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A note on the peeling theorem in higher dimensions

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Abstract

We demonstrate the 'peeling property' of the Weyl tensor in higher dimensions in the case of even dimensions (and with some additional assumptions), thereby providing a first step towards an understanding of the general peeling behaviour of the Weyl tensor, and the asymptotic structure at null infinity, in higher dimensions.

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1. Introduction

The study of higher dimensional manifolds in gravity theory is currently of great interest. A natural question arises whose results concerning four-dimensional gravity may be straighforwardly generalized to higher dimensions.

Recently the algebraic classification of the Weyl tensor in higher dimensional Lorentzian manifolds was presented by characterizing algebraically special Weyl tensors by means of the existence of aligned null vectors of various orders of alignment [1, 2]. This approach leads to a dimensionally independent classification scheme and reduces to the Petrov classification in four dimensions (4d).

Now it is of interest to determine whether some sort of peeling theorem is also valid in higher dimensions. Asymptotic 'peeling properties' of the Weyl tensor in physical four-dimensional spacetimes in general relativity can preferably be studied within the framework of conformal null infinity [3, 4], which was recently introduced also for higher even dimensions in [5] (see also [6] for a discussion of odd dimensions).

The purpose of this paper is to point out that under certain assumptions the peeling theorem is also valid in higher dimensions. Let us emphasize that in 4d the peeling theorem can be rigorously derived for asymptotically simple spacetimes using Einstein's equations. Here we simply assume that certain asymptotic properties of the spacetime are satisfied, which may be true only in particular situations (e.g., even dimensions). A more rigorous analysis of the peeling theorem in higher dimensions would be desirable. However, many necessary

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related concepts in higher dimensions are not well understood at present and even the physical importance of the concept of conformal infinity in higher dimensions is unclear. Perhaps the most promising approach would be to generalize the Bondi method [7, 8] for higher dimensions, but this may require a study of each dimension separately. This note should thus be regarded as a first step in the analysis of the validity of the peeling theorem in higher dimensions.

2. Peeling property of the Weyl tensor

We consider a D-dimensional spacetime (M, g_{ab}), D even, that is weakly asymptotically simple at null infinity [5]. The metric of an unphysical manifold $(\tilde{M}, \tilde{g}_{ab})$ with boundary $\Im$, is related to the physical metric by a conformal transformation $\tilde{g}_{ab} = \Omega^2 g_{ab}$, where $\Omega = 0$ and $\Omega_{;a} \neq 0$ is null at $\Im$. We note that such a spacetime is vacuum near $\Im$.

We further assume that components of the unphysical Weyl tensor with respect to the unphysical tetrad (see below) are of order $\mathcal{O}(\Omega^q)$ (with $q \geqslant 1$) in the neighbourhood of $\Im$. In 4D this follows from Einstein's equations (with $q = 1$). At this stage it is just a natural assumption from which 'peeling behaviour' in higher dimensions follows, and we do not study here to what extent it is satisfied in general. One possible method for studying the justifiability of this assumption in a given dimension is to use a generalized Bondi metric [5] for analysing asymptotic behaviour of the Weyl tensor. However, even in the simplest, six-dimensional, case this approach leads to very complicated calculations.

Let $\tilde{\gamma} \subset (\tilde{M}, \tilde{g}_{ab})$ be a null geodesic in the unphysical manifold that has an affine parameter $\tilde{r} \sim -\Omega$ near $\Im$ and a tangent vector $\tilde{\boldsymbol{\ell}}$ and $\gamma \subset (M, g_{ab})$ a corresponding null geodesic in the physical manifold with an affine parameter $r \sim 1/\Omega$ near $\Im$ and a tangent vector $\boldsymbol{\ell}$. In the physical spacetime we will use the frame $(\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}^{(i)})$ parallelly propagated along γ with respect to g_{ab}. $\boldsymbol{n}$ and $\boldsymbol{\ell}$ are null vectors satisfying $\ell^a n_a = 1$, and $\boldsymbol{m}^{(i)}$ are orthonormal spacelike vectors $(i, j = 2, \ldots, D-1, a, b, c, d = 0, \ldots, D-1)$. We choose the corresponding frame $(\tilde{\boldsymbol{\ell}}, \tilde{\boldsymbol{n}}, \tilde{\boldsymbol{m}}^{(i)})$ in the unphysical spacetime to be related to the physical one by

$$\begin{aligned} &\tilde{\ell}_a = \ell_a, && \tilde{m}^{(i)}_a = \Omega m^{(i)}_a, && \tilde{n}_a = \Omega^2 n_a \\ &\rightarrow \tilde{\ell}^a = \Omega^{-2}\ell^a, && \tilde{m}^{(i)a} = \Omega^{-1} m^{(i)a}, && \tilde{n}^a = n^a. \end{aligned} \tag{1}$$

The physical metric has the form

$$g_{ab} = 2\ell_{(a}n_{b)} + \delta_{ij} m^{(i)}_a m^{(j)}_b \tag{2}$$

that is preserved by null rotations

$$\hat{\boldsymbol{\ell}} = \boldsymbol{\ell} + z_i \boldsymbol{m}^{(i)} - \tfrac{1}{2} z^i z_i \boldsymbol{n}, \qquad \hat{\boldsymbol{n}} = \boldsymbol{n}, \quad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)} - z_i \boldsymbol{n}, \tag{3}$$

spins and boosts

$$\hat{\boldsymbol{\ell}} = \boldsymbol{\ell}, \quad \hat{\boldsymbol{n}} = \boldsymbol{n}, \quad \hat{\boldsymbol{m}}^{(i)} = X^i{}_j \boldsymbol{m}^{(j)}; \qquad \hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell}, \quad \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n}, \quad \hat{\boldsymbol{m}}^{(i)} = \boldsymbol{m}^{(i)}.$$

A quantity q is said to have a boost weight b if it transforms under a boost according to $\hat{q} = \lambda^b q$.

Let us now define the operation $\{\}$

$$w_{\{a} x_b y_c z_{d\}} \equiv \tfrac{1}{2}(w_{[a} x_{b]} y_{[c} z_{d]} + w_{[c} x_{d]} y_{[a} z_{b]})$$

which allows us to construct a basis from $(\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}^{(i)})$ in a vector space of four-rank tensors with symmetries $T_{abcd} = \frac{1}{2}(T_{[ab][cd]} + T_{[cd][ab]})$. The Weyl tensor can then be decomposed in its frame components with respect to the frame ($\boldsymbol{\ell} = \boldsymbol{m}^{(1)}, \boldsymbol{n} = \boldsymbol{m}^{(0)}, \boldsymbol{m}^{(i)}$) and these components can be sorted by their boost weight [1, 2]

$$
\begin{aligned}
C_{abcd} = {} & \overbrace{4C_{0i0j}n_{\{a}m_b^{(i)}n_c m_{d\}}^{(j)}}^{2} + \overbrace{8C_{010i}n_{\{a}\ell_b n_c m_{d\}}^{(i)} + 4C_{0ijk}n_{\{a}m_b^{(i)}m_c^{(j)}m_{d\}}^{(k)}}^{1} \\
& \left.\begin{aligned} & + 4C_{0101}n_{\{a}\ell_b n_c \ell_{d\}} + 4C_{01ij}n_{\{a}\ell_b m_c^{(i)}m_{d\}}^{(j)} \\ & + 8C_{0i1j}n_{\{a}m_b^{(i)}\ell_c m_{d\}}^{(j)} + C_{ijkl}m_{\{a}^{(i)}m_b^{(j)}m_c^{(k)}m_{d\}}^{(l)} \end{aligned}\right\}^{0} \\
& + \overbrace{8C_{101i}\ell_{\{a}n_b\ell_c m_{d\}}^{(i)} + 4C_{1ijk}\ell_{\{a}m_b^{(i)}m_c^{(j)}m_{d\}}^{(k)}}^{-1} + \overbrace{4C_{1i1j}\ell_{\{a}m_b^{(i)}\ell_c m_{d\}}^{(j)}}^{-2}.
\end{aligned} \tag{4}
$$

Components C_{0i0j} have boost weight 2 since they are proportional to $C_{abcd}\ell^a m^{(i)b}\ell^c m^{(j)d}$, and boost weights of other components may be determined similarly. Note that the frame components of the Weyl tensor $C_{0i0j}, \ldots, C_{1i1j}$ are subject to a number of constraints following from additional symmetries of the Weyl tensor and its tracelessness [1, 2].

Boost order of a tensor $\mathbf{T}$ is defined as the maximum boost weight of its frame components, and it can be shown that it depends only on the choice of a null direction $\boldsymbol{\ell}$ [1, 2]. Boost order of the Weyl tensor in a generic case is 2, but in algebraically special cases, there exist preferred null directions for which the boost order of the Weyl tensor is less. In other words, in algebraically special spacetimes, one can set all components of boost weight 2, C_{0i0j}, to zero by an appropriate null rotation (3) (we will call this case type I). Note than in four dimensions this is always possible since the Weyl tensor in 4D always possesses principal null directions and thus in 4D type I is generic, while for $D \geqslant 5$ type I is an algebraically special subclass of the general class G. One can proceed further and say that the Weyl tensor at a given point is of type II, III, and N if there exists a frame in which boost order of the Weyl tensor is 0, -1, and -2, respectively.

For spacetimes satisfying the above-mentioned assumptions, the Weyl tensor's decomposition (4) and the relation (1) lead to

$$
\begin{aligned}
\tilde{C}_{abc}{}^{d} &= \tilde{g}^{de}\tilde{C}_{abce} = \tilde{g}^{de}\big[4\tilde{C}_{0i0j}\tilde{n}_{\{a}\tilde{m}_b^{(i)}\tilde{n}_c\tilde{m}_{e\}}^{(j)} \\
&\quad + 8\tilde{C}_{010i}\tilde{n}_{\{a}\tilde{\ell}_b\tilde{n}_c\tilde{m}_{e\}}^{(i)} + 4\tilde{C}_{0ijk}\tilde{n}_{\{a}\tilde{m}_b^{(i)}\tilde{m}_c^{(j)}\tilde{m}_{e\}}^{(k)} + 4\tilde{C}_{0101}\tilde{n}_{\{a}\tilde{\ell}_b\tilde{n}_c\tilde{\ell}_{e\}} + 4\tilde{C}_{01ij}\tilde{n}_{\{a}\tilde{\ell}_b\tilde{m}_c^{(i)}\tilde{m}_{e\}}^{(j)} \\
&\quad + 8\tilde{C}_{0i1j}\tilde{n}_{\{a}\tilde{m}_b^{(i)}\tilde{\ell}_c\tilde{m}_{e\}}^{(j)} + \tilde{C}_{ijkl}\tilde{m}_{\{a}^{(i)}\tilde{m}_b^{(j)}\tilde{m}_c^{(k)}\tilde{m}_{e\}}^{(l)} \\
&\quad + 8\tilde{C}_{101i}\tilde{\ell}_{\{a}\tilde{n}_b\tilde{\ell}_c\tilde{m}_{e\}}^{(i)} + 4\tilde{C}_{1ijk}\tilde{\ell}_{\{a}\tilde{m}_b^{(i)}\tilde{m}_c^{(j)}\tilde{m}_{e\}}^{(k)} + 4\tilde{C}_{1i1j}\tilde{\ell}_{\{a}\tilde{m}_b^{(i)}\tilde{\ell}_c\tilde{m}_{e\}}^{(j)}\big] \\
&= \Omega^{-2}g^{de}\big[\Omega^{2+1+2+1}4\tilde{C}_{0i0j}n_{\{a}m_b^{(i)}n_c m_{e\}}^{(j)} \\
&\quad + \Omega^{2+2+1}8\tilde{C}_{010i}n_{\{a}\ell_b n_c m_{e\}}^{(i)} + \Omega^{2+1+1+1}4\tilde{C}_{0ijk}n_{\{a}m_b^{(i)}m_c^{(j)}m_{e\}}^{(k)} \\
&\quad + \Omega^{2+2}4\tilde{C}_{0101}n_{\{a}\ell_b n_c\ell_{e\}} + \Omega^{2+1+1}4\tilde{C}_{01ij}n_{\{a}\ell_b m_c^{(i)}m_{e\}}^{(j)} \\
&\quad + \Omega^{2+1+1}8\tilde{C}_{0i1j}n_{\{a}m_b^{(i)}\ell_c m_{e\}}^{(j)} + \Omega^{1+1+1+1}\tilde{C}_{ijkl}m_{\{a}^{(i)}m_b^{(j)}m_c^{(k)}m_{e\}}^{(l)} \\
&\quad + \Omega^{2+1}8\tilde{C}_{101i}\ell_{\{a}n_b\ell_c m_{e\}}^{(i)} + \Omega^{1+1+1}4\tilde{C}_{1ijk}\ell_{\{a}m_b^{(i)}m_c^{(j)}m_{e\}}^{(k)} \\
&\quad + \Omega^{1+1}4\tilde{C}_{1i1j}\ell_{\{a}m_b^{(i)}\ell_c m_{e\}}^{(j)}\big] = C_{abc}{}^{d}.
\end{aligned}
$$

Since all unphysical components of the Weyl tensor $\tilde{C}_{1i1j}, \tilde{C}_{1ijk}, \tilde{C}_{101i}, \tilde{C}_{ijkl}, \tilde{C}_{0i1j}, \tilde{C}_{01ij}, \tilde{C}_{0101}, \tilde{C}_{0ijk}, \tilde{C}_{010i}, \tilde{C}_{0i0j}$ are assumed to be of order $\mathcal{O}(\Omega^q)$, each physical component is of order $\mathcal{O}(\Omega^{\text{boost weight}+2+q})$, i.e., we obtain the peeling property

$$
\begin{aligned}
&C_{1i1j} = \mathcal{O}(\Omega^q), \qquad C_{1ijk}, C_{101i} = \mathcal{O}(\Omega^{q+1}),\\
&C_{ijkl}, C_{0i1j}, C_{01ij}, C_{0101} = \mathcal{O}(\Omega^{q+2}),\\
&C_{0ijk}, C_{010i} = \mathcal{O}(\Omega^{q+3}), \qquad C_{0i0j} = \mathcal{O}(\Omega^{q+4}),
\end{aligned} \tag{5}
$$

and thus

$$
\begin{aligned}
C_{abc}{}^{d} = {}& \Omega^q C^{(N)}{}_{abc}{}^{d} + \Omega^{q+1} C^{(III)}{}_{abc}{}^{d} + \Omega^{q+2} C^{(II)}{}_{abc}{}^{d} + \Omega^{q+3} C^{(I)}{}_{abc}{}^{d}\\
&+ \Omega^{q+4} C^{(G)}{}_{abc}{}^{d} + \mathcal{O}(\Omega^{q+5}).
\end{aligned} \tag{6}
$$

We have thus shown that from the assumptions outlined above the 'peeling property' of the Weyl tensor in the case of even dimensions follows. We hope that this may provide a first step in proving a peeling theorem in more generality.

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References

[1] Milson R, Coley A, Pravda V and Pravdová A 2005 Alignment and algebraically special tensors in Lorentzian geometry *Int. J. Geom. Methods Mod. Phys.* **2** 41 (*Preprint* gr-qc/0401010)

[2] Coley A, Milson R, Pravda V and Pravdová A 2004 Classification of the Weyl tensor in higher dimensions *Class. Quantum Grav.* **21** L35

[3] Newman E T and Penrose R 1962 An approach to gravitational radiation by a method of spin coefficients *J. Math. Phys.* **3** 566

[4] Penrose R 1963 Asymptotic properties of fields and spacetimes *Phys. Rev. Lett.* **10** 66

[5] Hollands S and Ishibashi A 2005 Asymptotic flatness and Bondi energy in higher dimensional gravity *J. Math. Phys.* **46** 022503

[6] Hollands S and Wald R M 2004 Conformal null infinity does not exist for radiating solutions in odd spacetime dimensions *Class. Quantum Grav.* **21** 5139

[7] Bondi H, van der Burg M G J and Metzner A W K 1962 Gravitational waves in general relativity: VII. Waves from isolated axisymmetric systems *Proc. R. Soc.* A **269** 21

[8] Sachs R K 1962 Gravitational waves in general relativity: VIII. Waves in asymptotically flat space-time *Proc. R. Soc.* A **270** 103