order zero and $T^{(-)}$ has negative boost order.

Clearly $T^{(-)}$ does not affect any invariant of T, and thus this lemma is a direct consequence of lemma 13. □

Lemma 15. *If there exists a null frame in which a tensor T of rank **3** has pure boost order zero, then T possesses a non-vanishing invariant.*

Proof. We again prove this lemma by contradiction. A general form of a third rank tensor of pure boost order zero is

$$T_{abc} = T_{01i} n_a \ell_b m_c^i + T_{10i} \ell_a n_b m_c^i + T_{0i1} n_a m_b^i \ell_c + T_{1i0} \ell_a m_b^i n_c$$
$$+ T_{i01} m_a^i n_b \ell_c + T_{i10} m_a^i \ell_b n_c + T_{ijk} m_a^i m_b^j m_c^k. \tag{83}$$

Let us now construct from T several second rank pure boost order zero tensors. If T has vanishing algebraical invariants, then all of these tensors have to vanish according to lemma 13. In fact, we do not need to express these tensors fully, we just need some of their components:

$$T_{abc} T_d{}^{cb} \ell^a n^d = T_{01i} T_{1i0} + T_{0i1} T_{10i}, \tag{84}$$

$$T_{abc} T^a{}_d{}^b \ell^c n^d = T_{01i} T_{1i0} + T_{i10} T_{i10}, \tag{85}$$

$$T_{abc} T^a{}_d{}^b n^c \ell^d = T_{0i1} T_{10i} + T_{i01} T_{i01}. \tag{86}$$

An appropriate linear combination of equations (84)–(86) leads to

$$T_{i10} T_{i10} + T_{i01} T_{i01} = 0 \tag{87}$$

and thus

$$T_{i10} = T_{i01} = 0. \tag{88}$$

Other components of pure boost order zero tensors of rank 2 are

$$T_{abc} T^b{}_d{}^c \ell^a n^d = T_{01i} T_{01i} + T_{0i1} T_{i10}, \tag{89}$$

$$T_{abc} T^b{}_d{}^c n^a \ell^d = T_{10i} T_{10i} + T_{1i0} T_{i01}, \tag{90}$$

$$T_{abc} T_d{}^{ba} \ell^c n^d = T_{1i0} T_{1i0} + T_{i10} T_{10i}, \tag{91}$$

$$T_{abc} T_d{}^{ba} n^c \ell^d = T_{0i1} T_{0i1} + T_{i01} T_{01i}. \tag{92}$$

From the vanishing of equations (88)–(92), it follows that

$$T_{01i} = T_{10i} = T_{1i0} = T_{0i1} = 0. \tag{93}$$

Now we need only one last invariant

$$T_{abc} T^{abc} = 2T_{01i} T_{10i} + 2T_{0i1} T_{1i0} + 2T_{i01} T_{i10} + T_{ijk} T_{ijk} \tag{94}$$

which thanks to equations (88) and (93) vanishes only if

$$T_{ijk} = 0. \tag{95}$$

Thus, if T has vanishing algebraical invariants then T vanishes. □

4.3.1. PP-N spacetimes. For PP-N spacetimes the Ricci tensor has the form [7]

$$R_{ab} = K_i \left(\ell_a m_b^i + m^i{}_a \ell_b \right) + A \ell_a \ell_b, \tag{96}$$

with $K_i \neq 0$ for at least one value of i.

Now a tensor of rank 3

$$T_{abc} \equiv R_{ab;\sigma} R_c^\sigma \tag{97}$$

has boost order zero, and thus (lemma 15) for VSI spacetimes all of its components with boost weight zero have to vanish. By expressing the component

$$T_{ijk} = K_i L_{j0} K_k + L_{i0} K_j K_k, \tag{98}$$

and multiplying this equation by L_{i0} and by contracting k with j, we obtain

$$(K_i L_{i0})^2 + L_{i0} L_{i0} K_j K_j = 0 \tag{99}$$

which implies

$$L_{i0} = 0; \tag{100}$$

i.e., ℓ is geodesic, and thus also

$$\overset{i}{M}_{00} = 0. \tag{101}$$

Assuming that equations (100) and (101) are satisfied, we find that the first covariant derivative of the Ricci tensor $R_{ab;c}$ has the boost order 0 and thus for VSI spacetimes all of its components of boost weight zero must vanish. From

$$R_{ij;k} = L_{jk} K_i + L_{ik} K_j = 0, \tag{102}$$

and contracting i with j, it follows that

$$L_{ik} K_i = 0. \tag{103}$$

Multiplying equation (102) by K_j and using (103) leads to

$$K_j K_j L_{ik} = 0 \tag{104}$$

which implies that

$$L_{ij} = 0; \tag{105}$$

i.e., the expansion and twist matrices are zero.

4.3.2. PP-O spacetimes. For PP-O spacetimes the Ricci tensor has the form [7]

$$R_{ab} = A \ell_a \ell_b. \tag{106}$$

The covariant derivative $R_{ab;c}$, has boost order 0. The boost weight zero component

$$R_{1i;0} = A L_{i0} \tag{107}$$

has to vanish from the VSI condition and thus

$$L_{i0} = 0 \tag{108}$$

and ℓ is geodesic. We choose the affine parametrization with $L_{10} = 0$. Now, the second rank tensor $R_{ab;c}{}^c$ has boost order 0. Thus, the component with boost weight zero

$$R_{ij;A}{}^A = 2A L_{jk} L_{ik} \tag{109}$$

has to vanish from the VSI condition and consequently

$$L_{ij} = 0, \tag{110}$$

i.e., the expansion and twist matrices are zero.

5. Discussion

We have proven that in N-dimensional Lorentzian spacetimes the boost order of the Riemann tensor is negative along some aligned non-expanding, non-twisting, geodesic null direction ℓ^a if and only if all scalar curvature invariants vanish (generalizing a previous theorem in 4D [1]). We emphasize that even though our main focus has been the Lorentzian geometry case, some results from the more general theory (N-dimensional, real vector space equipped with an inner product g_{ab} with no assumption about the signature) have been developed.

In the Lorentzian case, we have provided strong evidence for the following conjecture (*the algebraic VSI conjecture*): any tensor with vanishing algebraic scalar invariants must necessarily have negative boost order along some aligned null direction. We have given a proof of this conjecture for arbitrary dimensions and for the following tensor types: bivectors in proposition 27, symmetric rank 2 tensors in corollary 24, and for the general class of curvature-like tensors in theorem 7. Lemmas 14 and 15 provide additional evidence for this conjecture.

We note that all of the VSI spacetimes have a shear-free, non-expanding, non-twisting geodesic null congruence $\ell = \partial_v$, and hence belong to the 'generalized Kundt' class [7].

There is a number of potentially important physical applications of VSI spacetimes. For example, it is known that a wide range of VSI spacetimes (in addition to the pp-wave spacetimes [15, 16]) are exact solutions in string theory (to all perturbative orders in the string tension) [17]. Recently, type IIB superstrings in pp-wave backgrounds with an RR five-form field were also shown to be exactly solvable [18]. Indeed, many authors [15, 19] have investigated string theory in pp-wave backgrounds in order to search for connections between quantum gravity and gauge theory dynamics.

In the context of string theory, it is of considerable interest to study Lorentzian spacetimes in higher dimensions. In particular, higher dimensional generalizations of pp-wave backgrounds have been considered [19, 20], including string models corresponding not only to the NS–NS but also to certain R–R backgrounds [21, 22], and pp-waves in 11- and 10-dimensional supergravity theory [23]. In addition, a number of classical solutions of branes [24] in higher dimensional pp-wave backgrounds have been studied in order to better understand the non-perturbative dynamics of string theories. In particular, a class of pp-wave string spacetimes supported by non-constant NS–NS H_3 or R–R F_p form fields were shown to be exact type II superstring solutions to all orders in the string tension [22, 25]. In this class of 10-dimensional superstring theory models the pp-wave metric, the NS–NS 2-form potential and the 3-form H_3 background, which depends on arbitrary harmonic functions $b_m(x)$ ($\partial^2 b_m = 0$, $m = 1, 2, \ldots$) of the transverse coordinates x_i, are given by [22]

$$\mathrm{d}s^2 = \mathrm{d}u\,\mathrm{d}v + K(x)\,\mathrm{d}u^2 + \mathrm{d}x_i^2 + \mathrm{d}y_m^2, \qquad i = 1, \ldots, d, \quad m = d + 1, \ldots, 8, \tag{111}$$

$$B_2 = b_m(x)\,\mathrm{d}u \wedge \mathrm{d}y_m, \qquad H_3 = \partial_i b_m(x)\,\mathrm{d}x_i \wedge \mathrm{d}u \wedge \mathrm{d}y_m \tag{112}$$

where the only non-zero component of the generalized curvature is

$$\hat{R}_{uiuj} = -\tfrac{1}{2}\partial_i\partial_j K - \tfrac{1}{2}\partial_i b_m \partial_j b_m. \tag{113}$$

These solutions are consequently of PP-type O and of principal (algebraic Weyl) type N.

Acknowledgments

Appendix A. Indefinite signature inner products

Let $0 < p < q$ be integers, and let $\mathbf{R}^{p,q}$ denote $\mathbf{R}^{p+q}$ equipped with a signature (p, q) inner product

$$\sum_{\lambda=1}^{p} X^\lambda Y^\lambda - \sum_{\lambda=p+1}^{p+q} X^\lambda Y^\lambda, \qquad X^\lambda,\ Y^\lambda \in \mathbf{R}^{p+q}.$$

Let $SO_{p,q}$ be the corresponding group of (p, q)-orthogonal transformations. We consider a collection $X_i{}^\lambda$ of mutually orthogonal null vectors. In other words, for all values of collection indices i, j we have

$$\sum_{\lambda=1}^{p} X_i{}^\lambda X_j{}^\lambda - \sum_{\lambda=p+1}^{p+q} X_i{}^\lambda X_j{}^\lambda = 0. \tag{114}$$

Proposition 16. *A collection of mutually orthogonal null vectors, $X_i{}^\lambda \in \mathbf{R}^{p,q}$, has at most p linear independent elements.*

Proposition 17. *Let $X_i{}^\lambda \in \mathbf{R}^{p,q}$ be a collection of p, or fewer, linear independent, mutually orthogonal null vectors. Then, there exists a (p, q) isometry $T^\lambda{}_\mu \in SO_{p,q}$ such that the transformed collection of mutually orthogonal vectors*

$$Y_i{}^\lambda = T^\lambda{}_\mu X_i{}^\mu$$

has the form $Y_i{}^i = Y_i{}^{i+p} = 1$, with all other components 0.

Corollary 18. *Let $X_i{}^\lambda \in \mathbf{R}^{p,q}$ be an arbitrary collection of mutually orthogonal null vectors. Then, there exists a (p, q) isometry $T^\lambda{}_\mu \in SO_{p,q}$ such that the transformed collection of mutually orthogonal vectors*

$$Y_i{}^\lambda = T^\lambda{}_\mu X_i{}^\mu$$

satisfies

$$Y_i{}^\lambda = Y_i{}^{\lambda+p}, \qquad \lambda = 1, \ldots, p, \qquad and \qquad Y_i{}^\lambda = 0, \quad \lambda > 2p.$$

Appendix B. The Petrov normal form

Even though our focus is Lorentzian geometry, some signature-independent results need to be developed. Let $g_{\delta\epsilon}$ be an N-dimensional, non-degenerate inner product; we make no assumptions about signature[4]. Let $\mathbf{T} = T_{\delta\epsilon}$ be a general, rank 2 tensor. For each $k = 1, 2, \ldots, N-1$ let

$$\sigma_k = T^{(k)}{}_\epsilon{}^\epsilon = T_{\epsilon_1}{}^{\epsilon_2} T_{\epsilon_2}{}^{\epsilon_3} \cdots T_{\epsilon_k}{}^{\epsilon_1}$$

denote the kth power invariant of $\mathbf{T}$. The following is well known [29]:

Theorem 19. *Every scalar invariant of $\mathbf{T}$ has a unique representation as a polynomial of the power invariants $\sigma_0, \ldots, \sigma_{N-1}$.*

---

[4] As a notational reminder we will use δ, ϵ, γ to index tensors in the more general setting, and reserve a, b, c as indices of tensors in a Lorentzian setting.

Corollary 20. *A rank 2 tensor* **T** *has vanishing zeroth-order invariants (i.e., is VSI_0) if and only if it is nilpotent; i.e.,* $\mathbf{T}^{(k)} = 0$ *for some* $k \geqslant 0$.

Next, we consider a symmetric, rank 2 tensor $Q_{\delta\epsilon} = Q_{\epsilon\delta}$. The normal forms described below are a specialization of the normal forms described by Petrov [12], and the proof of the following result can be found therein.

Theorem 21. *If a symmetric tensor $Q_{\delta\epsilon}$ belongs to the VSI_0 class, then there exists a basis,*

$$\overset{\lambda,\nu}{K}_{\epsilon}, \qquad \lambda = 1, \ldots, r; \qquad \nu = 1, \ldots, d_\lambda,$$

and a sequence of block signatures $\sigma_\lambda = \pm 1$, such that

$$g_{\delta\epsilon} = \sum_{\lambda,\nu} \sigma_\lambda \overset{\lambda,\nu}{K}_{(\delta} \overset{\lambda,\rho}{K}_{\epsilon)}, \tag{115}$$

$$Q_{\delta\epsilon} = \sum_{\substack{\lambda,\nu \\ d_\lambda>1}} \sigma_\lambda \overset{\lambda,\nu+1}{K}_{(\delta} \overset{\lambda,\rho}{K}_{\epsilon)}, \tag{116}$$

where $\lambda = 1, \ldots, r$, where $\nu = 1, \ldots, d_\lambda$, and we are letting $\rho = d_\lambda + 1 - \nu$.

Corollary 22. *Let $Q_{\delta\epsilon}$ be a symmetric, rank 2 tensor. If $Q^{(2)}{}_{\delta\epsilon} = 0$, then*

$$Q_{\delta\epsilon} = \sum_{\lambda=1}^{p} \overset{\lambda}{K}_{\delta} \overset{\lambda}{K}_{\epsilon} - \sum_{\lambda=p+1}^{p+q} \overset{\lambda}{K}_{\delta} \overset{\lambda}{K}_{\epsilon},$$

where the $\overset{\lambda}{K}_{\epsilon}$, $\lambda = 1, \ldots, p + q$, are mutually orthogonal null vectors; i.e., $\overset{p}{K}_{\epsilon} \overset{q}{K}^{\epsilon} = 0$.

Proof. Conditions (115) and (116) are equivalent to the statement that the linear transformation $Q_{\delta}{}^{\epsilon}$ has the action

$$\overset{\lambda,1}{K}_{\epsilon} \to \overset{\lambda,2}{K}_{\epsilon} \to \cdots \to \overset{\lambda,d_\lambda}{K}_{\epsilon} \to 0. \tag{117}$$

Since $Q^{(2)}{}_{\delta\epsilon} = 0$, we have that $d_\lambda = 1$ or 2 for all λ. We then rearrange our basis so that $\sigma_\lambda = 1, d_\lambda = 2$ for $\lambda = 1, \ldots, p$, so that $\sigma_\lambda = -1, d_\lambda = 2$ for $\lambda = p + 1, \ldots, p + q$, and so that $d_\lambda = 1$ for $\lambda > p + q$. We obtain the desired form by setting

$$\overset{\lambda}{K}_{\epsilon} = \overset{\lambda,2}{K}_{\epsilon}, \qquad \lambda = 1, \ldots, p + q. \qquad \qquad \square$$

Conditions (115) and (116) are equivalent to the assertion that $g_{\delta\epsilon}$ and $Q_{\delta\epsilon}$ can be simultaneously put into block diagonal form:

$$g_{\delta\epsilon} = \begin{pmatrix} \overset{1}{G} & & \\ & \ddots & \\ & & \overset{r}{G} \end{pmatrix}, \qquad Q_{\delta\epsilon} = \begin{pmatrix} \overset{1}{Q} & & \\ & \ddots & \\ & & \overset{r}{Q} \end{pmatrix},$$

such that the blocks have the form

$$\overset{\lambda}{G} = \begin{pmatrix} & & \sigma_\lambda \\ & \iddots & \\ \sigma_\lambda & & \end{pmatrix}, \qquad \overset{\lambda}{Q} = \begin{pmatrix} & & & \sigma_\lambda & 0 \\ & & \sigma_\lambda & 0 & \\ & \iddots & \iddots & & \\ \sigma_\lambda & 0 & & & \\ 0 & & & & \end{pmatrix}.$$

Thus, there are four kinds of blocks, depending on the block signature σ_p and on the block parity—a block being called even or odd according to whether d_λ is even or odd. The basis vectors of an even block are pairs of conjugate null vectors. The same is true for odd blocks, save that the middle vector is either a unit spacelike or a timelike vector depending on whether the signature σ_λ is +1 or −1, respectively. The following table summarizes the four possibilities, and the corresponding signatures of the blocks:

σ_λ	Parity	Signature $\overset{\lambda}{G}$	Signature $\overset{\lambda}{Q}$
+1	Even	$\left(\frac{1}{2}d_\lambda, \frac{1}{2}d_\lambda\right)$	$\left(\frac{1}{2}d_\lambda, \frac{1}{2}d_\lambda - 1\right)$
−1	Even	$\left(\frac{1}{2}d_\lambda, \frac{1}{2}d_\lambda\right)$	$\left(\frac{1}{2}d_\lambda - 1, \frac{1}{2}d_\lambda\right)$
+1	Odd	$\left(\frac{1}{2}(d_\lambda + 1), \frac{1}{2}(d_\lambda - 1)\right)$	$\left(\frac{1}{2}(d_\lambda - 1), \frac{1}{2}(d_\lambda - 1)\right)$
−1	Odd	$\left(\frac{1}{2}(d_\lambda - 1), \frac{1}{2}(d_\lambda + 1)\right)$	$\left(\frac{1}{2}(d_\lambda - 1), \frac{1}{2}(d_\lambda - 1)\right)$

Summing the signatures of all the blocks we arrive at the following result.

Proposition 23. *Suppose that a symmetric $Q_{\delta\epsilon}$ has vanishing zeroth-order invariants (i.e., is VSI_0). Then, the signature of the inner product $g_{\delta\epsilon}$ is given by*

$$\left(\frac{N + \sigma_o}{2}, \frac{N - \sigma_o}{2}\right), \qquad where \quad \sigma_o = \sum_{d_\lambda \text{ odd}} \sigma_\lambda; \tag{118}$$

and the signature of $Q_{\delta\epsilon}$ by

$$\left(\frac{N - \nu + \sigma_e}{2}, \frac{N - \nu - \sigma_e}{2}\right), \qquad where \quad \sigma_e = \sum_{d_\lambda \text{ even}} \sigma_\lambda. \tag{119}$$

As a particular case of the above proposition, suppose that g_{ab} has Lorentz signature, $(N - 1, 1)$. In this case, equation (118) is a very strong constraint on the size and number of odd and even blocks. Indeed, there can be at most one block of size 2 and signature (1, 1), or one block of size 3 and signature (2, 1). The possibilities are summarized below.

Corollary 24. *Suppose that g_{ab} has Lorentz signature. Then, there exists a null-frame $\ell^a, n^a, m_i{}^a$ relative to which a VSI_0 Q_{ab} takes on exactly one of the following normal forms:*

$$Q_{ab} = 0; \tag{120}$$

$$Q_{ab} = \pm \ell_a \ell_b; \tag{121}$$

$$Q_{ab} = \ell_{(a} m^2{}_{b)}. \tag{122}$$

Appendix C. Bivectors

Henceforth, we assume that g_{ab} has Lorentz signature. A bivector is a rank 2 skew-symmetric tensor. The vector inner product g_{ab} naturally induces a bivector inner product

$$g_{\alpha\beta} = \tfrac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc}), \qquad \alpha = (a, b), \quad \beta = (c, d).$$

We use $\alpha = (a, b), a < b$ to denote a bivector index, and henceforth use $\alpha, \beta, \gamma, \ldots$ to denote bivector indices. A bivector index can take on $N(N - 1)/2$ possible values; this

is the dimension of the vector space of all bivectors. The inner product of two bivectors $J_\alpha = J_{ab}$, $K_\alpha = K_{ab}$ can also be characterized as the total contraction

$$g_{\alpha\beta} J^\alpha K^\beta = J_\alpha K^\alpha = J_{ab} K^{ab}.$$

A bivector K_α will be called null if $K_\alpha K^\alpha = 0$. Every null-frame ℓ^a, n^a, $m_i{}^a$ induces a basis of bivectors consisting of the $N - 2$ pairs of conjugate, null bivectors $\ell_{[a} m^i{}_{b]}$, $n_{[a} m^i{}_{b]}$, the negative-norm bivector $\ell_{[a} n_{b]}$ and $\frac{1}{2}(N - 2)(N - 3)$ positive-norm bivectors $m^i_{[a} m^j{}_{b]}$. It follows that the bivector inner product $g_{\alpha\beta}$ has signature $\left(\frac{1}{2}(N - 1)(N - 2), N - 1\right)$.

It can be shown [27] that for even N a bivector K_α always admits at least one aligned, real null direction, while for odd N it is possible that there is no real aligned null direction. The boost weights of the components of K_{ab} given by

$$K_{ab} = \overbrace{2 K_{0i} n_{[a} m^i{}_{b]}}^{1} + \overbrace{2 K_{01} n_{[a} \ell_{b]} + K_{ij} m^i{}_{[a} m^j{}_{b]}}^{0} + \overbrace{2 K_{1i} \ell_{[a} m^i{}_{b]}}^{-1}.$$

For an aligned (or singly aligned) bivector we can set $K_{0i} = 0$ (but K_{1i} is not zero) and for a bi-aligned bivector we can set $K_{0i} = K_{1i} = 0$. We can classify bivectors into alignment types (using notation consistent with [8, 10]). We will say that a bivector is of type G if K_{0i} cannot be made to vanish and of type I if $K_{0i} = 0$ (but K_{01} does not vanish), and of algebraically special type II if $K_{0i} = K_{01} = K_{ij} = 0$. For type II, the bivector is aligned and K_{1i} cannot be made to vanish; i.e., there is no bi-aligned subclass. For type I, we will say that the bivector is bi-aligned if $K_{0i} = K_{1i} = 0$, and we shall refer to this case as type I_i (this case is akin to type II_{ii} in the classification of the Weyl tensor [9] and could perhaps also be referred to as type D). We also note that in 4D, types I_i and II are referred to as types I and N, respectively [28].

A collection of bivectors $\overset{\lambda}{K}_\alpha$ is null and mutually orthogonal if and only if

$$2 \overset{\lambda}{K}_{0i} \overset{\mu}{K}_1{}^i + 2 \overset{\lambda}{K}_{1i} \overset{\mu}{K}_0{}^i + \overset{\lambda}{K}_{ij} \overset{\mu}{K}^{ij} - 2 \overset{\lambda}{K}_{01} \overset{\mu}{K}_{01} = 0 \tag{123}$$

for all λ, μ.

Proposition 25. *Let $\overset{\lambda}{K}_\alpha$, $\lambda = 1, \ldots, r$ be a collection of null, mutually orthogonal, linearly independent bivectors. Then, $r \leqslant N - 1$.*

Proof. This follows directly from proposition 16. $\qquad\square$

Proposition 26. *Let $\overset{\lambda}{K}_\alpha$ be a collection of null, mutually orthogonal bivectors with a common alignment, i.e., $\overset{\lambda}{K}_{0i} = 0$. Then, there exists a null, bi-aligned bivector M_α and scalars C_λ such that $\overset{\lambda}{K}_{ij} = C_\lambda M_{ij}$ and $\overset{\lambda}{K}_{01} = C_\lambda M_{01}$.*

Proof. Consider the sequence of type I_i bivectors defined by

$$\overset{\lambda}{M}_{0i} = 0, \qquad \overset{\lambda}{M}_{01} = \overset{\lambda}{K}_{01}, \qquad \overset{\lambda}{M}_{ij} = \overset{\lambda}{K}_{ij}, \qquad \overset{\lambda}{M}_{1i} = 0.$$

For a fixed ℓ^a, the vector space of type I_i bivectors has signature $(1, (N - 2)(N - 3)/2)$. Hence, by proposition 16, the $\overset{\lambda}{M}_\alpha$ must be multiples of one another. $\qquad\square$

Proposition 27. *A bivector K_α has vanishing zeroth-order scalar invariants (VSI_0) if and only if it is of alignment type II.*

Proof. Since an invariant has boost weight zero, a K_{ab} with negative boost order must have vanishing scalar invariants. Let us now prove the converse; i.e., we assume that K_{ab} has vanishing scalar invariants and prove that necessarily the boost order is negative.

Let us consider the symmetric rank 2 covariant

$$R_{ab} = K_{ac}K_b{}^c.$$

By corollary 24, we can choose an aligned ℓ^a so that

$$R_{00} = R_{0i} = R_{01} = R_{ij} = 0.$$

Since

$$R_{00} = \sum_i (K_{0i})^2,$$

we have that $K_{0i} = 0$. From

$$R_{ii} = -2K_{0i}K_{1i} + \sum_k (K_{ik})^2, \qquad R_{01} = K_{01}^2 - \sum_i K_{0i}K_{1i}.$$

we infer that $K_{ij} = K_{01} = 0$, as was to be shown.   □

We note that these results may be of importance in the study of higher dimensional spacetimes with Maxwell-like fields [26].

Appendix D. The proofs of the lemmas

Proof of lemma 8. Let R_{abcd} be a curvature-like tensor with terms of zero boost weight only. We argue by contradiction, and suppose that R_{abcd} has vanishing scalar invariants. By proposition 20,

$$R^{(k)}{}_{abcd} = 0$$

for a sufficiently large k. Consequently, $R^{(j)}{}_{abcd}$ has vanishing zeroth-order invariants for all $j < k$. Since $R^{(j)}{}_{abcd}$ is of pure boost weight zero we may, without loss of generality, suppose that $R^{(2)}{}_{abcd} = 0$.

We decompose the curvature-like tensor as follows:

$$R_{abcd} = A_{abcd} + B_{abcd},$$

where

$$A_{abcd} = 8R_{0101}n_{\{a}\ell_b n_c \ell_{d\}} + 4R_{01ij}n_{\{a}\ell_b m^i{}_c m^j{}_{d\}} + R_{ijkl}m^i{}_{\{a}m^j{}_b m^k{}_c m^l{}_{d\}};$$
$$B_{abcd} = 8R_{0i1j}n_{\{a}m^i{}_b \ell_c m^j{}_{d\}}.$$

Evidently,

$$A_{\alpha\beta}B^\beta{}_\gamma = B_{\alpha\beta}A^\beta{}_\gamma = 0,$$

and hence,

$$A^{(2)}{}_{abcd} = B^{(2)}{}_{abcd} = 0. \tag{124}$$

Now, there are two cases to consider; either $A_{\alpha\beta}$ vanishes, or it does not. If it does vanish, then

$$R_{abcd}R^{adcb} = 4\sum_{ij}(R_{0i1j})^2$$

is a non-vanishing invariant. Thus, without loss of generality, $A_{\alpha\beta} \neq 0$.

Note that $A_{\alpha\beta}$ is a quadratic combination of type I_i bivectors. In appendix C, we showed that the vector space of type I_i bivectors has Lorentz signature. Hence, by equations (124) and corollary 22,

$$A_{\alpha\beta} = \pm K_\alpha K_\beta,$$

where K_α is a null, type I_i bivector; i.e.,

$$-2(K_{01})^2 + K_{ij} K^{ij} = 0.$$

Consequently, $K_{01} \neq 0$, and hence,

$$R_{0101} = A_{0101} = \pm(K_{01})^2 \neq 0.$$

The matrix $X_{ij} = B_{0i1j}$ is nilpotent by (124), and hence, $X_i{}^i = 0$. Let $R_{ab} = R_{acb}{}^c$ be the Ricci covariant. We have

$$R_{01} = -R_{0101} + R_{0i1}{}^i = -A_{0101} + X_i{}^i \neq 0.$$

Hence,

$$R_{ab} R^{ab} = (R_{01})^2 + R_{ij} R^{ij}$$

is a non-vanishing invariant. We have established a contradiction and hence proved the lemma.

$\square$

Proof of lemma 9. Corollary 24 gives normal forms for R_{ab} with vanishing invariants. We choose an aligned ℓ^a so that

$$R_{00} = R_{0i} = R_{01} = R_{ij} = 0. \tag{125}$$

If $R_{ab} = 0$, then all contractions vanish. Assuming that $R_{ab} \neq 0$, we can construct a non-vanishing covariant of the form

$$Q_{abcd} = \ell_a \ell_b \ell_c \ell_d.$$

To obtain this covariant we use $R_{ab} R_{cd}$ or $R_{ae} R^e{}_b R_{cf} R^f{}_d$, depending on whether R_{ab} has the form (121) or the form (122), respectively. The vanishing of the invariant

$$Q^{abcd} R_{aebf} R_c{}^e{}_d{}^f = \sum_{ij} (R_{0i0j})^2$$

implies that

$$R_{0i0j} = 0. \tag{126}$$

We set $T_{\alpha\beta} = R^{(2)}{}_{\alpha\beta}$ and note that the vanishing of the invariant

$$Q^{abcd} T_{aebf} T_c{}^e{}_d{}^f = \sum_{ij} (T_{0i0j})^2$$

implies that

$$T_{0i0j} = R_{0iab} R_{0j}{}^{ab} = 0. \tag{127}$$

We define the following sequence of bivectors:

$$\overset{i}{K}_{ab} = R_{0iab};$$

these are aligned because of equation (126). Also, by equation (127), we have that

$$T_{0i0j} = \overset{i}{K}_\alpha \overset{j}{K}{}^\alpha = 0.$$

By proposition 26, there exist M_{jk}, M_{01} and C_i such that

$$M_{jk}M^{jk} - 2M_{01}{}^2 = 0,\tag{128}$$

and such that

$$R_{0ijk} = \overset{i}{K}_{jk} = C_i M_{jk}, \qquad R_{0i01} = \overset{i}{K}_{01} = C_i M_{01}.\tag{129}$$

By equation (125),

$$R_{0j} = R_{01j0} + R_{0ij}{}^i = -C_j M_{01} + C_i M_j{}^i = 0.$$

Hence, since M_i is skew-symmetric, we have

$$C^j R_{0j} = -C^j C_j M_{01} + C_j C_i M^{ij} = -M_{01}\sum_j (C_j)^2 = 0.$$

Hence either $C_j = 0$ or $M_{01} = 0$. In the second case, $M_{ij} = 0$ and equation (128) is satisfied. In both cases, by equation (129)

$$R_{0i01} = R_{0ijk} = 0.$$

Since components of negative weight cannot contribute to an invariant, lemma 8 implies that the weight zero components of R_{abcd} also vanish. $\qquad\square$

The *proof of lemma 10* is given in [30].

Proof of lemma 11. Setting

$$S_{abcd} = R_{(a|ef|b}R_c{}^{ef}{}_{d)},$$

we have

$$S_{0000} = \sum_{ij}(R_{0i0j})^2.$$

Using lemma 10, we choose ℓ^a such that $S_{0000} = 0$, and hence

$$R_{0i0j} = 0.\tag{130}$$

By corollary 22,

$$R_{\alpha\beta} = \sum_{\lambda=1}^{p}\overset{\lambda}{K}_\alpha\overset{\lambda}{K}_\beta - \sum_{\lambda=p+1}^{p+q}\overset{\lambda}{K}_\alpha\overset{\lambda}{K}_\beta,\tag{131}$$

where, without loss of generality, $p < q$, and where the generating bivectors $\overset{\lambda}{K}_\alpha$ are linearly independent, null and mutually orthogonal (123). Set

$$X_i{}^\lambda = \overset{\lambda}{K}_{0i}, \qquad \lambda = 1, \ldots, p+q, \quad i = 2, \ldots, n-1,$$

and note that, by equation (130),

$$R_{0i0j} = \sum_{\lambda=1}^{p} X_i{}^\lambda X_j{}^\lambda - \sum_{\lambda=p+1}^{p+q} X_i{}^\lambda X_j{}^\lambda = 0,$$

for all $i, j = 2, \ldots, N-1$. Hence, by corollary 18 we may, without loss of generality, assume that

$$\overset{\lambda}{K}_{0i} = \overset{\lambda+p}{K}_{0i}, \quad \lambda = 1, \ldots p, \qquad \text{and} \qquad \overset{\lambda}{K}_{0i} = 0, \quad \lambda > 2p.\tag{132}$$

For $\lambda = 1, \ldots, p$, we set

$$\overset{\lambda}{E}_\alpha = \overset{\lambda}{K}_\alpha + \overset{\lambda+p}{K}_\alpha, \qquad \overset{\lambda}{F}_\alpha = \overset{\lambda}{K}_\alpha - \overset{\lambda+p}{K}_\alpha;$$

for $\lambda = p + 1, \ldots, q$ we set

$$\overset{\lambda}{F}_\alpha = \overset{\lambda+p}{K}_\alpha.$$

Now, equation (131) may be re-expressed as

$$R_{\alpha\beta} = \sum_{\lambda=1}^{p} 2 \overset{\lambda}{E}_{(\alpha} \overset{\lambda}{F}_{\beta)} + \sum_{\lambda=p+1}^{q} \overset{\lambda}{F}_\alpha \overset{\lambda}{F}_\beta. \tag{133}$$

By equation (132) the $\overset{\lambda}{F}_\alpha$ are aligned. Since they are null and mutually orthogonal, we have by proposition 26 that there exist F_{ij}, F_{01} and C_λ such that

$$F_{ij} F^{ij} - 2F_{01}{}^2 = 0, \tag{134}$$

and such that

$$\overset{\lambda}{F}_{ij} = C_\lambda F_{ij}, \qquad \overset{\lambda}{F}_{01} = C_\lambda F_{01}.$$

Setting

$$E_\alpha = \sum_{\lambda=1}^{p} C_\lambda \overset{\lambda}{E}_\alpha,$$

we have, by equation (133),

$$R_{0i01} = E_{0i} F_{01}, \qquad R_{0ijk} = E_{0i} F_{jk}.$$

The assumption $R_{ab} = 0$ implies that

$$R_{0i} = -R_{0i01} + R_{0ji}{}^j - R_{0i01} = -E_{0i} F_{01} + E_{0j} F_i{}^j = 0.$$

However, F_{ij} is skew-symmetric, and hence

$$E_{0i} R_0{}^i = -E_{0i} E_0{}^i F_{01} + E_{0i} E_{0j} F^{ij} = -F_{01} \sum_i (E_{0i})^2 = 0.$$

Therefore, either $E_{0i} = 0$, or $F_{01} = 0$. In the latter case, by equation (128) $F_{ij} = 0$ as well. In either case, the components of weight 1 necessarily vanish: $R_{0i01} = R_{0ijk} = 0$. Hence, by lemma 8, R_{abcd} has negative boost order. $\square$

Proof of lemma 12. We define the following sequence of bivectors:

$$\overset{i}{K}_{ab} = R_{1iab};$$

these are aligned because of the assumption of negative boost order. Also, by assumption, we have

$$R^{(2)}{}_{1i1j} = \overset{i}{K}_\alpha \overset{j}{K}{}^\alpha = 0.$$

Since $R_{1j} = 0$, we can adapt the argument at the end of lemma 9 to establish that

$$R_{1i01} = R_{1ijk} = 0. \qquad \square$$

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