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Curvature tensors on distorted Killing horizons and their algebraic classification

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Abstract

We consider generic static spacetimes with Killing horizons and study properties of curvature tensors in the horizon limit. It is determined that the Weyl, Ricci, Riemann and Einstein tensors are algebraically special and mutually aligned on the horizon. It is also pointed out that results obtained in the tetrad adjusted to a static observer in general differ from those obtained in a free-falling frame. This is connected to the fact that a static observer becomes null on the horizon. It is also shown that finiteness of the Kretschmann scalar on the horizon is compatible with the divergence of the Weyl component Ψ_3 or Ψ_4 in the freely falling frame. Furthermore, finiteness of these components is compatible with divergence of curvature invariants constructed from second derivatives of the Riemann tensor. We call the objects with finite Kretschmann scalar but infinite Ψ_4 or Ψ_3 'truly naked black holes'. In the (ultra)extremal versions of these objects the structure of the Einstein tensor on the horizon changes due to extra terms as compared to the usual horizons, the null energy condition being violated at some portions of the horizon surface. The demand to rule out such divergences leads to the constancy of the factor that governs the leading term in the asymptotics of the lapse function and in this sense represents a formal analogue of the zeroth law of mechanics of non-extremal black holes. In doing so, all extra terms in the Einstein tensor automatically vanish.

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1. Introduction

The outstanding role of black holes in general relativity makes the investigation of properties of event horizons especially important. Findings and developments in this area up to the 1970s were summarized in fundamental surveys [1, 2]. Nowadays, the interest in this topic has been revived due to the appearance of new notions such as isolated and dynamical horizons (see [3] for review and references), the issue of black hole entropy, including conformal field theory in the near-horizon region [4], etc.

Recently, it has been derived by purely geometrical means that for a generic non-extremal static black hole the Einstein tensor has a high degree of symmetry in the horizon limit [5]. In the present paper, it is shown that for a generic non-extremal static black hole the Weyl, Riemann, Ricci and Einstein tensors are algebraically special and mutually aligned on the horizon and this analysis is also extended to the extremal and ultraextremal cases.

Conditions for the regularity of the horizon are also studied. It turns out that one can distinguish several classes by imposing various regularity conditions. Throughout the paper the regularity condition from [5], that the Kretschmann invariant and all other polynomial invariants of the Riemann tensor are regular on the horizon, is used. However it turns out that even after imposing the above regularity condition the Weyl components Ψ_3 and/or Ψ_4 in a parallelly propagated frame may still diverge on the horizon[3] and thus a parallelly propagated curvature singularity [6] may be located there. Even if both regularity conditions mentioned above (finiteness of Kretschmann invariant, Ψ_3 and Ψ_4) are satisfied, there are still cases for which curvature invariants constructed from second derivatives of the Riemann tensor diverge on the horizon.

Up to now, only partial results concerning classification of the curvature tensors on the horizon of a generic[4] static black hole were obtained. In [7], it was shown that on the horizon of an axially symmetric black hole the Petrov type of the Weyl tensor is D. However, this result was obtained in the static frame which becomes singular in the horizon limit and thus is not suitable for this purpose[5]. We consider also a frame attached to a freely falling observer and point out that the horizon limits of the Petrov type in both frames are, in general, different. In the frame of a free-falling observer the resulting Petrov type is II or more special. It was found in [8, 9] that for a generic isolated horizon the Petrov type is II or more special, but without further specification.

Since for physical reasons, physical properties of the black holes are often analysed in the static frame, which is regular everywhere in the outer region but becomes null on the horizon, we also perform calculations in the static frame and compare them with results obtained in the frame attached to the freely falling observer.

The paper is organized as follows. In section 2 we describe the choice of the null tetrad and, using 2 + 1 + 1 decomposition of the metric and curvature, list basic formulae for the Weyl scalars $\Psi_0, \ldots, \Psi_4$. We apply them to the static frame and consider separately non-extremal, extremal and ultraextremal horizons. In section 3 we consider a free-falling frame. As explicit examples, we discuss the Ernst metric describing a Schwarzschild-like black hole in a magnetic field and the Bonnor–Swaminarayan metric. In section 4 we show that, in general, a free-falling observer can register divergence of the Weyl scalars Ψ_4 or Ψ_3 near the horizon in spite of finiteness of the Kretschmann invariant. We discuss conditions that rule out such objects that we call 'truly naked black holes'. In section 5 we determine an algebraic type of the Ricci tensor on the horizon. Section 6 contains a list of main results and their brief discussion.

[3] For simplicity, the surface $N = 0$ is referred to as a horizon even in the case when a singularity is located there.

[4] We do not need to assume that energy conditions or the Einstein equations are satisfied. We also have no assumptions concerning the topology of the horizon.

[5] This has also been noted by M Ortaggio (private communication).

2. Static observer

Let us consider a generic static spacetime. We follow the technique of decomposition of the metric and curvature [5, 10]. The metric can be written in the 3 + 1 decomposed form with a subsequent 2 + 1 decomposition on the basis of the Gauss normal coordinates

$$ds^2 = -dt^2 N^2 + dn^2 + \gamma_{ab}\, dx^a\, dx^b, \tag{1}$$

where $x^1 = n, a = 2, 3$. As our spacetime is static, there exists a Killing vector which is timelike in the outer region, $\xi^\mu = (1, 0, 0, 0)$, its norm $\xi^\mu \xi_\mu \equiv \xi^2 = -N^2 < 0$. We also suppose that there exists a two-dimensional regular 2-surface obtained by the limiting transition $N^2 = c, c \to +0$ (c is a constant) on which the Killing vector becomes null, $\xi^2 = 0$. This surface separates the outer region in which this vector is timelike from that where it is spacelike ($\xi^2 > 0$). In other words, this is the so-called Killing horizon. It acts also as a surface of an infinite redshift. Its significance in black hole context is connected, in particular, with the fact that for any black hole solution in the stationary (in our case static) spacetime, with matter satisfying suitable hyperbolic equations, the event horizon is a Killing horizon [6] (prop. 9.3.6), [26] (section 6.3.1). A rigorous definition of a black hole usually assumes asymptotical flatness. Then a black hole is a region from which no causal signal can reach future null infinity (see, e.g., section 5.2.1 of [26]). However, we would like to stress that for our purposes we do not distinguish black hole horizons and acceleration horizons (but we speak sometimes about a black hole for the sake of simplicity and definiteness), do not require asymptotic flatness, etc. All we need is the local properties of the metric and curvature tensors that follow from the fact that $N^2 = 0$ and the regularity of spacetime (see details below).

Our goal is to elucidate to what extent the presence of the Killing horizon restricts the Petrov type of the gravitational field on the horizon, and find which types are possible there. Our determination of the Petrov type is based on studying curvature invariants I, J and coefficients K, L, N, in certain covariants (see chapter 9.3 in [14]), constructed from so-called Weyl scalars (see their exact definition below):

$$I = \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2, \qquad J = \det\begin{pmatrix}\Psi_4 & \Psi_3 & \Psi_2\\ \Psi_3 & \Psi_2 & \Psi_1\\ \Psi_2 & \Psi_1 & \Psi_0\end{pmatrix}, \tag{2}$$

$$K = \Psi_1\Psi_4^2 - 3\Psi_2\Psi_3\Psi_4 + 2\Psi_3^3, L = \Psi_2\Psi_4 - \Psi_3^2, \qquad N = 12L^2 - \Psi_4^2 \mathrm{I}. \tag{3}$$

The algorithm for determining the Petrov type of the Weyl tensor is based on whether or not equalities

$$I^3 = 27J^2, \quad I = J = 0, \quad K = N = 0, \quad K = L = 0 \tag{4}$$

are satisfied (see, e.g., [14]). Our strategy can be described as follows.

(1) We choose the complex tetrad frame and, with its help, define Weyl scalars;
(2) We use 2 + 1 + 1 splitting of the metric of the static spacetime using Gauss normal coordinates and find general expressions for Weyl scalars;
(3) The conditions of the regularity of spacetime on the horizon impose severe restrictions on the asymptotic form of the metric; we substitute this asymptotics into the formulae for Weyl scalars and find their near-horizon values;
(4) Compare the result with the conditions that define the Petrov type;
(5) Carry out this procedure for non-extremal and (ultra)extremal horizons separately;
(6) Repeat it for a tetrad that corresponds to a free-falling observer.

Let us construct the complex null tetrad from a usual orthonormal frame u^μ, e^μ, a^μ, b^μ, where u^μ is the 4-velocity of an observer, e^μ is a vector aligned along the n-direction, a^μ and b^μ lie in the x^2–x^3 subspace. We define

$$l^\mu = \frac{u^\mu + e^\mu}{\sqrt{2}}, \qquad n^\mu = \frac{u^\mu - e^\mu}{\sqrt{2}}, \qquad m^\mu = \frac{a^\mu + \mathrm{i}b^\mu}{\sqrt{2}}, \qquad \bar{m}^\mu = \frac{a^\mu - \mathrm{i}b^\mu}{\sqrt{2}}. \tag{5}$$

Now $l^\mu n_\mu = -1$, $m^a \bar{m}_a = 1$, all other contractions vanish. We use the standard definition of the Weyl scalars

$$\begin{aligned} &\Psi_0 = C_\alpha\beta\gamma\delta l^\alpha m^\beta l^\gamma m^\delta, \qquad \Psi_1 = C_\alpha\beta\gamma\delta l^\alpha m^\beta l^\gamma n^\delta, \qquad \Psi_2 = -C_\alpha\beta\gamma\delta l^\alpha m^\beta n^\gamma \bar{m}^\delta, \\ &\Psi_3 = C_\alpha\beta\gamma\delta l^\alpha n^\beta \bar{m}^\gamma n^\delta, \qquad \Psi_4 = C_\alpha\beta\gamma\delta n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta. \end{aligned} \tag{6}$$

Here the Weyl tensor

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - R_{\gamma[\alpha} g_{\beta]\delta} + R_{\delta[\alpha} g_{\beta]\gamma} + \frac{R}{3} g_{\gamma[\alpha} g_{\beta]\delta}, \tag{7}$$

where $R_{\alpha\beta\gamma\delta}$ is the curvature tensor, R is the scalar curvature. Our first goal is to find behaviour of these scalars near the horizon whence we will be able to extract information about the Petrov type. In what follows, we will be dealing with two types of such tetrads corresponding to a static observer (SO) and a freely falling one (FFO).

In the present section, we consider the first case (SO). Then

$$u^{(0)\mu} = (N^{-1}, 0, 0, 0), \qquad e^{(0)\mu} = (0, 1, 0, 0), \tag{8}$$

where the superscript (0) refers to the static frame, so that

$$l^\mu = \frac{1}{\sqrt{2}}\left(\frac{1}{N}, 1, 0, 0\right), \qquad n^\mu = \frac{1}{\sqrt{2}}\left(\frac{1}{N}, -1, 0, 0\right), \qquad m^\mu = (0, 0, m^a). \tag{9}$$

With this choice of the tetrad,

$$\Psi_4 = \bar{\Psi}_0, \qquad \Psi_3 = -\bar{\Psi}_1. \tag{10}$$

Thus, it is sufficient to determine Ψ_0, Ψ_1, Ψ_2.

In what follows we will extensively use the convenient representation of the curvature tensor based on 2 + 1 + 1 decomposition of the metric (1) (details can be found in section 2 of [5]):

$$R_{abcd} = \frac{R_\|}{2}(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc}) + K_{ad}K_{bc} - K_{ac}K_{bd} \tag{11}$$

$$R_{1abc} = K_{ac;b} - K_{ab;c} \tag{12}$$

$$R_{1a1b} = \frac{\partial K_{ab}}{\partial n} + (K^2)_{ab} \tag{13}$$

$$\frac{R_{0a0b}}{N^2} = \frac{N_{;a;b} - K_{ab}N'}{N} \tag{14}$$

$$\frac{R_{0101}}{N^2} = \frac{N''}{N} = -\frac{1}{2}R_\perp, \tag{15}$$

$$\frac{R_{010a}}{N^2} = \frac{\partial_n N_{;a} + K_a^b N_{;b}}{N}, \tag{16}$$

where K_{ab} is the extrinsic curvature tensor for the surface t = const, n = const embedded in the outer 3-space, for our metric (1)

$$K_{ab} = -\frac{1}{2}\frac{\partial \gamma_{ab}}{\partial n}, \tag{17}$$

$K = K^a_a$, the prime denotes differentiation with respect to n, quantities like $N_{|i|j}$ represent covariant derivatives with respect to the 3-metric and $N_{;b;d}$ correspond to covariant derivatives with respect to the 2-metric γ_{ab}, $R_\parallel$ represents the two-dimensional Ricci scalar for the surface $t =$ const, $n =$ const, $R_\perp$ is the similar quantity for the n–t subspace.

In a similar way, for the Ricci tensor one has

$$\frac{R_{00}}{N^2} = \frac{\Delta_2 N - K N' + N''}{N}, \tag{18}$$

$$R_{11} = K' - \mathrm{Sp}K^2 - \frac{N''}{N}, \tag{19}$$

$$R_{1a} = K_{;a} - K^b_{a;b} - \frac{\partial_n N_{;a} + K^b_a N_{;b}}{N}, \tag{20}$$

$$R_{ab} = \frac{R_\parallel}{2}\gamma_{ab} + \frac{\partial K_{ab}}{\partial n} + 2(K^2)_{ab} - K K_{ab} - \frac{N_{;a;b} - K_{ab}N'}{N}, \tag{21}$$

where Δ_2 is the two-dimensional Laplacian with respect to the metric γ_{ab}. The scalar curvature

$$R = R_\parallel + 2K' - \mathrm{Sp}K^2 - K^2 - 2\frac{\Delta_2 N - K N' + N''}{N}. \tag{22}$$

Now equation (6) leads to

$$\Psi_0 = \frac{1}{2} m^b m^d \left(\frac{C_{0b0d}}{N^2} + C_{1b1d}\right), \tag{23}$$

where C_{0b0d} and C_{1b1d} may be obtained from the definition of the Weyl tensor (7):

$$C_{0b0d} = R_{0b0d} - \frac{R_{00}}{2}\gamma_{bd} + \frac{N^2}{2}R_{bd} - \frac{R}{6}N^2\gamma_{bd}, \tag{24}$$

$$C_{1b1d} = R_{1b1d} - \frac{R_{11}}{2}\gamma_{bd} - \frac{R_{bd}}{2} + \frac{R}{6}\gamma_{bd}. \tag{25}$$

Since $\gamma_{ab}m^a m^b = 0$,

$$\Psi_0 = \frac{1}{2} m^b m^d \tilde{S}_{bd}, \qquad \text{where} \quad \tilde{S}_{bd} = \frac{R_{0b0d}}{N^2} + R_{1b1d}. \tag{26}$$

Consequently

$$\Psi_0 = \frac{1}{2} m^b m^d \left[\frac{\partial K_{bd}}{\partial n} - K_{bd}\frac{N'}{N} + (K^2)_{bd} + \frac{N_{;b;d}}{N}\right]. \tag{27}$$

We also have

$$\Psi_1 = -\frac{m^b}{\sqrt{2}}\frac{C_{0b01}}{N^2}, \tag{28}$$

where

$$\frac{C_{0b01}}{N^2} = \frac{R_{0b01}}{N^2} + \frac{1}{2}R_{1b}, \tag{29}$$

whence

$$\Psi_1 = -\frac{m^b}{2\sqrt{2}}\left(N^{-1}\frac{\partial N_{;b}}{\partial n} + \frac{K^c_b N_{;c}}{N} + K_{;b} - \gamma^{cd}K_{bc;d}\right). \tag{30}$$

One also finds

$$-2\Psi_2 = m^b \bar{m}^d \left[\frac{R_\parallel}{2}\gamma_{bd} + (K^2)_{bd} - K_{bd}K\right] + \frac{R_{11}}{2} - \frac{R}{3} - \frac{R_{00}}{2N^2}. \tag{31}$$

After expressing this quantity in terms of two-dimensional geometry we obtain

$$-2\Psi_2 = \frac{R_\parallel + R_\perp}{6} + \frac{K^2}{3} - \frac{SpK^2}{6} - \frac{K'}{6} - \frac{KN'}{6N} + \frac{\Delta_2 N}{6N} + m^b\bar{m}^d[(K^2)_{bd} - K_{bd}K]. \quad (32)$$

Now different types of horizons should be considered separately.

2.1. Non-extremal case

The necessary condition of absence of singularities is the finiteness of the Kretschmann invariant

$$Kr \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}. \quad (33)$$

For the static spacetimes (1) it is easy to show [5, 10] that $Kr = {}^{(3)}R_{ijkl}{}^{(3)}R^{ijkl} + 4\frac{N_{|i|j}N^{N|i|j}}{N^2}$. The first term here is calculated with respect to the space with positively defined 3-metric and is positive, if we want to have a regular spacetime, the second term (which is also positive) should be finite. Then, if the surface gravity $\kappa_H \neq 0$ (κ_H is equal to the horizon limit of the derivative $\frac{\partial N}{\partial n}$), one derives the general form of the asymptotic behaviour of the lapse function for small n [5]:

$$N = \kappa_H n + \frac{\kappa_2(x^2, x^3)}{3!}n^3 + \frac{\kappa_3(x^2, x^3)}{4!}n^4 + O(n^5), \quad (34)$$

$$\gamma_{ab} = [\gamma_H]_{ab}(x^2, x^3) + \frac{[\gamma_2]_{ab}(x^2, x^3)}{2!}n^2 + \frac{[\gamma_3]_{ab}(x^2, x^3)}{3!}n^3 + O(n^4), \quad (35)$$

where κ_H is a constant on the horizon. From these expansions we obtain that in general near the horizon

$$\Psi_0, \Psi_1, \Psi_3, \Psi_4 \sim n. \quad (36)$$

In the particular case with $[\gamma_3]_{ab}(x, y) = 0$ which will be clarified later we obtain

$$\Psi_0, \Psi_4 \sim n^2, \Psi_1, \Psi_3 \sim n. \quad (37)$$

It also follows from (32) that the horizon value of Ψ_2 is equal to

$$\Psi_2^H = -\frac{R_\parallel^H + R_\perp^H}{12}. \quad (38)$$

Thus, there exist only two possibilities on the horizon: (i) $\Psi_0, \Psi_1, \Psi_3, \Psi_4$ vanish, $\Psi_2 \neq 0$, (ii) all components of the Weyl tensor vanish. Case (i) corresponds to the Petrov type D and case (ii) to the Petrov type O. We must make a reservation here. As the static frame becomes singular on the horizon (we discuss it in more detail below), in this section, by the Petrov type on the horizon we simply mean the type obtained by taking the horizon limit from the outer region.

As an example of case (ii), we can mention the Bertotti–Robinson (BR) metric for which $R_\parallel + R_\perp = 0$ everywhere and so it is of type O. However, quantum backreaction of massless conformally invariant fields on spacetimes of the type $AdS_2 \times S_2$ (which the BR metric belongs to) violates this condition [11]. In contrast, backreaction of massive fields retains its validity [12]. Thus, as far as the role of quantum backreaction is concerned, conformal fields change the Petrov type of the metric on the horizon from O to D, whereas massive fields leave it intact.

2.2. Extremal case

Let $\kappa_H = 0$. In this case the horizon is situated at the infinite proper distance from any point. In [5] this was shown for the function N that has the power-like asymptotics at large n. This corresponds to what is usually called 'ultraextremal' horizon in the spherically-symmetrical case (see, e.g., [13]), when $N^2 \sim (r - r_+)^m$, $m \geqslant 3$, r is the Schwarzschild-like coordinate, r_+ is the position of the horizon. For the case of the usual extremal black hole $m = 2$. The extremal Reissner–Nordström metric can serve as example, then $N \sim (r - r_+) \sim \exp\left(-\frac{n}{r_+}\right)$. Thus, there are two qualitatively different cases of the asymptotic behaviour of the lapse function N. A similar division occurs also in a general non-spherical case. It turns out that finiteness of the Kretschmann invariant on the horizon requires that $\frac{N_{|i|j}}{N}$ be finite there [10], $i, j = 1, 2, 3$. It makes sense to distinguish two cases: (i) $\frac{N_{|1|1}}{N} = C(x^2, x^3) \neq 0$ on the horizon, (ii) $\frac{N_{|1|1}}{N} = 0$ on the horizon.

In case (i) we obtain the asymptotic behaviour

$$N = B(x^2, x^3)\exp\left(-\frac{n}{n_0}\right) + O\left(\mathrm{e}^{-\frac{2n}{n_0}}\right), \qquad n_0 > 0, \quad \frac{1}{n_0^2} = C \tag{39}$$

for $n \to \infty$. Then the dependence $n_0(x^2, x^3)$ would produce in $\frac{N_{|b|d}}{N} = \frac{N_{;b;d}}{N} - K_{bd}\frac{N'}{N}$ terms proportional to n^2 that diverge on the horizon. Therefore, n_0 should be constant. Expanding the two-dimensional metric and the extrinsic curvature tensor

$$\gamma_{ab} = \gamma_{ab}^{(0)} + \gamma_{ab}^{(1)}\exp\left(-\frac{n}{n_0}\right) + O\left(\mathrm{e}^{-\frac{2n}{n_0}}\right), \qquad K_{ab} = K_{ab}^{(1)}\exp\left(-\frac{n}{n_0}\right) + O\left(\mathrm{e}^{-\frac{2n}{n_0}}\right) \tag{40}$$

and calculating the Weyl scalars, we obtain that

$$\Psi_0^H = \frac{B_{;b;d}}{2B}m^b m^d, \qquad \Psi_1^H = \frac{m^b}{2n_0\sqrt{2}}\frac{B_{;b}}{B}, \tag{41}$$

$$\Psi_2^H = -\frac{1}{12}\left(R_\parallel + R_\perp + \frac{\Delta_2 B}{B}\right), \tag{42}$$

where the right-hand sides are evaluated on the horizon.

2.3. Ultraextremal case

In case (ii)

$$N = \frac{A}{n^m} + O(n^{-m-1}), \qquad n \to \infty, \quad m > 0 \tag{43}$$

$$\gamma_{ab} = \gamma_{ab}^{(0)} + \frac{\gamma_{ab}^{(1)}}{n^s} + O(n^{-s-1}), \qquad s > 0 \tag{44}$$

$$K_{ab} = \frac{K_{ab}^{(1)}}{n^{s+1}} + O(n^{-s-2}), \tag{45}$$

$$\Psi_0^H = \frac{A_{;b;d}}{2A}m^b m^d, \qquad \Psi_1^H = 0, \tag{46}$$

$$\Psi_2^H = -\frac{1}{12}\left(R_\parallel^H + R_\perp^H + \frac{\Delta_2 A}{A}\right). \tag{47}$$

We postpone discussion of the meaning of the terms with derivatives like $B_{;b}$ or $A_{;a}$ and now pass to the description of the gravitational field from the viewpoint of the FFO.

3. Freely falling observer

Let u^μ and e^μ denote vectors attached to a FFO that moves towards the horizon. Then it follows from integration of equations of timelike geodesics that for a 'radial' motion $x^2 =$ const, $x^3 =$ const

$$l^\mu = \left(\frac{E}{N^2} - \frac{\sqrt{E^2-N^2}}{N^2}, \frac{E}{N} - \frac{\sqrt{E^2-N^2}}{N}, 0, 0\right) = \mathrm{e}^{-\alpha}\left(\frac{1}{N}, 1, 0, 0\right), \tag{48}$$

$$n^\mu = \left(\frac{E}{N^2} + \frac{\sqrt{E^2-N^2}}{N^2}, -\frac{E}{N} - \frac{\sqrt{E^2-N^2}}{N}, 0, 0\right) = \mathrm{e}^{\alpha}\left(\frac{1}{N}, -1, 0, 0\right), \tag{49}$$

$$\cosh\alpha = \frac{E}{N}, \quad \alpha > 0, \tag{50}$$

with $E = -u_0$ being the energy per unit mass.

Two pairs of vectors corresponding to SO and FFO are connected by the relationships

$$u^\mu = u^{\mu(0)}\cosh\alpha - e^{\mu(0)}\sinh\alpha, \tag{51}$$

$$e^\mu = e^{\mu(0)}\cosh\alpha - u^{\mu(0)}\sinh\alpha, \tag{52}$$

$$l^\mu = z l^{(0)\mu}, \qquad n^\mu = z^{-1} n^{(0)\mu}, \qquad \text{with} \quad z = \exp(-\alpha). \tag{53}$$

Under the boost (53) the Weyl scalars transform in the standard way:

$$\Psi_0 = z^2\Psi_0^{(0)}, \quad \Psi_1 = z\Psi_1^{(0)}, \quad \Psi_2 = \Psi_2^{(0)}, \quad \Psi_3 = z^{-1}\Psi_3^{(0)}, \quad \Psi_4 = z^{-2}\Psi_4^{(0)}, \tag{54}$$

curvature invariants obviously do not depend on the choice of the reference frame, $I = I^{(0)}$, $J = J^{(0)}$, and coefficients K, L, N in certain covariants (see chapter 9.3 in [14]) transform according to

$$K = z^{-3}K^{(0)}, \quad L = z^{-2}L^{(0)}, \quad N = z^{-4}N^{(0)}. \tag{55}$$

Here we use definitions (2), (3).

Usually, the parameter z entering the boost (53) is finite and non-vanishing, so that classification criteria (whether equation (4) is satisfied or not) are not affected by the boost and all timelike observers agree that the field belongs to the same type which is an invariant characteristic of a spacetime at a given point. The situation is qualitatively different on the horizon since $z \to 0$ and thus, in general, some of the quantities K, L, N that vanish in the static frame may or may not vanish in the freely falling one. This is obviously related to the fact that the SO becomes null on the horizon and the corresponding null frame is singular there. Consequently, only the results obtained in FFO's frame should be considered as physically relevant.

For completeness, let us consider different cases separately, assuming that all Ψ's are finite on the horizon for both SO and FFO.

(1) $\Psi_2^H \neq 0$. Then SO finds the gravitational field of type D on the horizon, while FFO sees in general type II. Only if the metric satisfies $K = N = 0$ on the horizon, is it of type D also in FFO's frame.

(2) $\Psi_2^H = 0$. Then SO sees type O on the horizon while FFO sees type III, N or O depending on the behaviour of Ψ_3 and Ψ_4 there.

Thus, in general, there exists a variety of situations depending on the relationship between invariants. It is convenient to summarize the set of possible situations in the table:

SO	FFO
D	II, D
O	III, N, O

It is worth stressing that the static frame becomes singular on the horizon and cannot be continued inside. This is one of the reasons why it is necessary to introduce the Kruskal-like coordinate system near the horizon to obtain the maximal analytical extension of the manifold. Note, e.g., that the fact that all Ψ's vanish on the horizon in the static frame (type O) does not necessarily mean that the Weyl tensor as such vanishes—rather, this is a pure coordinate effect because of a 'bad' frame and FFO (who uses a 'good' frame) would see it in general non-vanishing. However, in a small vicinity of the horizon the static frame is well-defined and, correspondingly, the horizon limit has a clear sense. In [7], where the Petrov type on the horizon for the axially symmetric case was considered, only the limit in the SO sense was exploited. However, as was pointed out above, such a procedure is not so 'innocent' and requires introducing FFO for a complete and correct description. To conclude this section, let us again emphasize that the horizon limits of Petrov type for SO and FFO in general do not coincide.

3.1. Examples—Ernst metric and Bonnor–Swaminarayan solution

Let us here present two examples of exact solutions which are in general of Petrov type I and become algebraically special on the event or acceleration horizon.

The Ernst metric [15]

$$\mathrm{d}s^2 = \Lambda^2\left[-\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2\right] + \Lambda^{-2}r^2\sin^2\theta\,\mathrm{d}\phi^2, \tag{56}$$

where

$$\Lambda = 1 + \tfrac{1}{4}B^2r^2\sin^2\theta, \tag{57}$$

represents a black hole immersed in an external magnetic field and for this reason it is also called the Schwarzschild–Melvin spacetime in the literature. The parameter B governs the strength of the magnetic field. Let us choose the frame (retaining the notation with prime of [15])

$$l'^{\mu} = \frac{1}{\sqrt{2}}\left(\frac{1}{\Lambda}\chi^{-1/2}, \frac{1}{\Lambda}\chi^{1/2}, 0, 0\right), \tag{58}$$

$$n'^{\mu} = \frac{1}{\sqrt{2}}\left(-\frac{1}{\Lambda}\chi^{-1/2}, \frac{1}{\Lambda}\chi^{1/2}, 0, 0\right), \tag{59}$$

$$m_{\mu} = \frac{r}{\sqrt{2}}\left(0, 0, -\Lambda, -\frac{\mathrm{i}\sin\theta}{\Lambda}\right), \tag{60}$$

with $\chi = \left(1-\frac{2m}{r}\right)$, which corresponds to SO.

Then it follows that [16]

$$\Psi'_0 = C\chi = \Psi'_4, \qquad C = \frac{3(\Lambda-1)(\Lambda-2)}{r^2\Lambda^4}, \tag{61}$$

$$\Psi'_1 = -\Psi'_3 = C\chi^{1/2}\cot\theta, \tag{62}$$

$$\Psi_2' = \frac{\Lambda - 2}{r^2\Lambda^4}\left[(2\cot^2\theta - 1)(\Lambda - 1) + \frac{m}{r}(3\Lambda - 2)\right], \tag{63}$$

$$I = \chi^2 C^2 + 4C^2\chi\cot^2\theta + 3\Psi_2'^2, \tag{64}$$

$$K' = \chi^{3/2}K, \qquad K = 3C^2\frac{\Lambda - 2}{r^2\Lambda^4}k\cot\theta, \tag{65}$$

$$k = \frac{m}{r}(3\Lambda - 2), \tag{66}$$

$$L' = \chi L, \qquad L = \frac{\Lambda - 2}{r^2\Lambda^4}CA, \tag{67}$$

$$A = -\frac{\Lambda - 1}{\sin^2\theta} + \frac{m}{r}(3\Lambda - 2). \tag{68}$$

$$N' = \chi^2 N, \qquad N = 12L^2 - C^2 I. \tag{69}$$

It is worth noting that there are no divergences in Ψ and quantities K, N, L at $\theta = 0$ and $\theta = \pi$ due to the factors C and $\Lambda - 1$ which are proportional to $\sin^2\theta$. In general, $I^3 \neq 27J^2$ and the metric is of generic type I [16]. As was pointed out in [17], the metric is of type II on the horizon. The fact that it is algebraically special on the horizon can be easily seen since $I^3 - 27J^2$ vanishes there. Since for the freely falling observer the quantities K and N do not vanish on the horizon the Petrov type is II there. In the static frame (58), (59), $K' = L' = N' = 0$ on the horizon and this could lead to an incorrect conclusion that the Petrov type on the horizon is D.

However, a special case arises on the axis of symmetry ($\theta = 0$ or $\theta = \pi$). It is seen from the above formulae that now all Ψ's vanish except Ψ_2 and both observers (a static and a freely falling one) will agree that the Petrov type is D. In the limit $B = 0$ we recover the Schwarzschild metric that is spherically symmetric and therefore we can rotate a coordinate frame to achieve any point to lie on the axis. As a result, the metric is of type D on the entire horizon that agrees with the fact that any spherically symmetric metric is of the type D everywhere ([22], p 187).

Another special case arises if $\Lambda = 2$. This can be achieved on the horizon at $\sin\theta = (Bm)^{-1}$, if $Bm \geqslant 1$. Then all Ψ's vanish for both SO and FFO and thus the Weyl tensor is of type O.

Another interesting example of exact solutions with algebraically special Weyl tensor on the horizon is the class of boost-rotation symmetric spacetimes [18, 19]. These spacetimes correspond to various types of uniformly accelerated 'particles' and they thus possess an acceleration horizon. The well-known exact solution belonging to this class is the C-metric representing two uniformly accelerated black holes that, however, does not suit our purposes since it is globally of type D. Another interesting exact solution in this class is the Bonnor–Swaminarayan solution [20], representing uniformly accelerated Curzon–Chazy particles. This solution is in general of Petrov type I and by evaluating $I^3 - 27J^2$ it can be checked that it becomes algebraically special on the acceleration horizon.

4. Truly naked black holes

4.1. Near-horizon behaviour of the Weyl scalars in the freely falling frame

If a null tetrad corresponds to the static frame (9), all Weyl scalars are finite and, moreover, as we saw above, they vanish on the horizon, except (possibly) Ψ_2. However, the appearance of the 'dangerous' factor z in the denominator opens the possibility that not only do some Weyl scalars not vanish but also they may even diverge on the horizon in the freely falling frame. To avoid such divergences representing parallelly propagated curvature singularity [6], some additional constraints should be imposed on the metric. To distinguish metrics with finite and infinite Ψ^H, let us now consider properties of the Weyl scalars in the freely falling frame in more detail.

4.1.1. Non-extremal case. Since in both frames Ψ_2 is the same and the horizon values of Ψ_0 and Ψ_1 vanish, we need only consider Ψ_3 and Ψ_4. Then it follows from (10), (30) and (54) that on the horizon

$$\Psi_3^H = \frac{\bar{m}^b E}{\sqrt{2}\kappa_H}\left(\frac{\kappa_{2:b}}{2\kappa_H} + K^{(1)}_{;b} - \gamma^{cd}K^{(1)}_{bc;d}\right) \tag{70}$$

is always finite.

Substituting the asymptotic expansion of the lapse function (34) and the expansion

$$K_{ab} = K^{(1)}_{ab}n + \frac{K^{(2)}_{ab}}{2}n^2 + \frac{K^{(3)}_{ab}}{6}n^3 + O(n^4) \tag{71}$$

into

$$\Psi_4 = \frac{\bar{m}^b\bar{m}^d}{2}\exp(2\alpha)\left[\frac{\partial K_{bd}}{\partial n} - K_{bd}\frac{N'}{N} + (K^2)_{bd} + \frac{N_{;b;d}}{N}\right] \tag{72}$$

leads to

$$\Psi_4 = 2E^2\frac{\bar{m}^a\bar{m}^b}{\kappa_H^2}\left[\frac{K^{(2)}_{ab}}{2n} + C_{ab}\right] + O(n), \tag{73}$$

where

$$C_{ab} = [K^{(1)2}]_{ab} + \frac{\kappa_{2a;b}}{6\kappa_H} - \frac{\kappa_2}{3\kappa_H}K^{(1)}_{ab} + \frac{K^{(3)}_{ab}}{3}. \tag{74}$$

Thus, if we want Ψ_4 to be finite on the horizon we must demand $K^{(2)}_{ab} = 0$. Then $\Psi_4^H = 2E^2\frac{\bar{m}^a\bar{m}^b}{\kappa_H^2}C_{ab}$. Note that for $K^{(2)}_{ab} = 0$ all components of the Riemann and Ricci tensors are also regular on the horizon.

4.1.2. Extremal case. Near the horizon

$$\Psi_3 = -\frac{\bar{m}^b E}{n_0\sqrt{2}}\frac{B_{;b}}{B^2}\exp\left(\frac{n}{n_0}\right). \tag{75}$$

Thus, if we demand Ψ_3 be finite on the horizon we must require

$$B_{;b} = 0. \tag{76}$$

The limiting value of Ψ_4 on the horizon is

$$\Psi_4^H = 2E^2\bar{m}^b\bar{m}^d\lim_{n\to\infty}\frac{1}{N^2}\left[\frac{\partial K_{bd}}{\partial n} - K_{bd}\frac{N'}{N} + (K^2)_{bd} + \frac{N_{;b;d}}{N}\right]. \tag{77}$$

Bearing in mind (76) we see that the most severe divergences near the horizon are absent because terms $\exp\left(\frac{2n}{n_0}\right)$ are absent.

To make sure that terms $\exp\left(\frac{n}{n_0}\right)$ are also absent, we must impose a constraint on the correction to N, demanding that in the expansion

$$N = \exp\left(-\frac{n}{n_0}\right)M, \qquad M = B + B_1\exp\left(-\frac{n}{n_0}\right) + B_2(x^2, x^3)\exp\left(-\frac{2n}{n_0}\right) + \cdots \tag{78}$$

not only B but also B_1 do not depend on x^2 and x^3, $B_{1;a} = 0$. In contrast to the non-extremal case, finiteness of Ψ_4 does not entail constraints on K_{ab}.

4.1.3. Ultraextremal case. Now near the horizon $n \to \infty$, $z^{-1} = \frac{2E}{N} + O(N)$ and it follows from (10), (28), (30), (27), (54) and (43)–(45) that

$$\Psi_3^H = \frac{\bar{m}^b E}{\sqrt{2}} \lim_{n\to\infty} \frac{1}{N} \left(N^{-1} \frac{\partial N_{;b}}{\partial n} + \frac{K_b^c N_{;c}}{N} + K_{;b} - \gamma^{cd} K_{bc;d} \right), \tag{79}$$

$$\Psi_3 = -\frac{\bar{m}^b E}{\sqrt{2}} \frac{n^{m-1}}{A^2} [m A_{;b} + O(n^{-1})], \tag{80}$$

$$\Psi_4 = \frac{2E^2}{A^2} \bar{m}^b \bar{m}^d n^{2m} \left[\frac{A_{;b;d}}{A} + O(n^{-1}) \right]. \tag{81}$$

The condition

$$A_{;b} = 0 \tag{82}$$

is necessary to ensure the finiteness of Ψ_3 and Ψ_4 on the horizon.

Let us point out that the conditions of constancy of B or A on the (ultra)extremal horizon also ensure that in the static frame Ψ_3 and Ψ_4 vanish and thus, similarly as in the non-extremal case, the field from SO's viewpoint is of type D or O.

4.2. Higher order curvature invariants

From the beginning we assumed that the Kretschmann invariant as well as other polynomial invariants of the Riemann tensor are regular on the horizon as was done in [5]. In the previous section we showed that even if this regularity condition is satisfied, Ψ_4 may diverge in the horizon limit in the parallelly propagated frame attached to FFO. Here we show that even when all components of the Weyl tensor in the parallelly propagated frame are regular, curvature invariants constructed from second derivatives of the Riemann tensor may still diverge on the horizon.

First let us point out that if Ψ_4 is finite on the horizon, then it turns out that the first-order[6] curvature invariant $I_{d1} = R_{\alpha\beta\gamma\delta;\epsilon} R^{\alpha\beta\gamma\delta;\epsilon}$ is regular on the horizon[7].

In order to express the second-order curvature invariant $I_{d2} = R_{\alpha\beta\gamma\delta;\epsilon\eta} R^{\alpha\beta\gamma\delta;\epsilon\eta}$ we invoke the remaining coordinate freedom and substitute $[\gamma_H]_{ab}(x, y) = \exp(2\theta(x, y))\delta_{ab}$ into the expansion (35) [5]. Then we obtain

$$I_{d2} = \left(\frac{9}{16} \exp(-4\theta(x, y)) \mathrm{Sp}([\gamma_3](x, y)^2) + \frac{1}{4{\kappa_H}^2} \kappa_3(x, y)^2 \right) n^{-2} + O(n^{-1}), \tag{83}$$

where $\mathrm{Sp}([\gamma_3]^2) = ([\gamma_3]_{11})^2 + 2([\gamma_3]_{12})^2 + ([\gamma_3]_{22})^2$. The term proportional to n^{-1} is quite complicated, but the only information we need is that it vanishes for $\mathrm{Sp}([\gamma_3]^2) = 0 = \kappa_3$. According to section 4.1.1, the first term $\mathrm{Sp}([\gamma_3]^2)$ in the invariant I_{d2} has to vanish if we want to avoid parallelly propagated curvature singularity. It is worthwhile noting that $[\gamma_3] \sim K_{ab}^{(2)}$. As is shown in preceding subsections, this quantity vanishes if all Weyl scalars are finite on the horizon. However, for $\kappa_3(x, y) \neq 0$ the invariant I_{d2} still diverges on the horizon. The case with diverging higher order curvature invariants is often also considered as a curvature singularity (see, e.g., [22]), but we see no reason why timelike geodesics could not be extended through the horizon since all zeroth- and first-order curvature invariants are regular on the horizon. Furthermore, either frame components of the Weyl, Ricci and Riemann tensors as

[6] A curvature invariant is said to be of order n if it contains covariant derivatives of the Riemann tensor up to n.

[7] Recently, certain constraints on the expansion (34), (35) of the metric in the vicinity of the horizon were obtained in [21] by demanding regularity of a first-order curvature invariant instead of regularity of the components of the Weyl tensor in a parallelly propagated frame.

well as the energy density measured by FFO in the parallelly propagated frame are also finite there or we are faced with a new type of a horizon which is considered in the next subsection and that we now turn to (see also section 4.4).

4.3. Regularity

We have seen that finiteness of the Kretschmann invariant $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ does not guarantee by itself finiteness of all curvature components since different terms can enter this expression with different signs due to the Lorentz signature. This does not happen in the static frame where all non-vanishing contributions to the Kretschmann invariant have the same sign but it occurs in the freely falling frame where some divergences in particular components can be mutually cancelled. The same is true with respect to higher order curvature invariants. To gain insight, it is instructive to consider the simplest spherically symmetric case where the metric can be cast in the form

$$\mathrm{d}s^2 = -\frac{F}{G}\,\mathrm{d}t^2 + \frac{\mathrm{d}\rho^2}{F} + r^2(\rho)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{84}$$

It was observed in [23, 24] that the transition from a static frame to a freely falling one leads, in general, to a significant enhancement of some curvature components due to the divergent factors such as $\cosh\alpha$. In particular, in the orthonormal static frame

$$R^{(0)}_{\hat{0}\hat{2}\hat{0}\hat{2}} + R^{(0)}_{\hat{1}\hat{2}\hat{1}\hat{2}} = \frac{F}{r}\left[-r''(\rho) - \frac{G'}{2G}r'(\rho)\right], \tag{85}$$

while in the freely falling frame

$$R_{\hat{1}\hat{2}\hat{1}\hat{2}} + R_{\hat{0}\hat{2}\hat{0}\hat{2}} = (2\cosh^2\alpha - 1)\big(R^{(0)}_{\hat{1}\hat{2}\hat{1}\hat{2}} + R^{(0)}_{\hat{0}\hat{2}\hat{0}\hat{2}}\big). \tag{86}$$

As one approaches the horizon $F(\rho_+) = 0$, the factor $\cosh\alpha$ diverges but the combination (85) tends to zero and, as a result of the competition of these two factors, $R_{\hat{1}\hat{2}\hat{1}\hat{2}} + R_{\hat{0}\hat{2}\hat{0}\hat{2}}$ remains finite (see, e.g., equation (2.12) of [23]). This is, however, not necessarily the case for the distorted horizon. In turn, divergent components of the Riemann curvature tensor lead to divergences in some Weyl components. In particular, for the non-extremal horizon we have seen that the component Ψ_4 remains finite only provided the terms of the order n^2 in K_{ab} vanish near the horizon (equivalently, the terms of the order n^3 vanish in γ_{ab}). In a spherically symmetric spacetime such a condition simply follows from the analyticity of $F(\rho)$, $G(\rho)$ and $r(\rho)$ since the expansion in $\rho - \rho_+$ is equivalent to the expansion in even powers of n. Alternatively, for spherically symmetric spacetimes all Ψ's except Ψ_2 vanish everywhere, so that the transformation (54) does not change this circumstance irrespective of the fact that on the horizon z vanishes. This is simply a manifestation of the known fact that all spherically symmetric spacetimes belong to type D [22].

Thus, as a matter of fact, 'naked black holes' [23, 24] are not naked in the sense that in both frames curvature components are finite. Meanwhile, for a distorted horizon a new possibility opens when curvature components are finite and non-zero in the static frame but some of them become infinite in the freely falling one. As the term 'naked black holes' is already reserved, such objects can be called 'truly naked black holes' (TNBH). We cannot indicate concrete examples of such objects because of obvious difficulties connected with finding exact solutions without spherical symmetry but, in any case, we see no reason why they should be rejected in advance. Apart from this, it is just the condition for absence of TNBH that enables us to understand better the conditions for the curvature components that should hold on the horizon to make it perfectly regular in any frame. It is natural to call it 'strong regularity' of the horizon. In a similar way, one can introduce the condition of regularity of invariants I_{dn} with nth derivatives of the Riemann or Weyl tensor. As was stressed in [25], establishing properties of a generic spacetime connected with analyticity and/or regularity is not an easy task. However, using the conditions of absence of TNBH and regularity of I_{dn}, we can suggest (by analogy with the spherically-symmetrical case) the generalization of the notion of regularity applicable to distorted spacetimes: if all curvature invariants composed of the curvature components and their derivatives up to the order m are finite on the horizon, the metric can be said to be regular up to the order m.

4.4. Structure of Einstein and stress–energy tensors on the horizon

The typical feature of TNBH is that the structure of the Einstein tensor on the horizon changes as compared with usual black holes. Before discussing this point, we would like to comment on some generic properties of the horizon. It was observed in [5] that on the static regular horizon the Einstein tensor should obey the relationships

$$G_1^1 = G_0^0, \qquad G_{1a} = 0 \tag{87}$$

and (if Einstein's equations are fulfilled) similar ones for the stress–energy tensor T_μ^ν. Meanwhile, these equalities immediately follow from the general conditions on the non-extremal horizon

$$R_{\alpha\beta} l^\alpha l^\beta = 0 \tag{88}$$

and

$$R_{\alpha\beta} l^\alpha m^\beta = 0 \tag{89}$$

(see e.g. [26], equations (6.2.2) and an unnumbered equation after equation (6.3.29)). Indeed, substituting (9) into (88), (89), we immediately arrive at (87). On the other hand, the advantage of proof of equations (87) in [5] is that it does not use the weak and dominant energy conditions, on which equations (88), (89) usually rely [26], and arises as a pure geometrical property. (We recall that the constancy of the surface gravity for static Killing horizons was also demonstrated in [5] without using the energy conditions.)

Note that the properties (87) were established for the non-extremal horizons only. Let us see what happens in the case of (ultra)extremal horizons. It follows from equations (40), (42)–(44) of [5] that

$$G_{1a} = K_{;a} - K_{ab;c}\gamma^{cb} - K_a^b \frac{N_{;b}}{N} - \frac{\partial_n N_{;a}}{N}, \tag{90}$$

$$G_1^1 - G_0^0 = -\mathrm{Sp}K^2 - K\frac{N'}{N} + K' + \frac{\Delta_2 N}{N}. \tag{91}$$

On the horizon all terms with K vanish. With the asymptotic form of the lapse function (39) one obtains that on the extremal horizon

$$G_{1a}^H = \frac{B_{;a}}{n_{0B}}, \tag{92}$$

$$\left(G_1^1 - G_0^0\right)^H = \frac{\Delta_2 B}{B}. \tag{93}$$

In a similar way we obtain for the ultraextremal case (43) that

$$G_{1a}^H = 0, \tag{94}$$

$$\left(G_1^1 - G_0^0\right)^H = \frac{\Delta_2 A}{A}. \tag{95}$$

Thus, the block-diagonal structure of G_μ^ν and T_μ^ν, typical for non-extremal horizons, fails for an extremal horizon and the 'equation of state' $p_\parallel \equiv -\rho$ ($p_\parallel \equiv T_1^1$ is a longitudal pressure, $\rho = -T_0^0$ is the energy density) does not hold in that case. Moreover, additional stresses T_{1a} appear on the horizon. For the ultraextremal horizon, the block-diagonal structure retains its validity as is seen from (94) but the horizon 'equation of state' changes. It is worth stressing that it is the combined effect of non-sphericity and (ultra)extremality that leads to such changes. For spherically symmetric spacetimes $\Delta_2 A = \Delta_2 B = 0$ and the horizon structure of G_μ^ν and T_μ^ν coincides with that of non-extremal horizons.

Note that, as $\int \mathrm{d}^2x\sqrt{\gamma}\Delta_2 A = \int \mathrm{d}^2x\sqrt{\gamma}\Delta_2 B = 0$ over a closed surface, these quantities should change sign somewhere on the cross-section of the horizon ($t = \text{const}$, $n \to \infty$). It means that $\xi \equiv (p_\parallel + \rho)^H$ also changes sign, so that the null energy condition ($T_{\mu\nu}l^\mu l^\nu \geqslant 0$ for any null vector) is violated in some region on the horizon surface. In other words, there are regions on the (ultra)extremal distorted horizon where $\xi < 0$, so that the matter source becomes 'phantomic'. In recent years, such a type of source has been discussed intensively in cosmology (see, e.g., [27]) but we see that it arises in the black hole context as well.

The relationships (92), (93)–(95) are obtained from the regularity conditions in the static frame only. If, additionally, we assume that the horizon is not 'truly naked' then the equalities (76), (82) should hold. As a consequence, all extra terms in (92), (93)–(95) vanish and we return to the 'normal' relations (87) so that there is no difference at this point between the non-extremal and (ultra)extremal horizons. In a similar way, the extra terms vanish in Ψ_2^H in (42), (47).

One can also carry out the following analogy between non-extremal and (ultra)extremal horizons. In the first case, the leading term of the asymptotics of the lapse function N is determined by the surface gravity κ_H according to (34). The demand of regularity of the Kretschmann invariant on the horizon leads to the constancy of the surface gravity there: $\kappa_{H;a} = 0$. In the second case $\kappa_H = 0$ and the main term in the asymptotics is determined by the coefficient B (or A). If we demand the regularity of not only the Kretschmann scalar but also all curvature components in the freely falling frame, we obtain a similar condition $B_{;a} = 0$ (or $A_{;a} = 0$). In this sense, the condition of absence of truly naked extremal horizons looks formally like the analogue of the zeroth law of mechanics of non-extremal black holes.

The condition of finiteness of all components of the Riemann (or Weyl) tensors on the horizon in the freely falling frame can be also understood as follows. The energy measured by a freely falling observer is equal to

$$\varepsilon = T_{\mu\nu}u^\mu u^\nu = \cosh^2\alpha\,\varepsilon^{(0)} + \sinh^2\alpha\, p_\parallel^{(0)} = \frac{E^2}{N^2}\left[\varepsilon^{(0)} + p_\parallel^{(0)}\right] - p_\parallel^{(0)}, \tag{96}$$

where we used (51), (52) and (8). It follows from the Einstein equations $\varepsilon^{(0)} + p_\parallel^{(0)} = \frac{G_1^1 - G_0^0}{8\pi}$, where the latter quantity is given by equation (91). Meanwhile, the Ricci component in the FFO's frame $R_{\alpha\beta}n^\alpha n^\beta = \frac{G_1^1 - G_0^0}{2z^2}$, where near the horizon $z^{-2} = \frac{4E^2}{N^2}$. Thus, near the horizon equation (96) can be rewritten also as $\varepsilon = \frac{R_{\alpha\beta}n^\alpha n^\beta}{16\pi} - p_\parallel^{(0)}$.

Consider the non-extremal case. In the limit $n \to 0$ we obtain that $\varepsilon = \frac{E^2 K_a^{a(2)}}{16\pi\kappa_H^2}n^{-1} + O(1)$ so that ε diverges on the horizon unless $K_a^{a(2)} = 0$. Thus, the condition $K_{ab}^{(2)} = 0$ necessary for the finiteness of Ψ_4 ensures also the finiteness of ε on the horizon.

Consider now the extremal case and substitute the expansion (78) into (96). Then near the horizon

$$\varepsilon = \frac{E^2}{8\pi M^3}\left[\Delta B \exp\left(2\frac{n}{n_0}\right) + \Delta B_1 \exp\left(\frac{n}{n_0}\right) + \Delta B_2\right] + \cdots \tag{97}$$

and the conditions $B_{;a} = 0 = B_{1;a}$ ensuring the finiteness of Ψ_4 ensure also the absence of singular terms in ε. A similar result takes place for ultraextremal black holes. In other words, in all three cases either we have a regular horizon without TNBH and finite ε or TNBH and infinite ε.

5. Ricci tensor

There are several methods of classification of the Ricci tensor (see, e.g., [14, 28–31]). One possible method is to construct the so-called Plebański tensor, with the same symmetries as the Weyl tensor, from the traceless part of the Ricci tensor, $S_{\alpha\beta} = R_{\alpha\beta} - \frac{R}{4} g_{\alpha\beta}$. Then one can classify the Ricci tensor according to the Petrov type of the Plebański tensor. The corresponding algebraic type is then called Petrov-Plebański (PP) type. Another more detailed classification is the Segre classification based on geometric multiplicities of eigenvalues of $S_{\alpha\beta}$.

For determining to which class the Ricci tensor on the horizon belongs, we will use the classification algorithm and notation given in [30]. We will thus need to express curvature invariants I_6, I_7 and I_8 and analyse whether certain syzygies between them are satisfied.

First step is to determine whether

$$Q \equiv {I_R}^3 - 27{J_R}^2 = 0, \tag{98}$$

where

$$I_R = \frac{1}{48}\left(7{I_6}^2 - 12 I_8\right), \tag{99}$$

$$J_R = \frac{1}{1728}\left(36 I_6 I_8 - 17 {I_6}^3 - 12{I_7}^2\right), \tag{100}$$

$$I_6 = S_\mu^\nu S_\nu^\mu, \qquad I_7 = S_\mu^\nu S_\alpha^\mu S_\nu^\alpha, \qquad I_8 = S_\mu^\nu S_\alpha^\mu S_\beta^\alpha S_\nu^\beta. \tag{101}$$

Using the formulae of $2+1+1$ decomposition (18)–(21), it is straightforward to find that

$$S_0^0 = \frac{1}{4}(\mathrm{Sp}K^2 + K^2 - R_\parallel) + \frac{1}{2}\left(K\frac{N'}{N} - \frac{N''}{N} - K' - \frac{\Delta_2 N}{N}\right), \tag{102}$$

$$S_1^1 = \frac{K'}{2} - \frac{3}{4}\mathrm{Sp}K^2 - \frac{1}{2}\frac{N''}{N} - \frac{1}{4}\left(R_\parallel - K^2 - \frac{2\Delta_2 N}{N} + 2K\frac{N'}{N}\right), \tag{103}$$

$$S_{ab} = \frac{\partial K_{ab}}{\partial n} + 2(K^2)_{ab} - K_{ab}K + \frac{1}{N}\left(K_{ab}N' - N_{;ab}\right) + \gamma_{ab}\frac{L}{4}, \tag{104}$$

$$L = R_\parallel - R_\perp - 2K' + \mathrm{Sp}K^2 + K^2 + \frac{2\Delta_2 N}{N} - 2K\frac{N'}{N}, \tag{105}$$

$$S_{1a} = R_{1a} = K_{;a} - K_{ab;c}\gamma^{bc} - K_a^b\frac{N_{;b}}{N} - \frac{\partial_n N_{;a}}{N}. \tag{106}$$

On the non-extremal horizon

$$S_{1a}^H = 0, \tag{107}$$

$$S_1^{1(H)} = S_0^0(H) \equiv -\alpha, \tag{108}$$

$$S_{ab}^H = \gamma_{ab}\alpha + \mu_{ab}, \qquad \mu_{ab} \equiv 2K_{ab}^{(1)} - \gamma_{ab}K^{(1)}, \qquad \mathrm{Sp}\hat{\mu} = \mu_a^a = 0. \tag{109}$$

It is worth noting that, for our choice (9), equation (108) is equivalent to (88) and equation (107) is equivalent to (89).

The expression for 2×2 matrix S_{ab} can be rewritten in the form

$$\hat{S} = \alpha\hat{I} + \hat{\mu}. \tag{110}$$

It is convenient to represent the traceless symmetrical $2 \times 2\hat{\mu}$ matrix in the form

$$\hat{\mu} = a\sigma_z + b\sigma_x, \tag{111}$$

where σ_z and σ_x are Pauli matrices and $a = \mu_{11}$, $b = \mu_{12}$. Then, using the properties of Pauli matrices $\sigma_x^2 = \sigma_z^2 = 1$ and $\mathrm{Sp}\,\sigma_x = \mathrm{Sp}\,\sigma_z = 0$, it is easy to find that

$$I_6 = 2(2\alpha^2 + \beta^2), \qquad \beta^2 \equiv a^2 + b^2, \tag{112}$$

$$I_7 = 6\alpha\beta^2, \tag{113}$$

$$I_8 = 2(2\alpha^4 + 6\alpha^2\beta^2 + \beta^4), \tag{114}$$

$$I_R = \frac{A}{48}, \qquad A = 4(4\alpha^2 - \beta^2)^2, \tag{115}$$

$$J_R = \frac{(-64\alpha^6 + 48\alpha^4\beta^2 - 12\alpha^2\beta^4 + \beta^6)}{216} \tag{116}$$

and direct comparison shows that, indeed, $Q = 0$. Since I_7 is in general non-vanishing, we can conclude that the Ricci tensor has in the generic case one pair of equal eigenvalues and thus the list of its algebraic multiplicities is {112}. Corresponding Segre types are [112] (PP-type II) and [(11)1, 1], [11(1, 1)] and $[(11)Z\bar{Z}]$ (PP-types D).

Similarly, the same result may be obtained for (ultra)extremal horizons.

Note also that, if the TNBH is excluded, the Ricci components $\Phi_{00} = \frac{1}{2}S_{\alpha\beta}l^\alpha l^\beta$ and $\Phi_{01} = \frac{1}{2}S_{\alpha\beta}l^\alpha m^\beta$ vanish on the horizon both in SO's and FFO's frame (this immediately implies that the PP-type is algebraically special, but we still need invariant I_7 to rule out more special cases).

Now, let us say that a quantity x which under a boost $l'^\alpha = zl^\alpha$, $n'^\alpha = z^{-1}n^\alpha$ changes according to $x' = z^q x$ has a boost weight q. We may conclude that on the horizon all components of the Weyl, Ricci and also Riemann and Einstein tensors with positive boost weight vanish. This implies that all these tensors are aligned[8], algebraically special on the horizon in the sense of [32] and of the alignment type (2).

6. Summary and conclusions

We have analysed various properties of the curvature tensors in the vicinity of a generic static Killing horizon.

For both SO and FFO the Weyl, Ricci, Riemann and Einstein tensors on the horizon are algebraically special of alignment type (2) with a common aligned null direction. However, for FFO and SO the horizon limits in general do not coincide since SO becomes null and the corresponding frame is singular on the horizon. Consequently, only the results obtained in the freely falling frame should be regarded as relevant.

It turns out that the horizon of a generic static black hole is in general of Petrov type II. More special types (D, III, N, O) are also possible. Further details depend on relationships between quantities J, K, L, N (see section 2), that do not follow directly from the properties of the Killing horizon and, therefore, cannot be determined in a general form.

Possible Segre types of the Ricci tensor on the horizon are [112], [(11)1, 1], [11(1, 1)] and $[(11)Z\bar{Z}]$ and more special.

[8] I.e. they have common aligned null direction defined in [32].

It is found that the notion of regularity on the horizon requires a more subtle definition. Due to the Lorentz signature, finiteness of the Kretschmann invariant on the horizon does not exclude divergences in some components of the Weyl tensor in the freely falling frame or, equivalently, infinite tidal forces or energy density as measured by FFO. It turns out that the horizon may look regular from the viewpoint of SO but singular from the viewpoint of FFO and the conditions ('strong regularity' of a horizon) that exclude such exotic objects ('truly naked black holes'—TNBH) are given. In doing so, for the (ultra)extremal case we obtained a formal analogue of the zeroth law of mechanics of non-extremal black holes.

It has also turned out that for non-extremal horizons it is necessary to apply an additional restriction on expansion of the lapse function together with strong regularity conditions in order to guarantee that a curvature invariant composed of the second derivatives of the Riemann tensor is also finite.

It is shown that the structure of a stress–energy tensor for distorted truly naked horizons differs from the non-extremal case, so that the block-diagonal structure of the stress–energy tensor fails. In doing so, the stress–energy tensor should somewhere on the horizon have a phantomic-like equation of state. This reveals itself for non-spherical horizons only as a combined effect of non-sphericity, extremality and presence of infinite tidal forces for FFOs. Once we impose the condition of strong regularity, this effect immediately disappears.

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