BOOTSTRAPPING ON CUTS IN Q

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ABSTRACT. In this brief note, I sketch the first stage of the interpretation of $\mathrm{I}\Delta_0$ on a cut in Q. Many of the details are already in Rachel Sterken's masters thesis [Ste08].

1. WHAT IS Q?

The original formalisation of Q in Tarski, Mostowski, Robinson [TMR53] does not have the symbol $\leqslant$ in the language, but treats it as a defined symbol. Nelson [Nel86] has the same axioms and so do Boolos & Jeffrey (2nd Edition) [BJ02]. Hájek & Pudlák [HP93] have $\leqslant$ in the language and add a defining axiom. In both treatments, we have $\vdash x \leqslant y \leftrightarrow \exists z\, z + x = y$. We will treat $\leqslant$ as defined here. We write $x < y$ for $x \leqslant y \wedge y \not\leqslant x$.

We note that the choice of definition of $\leqslant$ works out very differently from the alternative $x \leqslant y :\leftrightarrow \exists z\, x + z = y$.

Boolos, Jeffrey & Burgess (4th edition) [BBJ02] has as theory called Q which is our theory Q in the language extended with with $<$ plus axioms $\neg x < 0$ and $x < \mathsf{S}y \leftrightarrow (x < y \vee x = y)$ and $x < y \vee x = y \vee y < x$. I strongly disapprove of the renaming. On the other hand, we can perhaps find a weaker theory than Q where $<$ or $\leqslant$ is axiomatised using universal axioms that support the bootstrap. That would evade a number of problems. Of course, the BJB version of Q is not weaker than Q, so that is not the right choice.

2. ELEMENTS OF THE BOOTSTRAP

Here is a new version of the bootstap going from Q to a more convenient starting point for further bootstrap.

Apart from some easily repairable small gaps in the Hájek & Pudlák argument, there are two larger ones. First, they do not create a situation in which $\leqslant$ becomes absolute as we do here using $\mathcal{A}_2$. Secondly, the argument that downward closure is preserved by the addition of addition is entirely lacking.

A class will be a virtual class in the sense of Quine. It is really simply a formula that we conceptually view as a class, however we allow ourselves the luxury of writing $x \in \mathcal{A}$, etc.

A class is *inductive* if it is closed under 0 and successor. A class is *a cut* if it is inductive and downward closed under $\leqslant$. We note that a cut is not automatically closed under predecessor, since Q does not prove $x \leqslant \mathsf{S}x$.

2.1. **Output/Value Bootstraps.** In this subsection, we execute a number of bootstraps where the variable of the defined class is the value z of an equation of the form $t = z$. I conjecture that any bootstrap leading to a cut-interpretation of a sufficiently large portion of $\mathrm{I}\Delta_0$ will need at least one or two of such classes.

Let $\mathcal{A}_0$ be the class of z such that, for all x, y, if $x + \mathsf{S}y = z$, then $\mathsf{S}x + y = z$.

Theorem 2.1 (Q). *$\mathcal{A}_0$ is inductive.*

Proof. Clearly 0 is in $\mathcal{A}_0$.

Suppose z is in $\mathcal{A}_0$ and $x + \mathsf{S}y = \mathsf{S}z$. Then, $\mathsf{S}(x + y) = \mathsf{S}z$ and, hence, $x + y = z$. If $y = 0$, we find $x = z$, and, thus, $\mathsf{S}x + 0 = \mathsf{S}z$. Suppose $y = \mathsf{S}y'$. Applying that $z \in \mathcal{A}_0$, we obtain: $\mathsf{S}z = x+\mathsf{S}y = x+\mathsf{S}\mathsf{S}y' = \mathsf{S}(x+\mathsf{S}y')$. So, $z = x+\mathsf{S}y'$. It follows that $z = \mathsf{S}x + y'$. Hence, $\mathsf{S}z = \mathsf{S}x + y$ ❑

Let $\mathcal{A}_1$ be the class of z such that $0 + z = z$.

Theorem 2.2 (Q). *$\mathcal{A}_1$ is inductive.*

The proof is trivial.

Let $\mathcal{A}_2$ be the class of all z such that $z \in \mathcal{A}_0 \cap \mathcal{A}_1$ and, for all x, y, if $x + y = z$, then $y + x = z$.

Theorem 2.3 (Q). *$\mathcal{A}_2$ is inductive.*

Proof. It is easy to see that 0 is in $\mathcal{A}_2$.

Suppose z is in $\mathcal{A}_2$ and $x+y = \mathsf{S}z$. If $y = 0$, we find $x = \mathsf{S}z$. Since $z \in \mathcal{A}_1$, we have $x = \mathsf{S}z \in A_1$, so $x + 0 = 0 + x$. Suppose $y = \mathsf{S}y'$. We have, using that $\mathsf{S}z \in \mathcal{A}_0$,

$$\begin{aligned} x + y = \mathsf{S}z &\to x + \mathsf{S}y' = \mathsf{S}z \\ &\to x + y' = z \\ &\to y' + x = z \\ &\to y' + \mathsf{S}x = \mathsf{S}z \\ &\to \mathsf{S}y' + x = \mathsf{S}z \\ &\to y + x = \mathsf{S}z \end{aligned}$$

❑

We note that if we are given that, for all z in $\mathcal{X}$ and, for all x and y, we have $x+y = z$ implies $y+x = z$, and that $\mathcal{X}$ is closed under 0, successor and predecessor, it follows that $\mathcal{X}$ is contained in $\mathcal{A}_0$ and $\mathcal{A}_1$ and, hence, in $\mathcal{A}_2$.

Theorem 2.4 (Q). *Suppose $z \in \mathcal{A}_2$. Then, $z \leqslant z + \underline{n}$.*

Proof. Suppose $z \in \mathcal{A}_2$. Then, $z + \underline{n} \in \mathcal{A}_2$. So, $z + \underline{n} = \underline{n} + z$. it follows that $z \leqslant z + \underline{n}$. ❑

Theorem 2.5 (Q). *Suppose* $\mathcal{B} \subseteq \mathcal{A}_2$ *and* $\mathcal{B}$ *is a cut. Then,* $\mathcal{B}$ *is closed under predecessor.*

Proof. Suppose $\mathsf{S}z \in \mathcal{B}$. Then, $z \in \mathcal{B}$, since $z \leqslant \mathsf{S}z$. ❑

The next theorem gives our condition under which $\leqslant$ becomes absolute.

Theorem 2.6 (Q). *Suppose* x *and* y *are in* $\mathcal{A}_2$. *Then,* $x \leqslant y$ *iff* $(x \leqslant y)^{\mathcal{A}_2}$.

Proof. Suppose $y \in \mathcal{A}_2$. The right-teft direction is immediate. Suppose $x \leqslant y$. Then, for some u, we have $u + x = y$. It follows that $x + u = y$ and, thus, $u \leqslant y$. So, $u \in \mathcal{A}_2$. ❑

Let $\mathcal{A}_3$ be the class of z such that, for all w, x, y, we have $w + (x + y) = z$ iff $(w + x) + y = z$. We call the elements of $\mathcal{A}_3$ the *associators.*

Theorem 2.7 (Q). $\mathcal{A}_3$ *is inductive.*

Proof. Clearly 0 is in $\mathcal{A}_3$.

Suppose $z \in \mathcal{A}_3$ and $w + (x + y) = \mathsf{S}z$. If $y = 0$, we are immediately done. Let $y = \mathsf{S}y'$. Then, $w + (x + y') = z$ and, thus, $(w + x) + y' = z$. It follows that $(w + x) + y = \mathsf{S}z$.

The right-to-left direction of the *iff* is similar. ❑

Theorem 2.8 (Q). *Suppose* $x \leqslant y \leqslant z$ *and* $z \in \mathcal{A}_3$. *Then* $x \leqslant z$.

Proof. For some u and v, we have $u + x = y$ and $v + y = z$. Hence $v + (u + x) = z$. Ergo, $(v + u) + x = z$ and, hence, $x \leqslant z$. ❑

We define $\mathsf{dc}(\mathcal{B})$ as the class of the z such that, for all $y \leqslant z$, $y \in \mathcal{B}$.

Theorem 2.9 (Q). *Suppose* $\mathcal{B} \subseteq \mathcal{A}_2 \cap \mathcal{A}_3$ *and* $\mathcal{B}$ *is inductive. Then,* $\mathsf{dc}(\mathcal{B})$ *is a subclass of* $\mathcal{B}$ *and* $\mathcal{B}$ *is a cut.*

Proof. $\mathsf{dc}(\mathcal{B})$ is a subclass of $\mathcal{B}$ since for all elements x of $\mathcal{A}_2$, we have $x \leqslant x$ (by Theorem 2.4).

This step is lacking in H&P. Of course they could have adapted the definition of dc by intersecting our $\mathsf{dc}(\mathcal{B})$ with $\mathcal{B}$. There is, for example, a model of Q in which $\leqslant$ is transitive with a set $\mathcal{X}$ of $2^{\aleph_0}$ elements a such that, for no b, $b \leqslant a$. The downward closure in the style of H&P would contain $\mathcal{X}$.

Clearly 0 is in $\mathsf{dc}(\mathcal{B})$.

Suppose $z \in \mathsf{dc}(\mathcal{B})$. Then z is in $\mathcal{B}$ and, hence, $\mathsf{S}z$ is in $\mathcal{B}$. Consider any $y \leqslant \mathsf{S}z$. If $y = 0$, we are immediately done. Suppose $y = \mathsf{S}y'$. Then, $\mathsf{S}y' \leqslant \mathsf{S}z$, so $y' \leqslant z$ and, hence, $y' \in \mathcal{B}$. So, $y \in \mathcal{B}$.

Suppose $z \in \mathsf{dc}(\mathcal{B})$ and $y \leqslant z$. Suppose $x \leqslant y$. Then, since z is in $\mathcal{A}_3$, we find that $x \leqslant z$ and, so, $x \in \mathcal{B}$. ❑

We define the additive closure of $\mathcal{B}$, $\mathsf{add}(\mathcal{B})$, as the class of all z such that, for all y in $\mathcal{B}$, we have $y + z \in \mathcal{B}$.

Theorem 2.10 (Q). *Suppose $\mathcal{B}$ is inductive and $\mathcal{B}$ is contained in $\mathcal{A}_2 \cap \mathcal{A}_3$. Then, $\mathcal{B}$ is contained in $\mathsf{add}(\mathcal{B})$ and $\mathsf{add}(\mathcal{B})$ is is an inductive class that is closed under $+$. Moreover, if $\mathcal{B}$ is also a cut, then $\mathsf{add}(\mathcal{B})$ is a cut.*

Proof. Suppose $z \in \mathsf{add}(\mathcal{B})$. Then, $z = 0 + z \in \mathcal{B}$. This uses that $z \in \mathcal{A}_2$.

Clearly, 0 is in $\mathsf{add}(\mathcal{B})$.

Suppose $z \in \mathsf{add}(\mathcal{B})$. Then, $\mathsf{S}z \in \mathcal{A}_2 \cap \mathcal{A}_3$. Suppose $y \in \mathcal{B}$. Then, $y + z \in \mathcal{B}$ and hence $y + \mathsf{S}z = \mathsf{S}(y + z) \in \mathcal{B}$.

Suppose w and z are in $\mathsf{add}(\mathcal{B})$ and $y \in \mathcal{B}$. Then, $(y+w) \in \mathcal{B}$ and, so, $((y+w)+z) \in \mathcal{B}$. Since, the $\mathcal{B}$ are associators, we find $(y + (w + z)) \in \mathcal{B}$.

Let $\mathcal{B}$ be a cut. Suppose $z \in \mathsf{add}(\mathcal{B})$ and $w \leqslant z$ and $y \in \mathcal{B}$. Say, $u+w = z$. We have $(y + (u + w)) \in \mathcal{B}$. Since the elements of $\mathcal{B}$ are both commutators and distributors and $\mathcal{B}$ is downward closed, we find $((y + u) + w) \in \mathcal{B}$, so $(w + (y + u)) \in \mathcal{B}$, hence, $(y+u) \in \mathcal{B}$, and, thus $y+u = u+y$. It follows that $((u+y)+w) \in \mathcal{B}$ and, therefore, $(u + (y + w)) \in \mathcal{B}$. We may conclude $(y + w) \in \mathcal{B}$. ❑

No consideration like the last paragraph of the above proof can be found in H&P.

2.2. Bootstraps using the Induction Variable.

We now turn to the bootstrap over the variable that is active in the inductive definition of a connective.

We define $\mathcal{A}_4$ as the class of z such that, for all x and y, we have $x + (y + z) = (x + y) + z$.

Theorem 2.11 (Q). *$\mathcal{A}_4$ is inductive.*

The proof is entirely trivial.

We define $\mathcal{A}_5$ as the class of z such that, for all x in $\mathcal{A}_4$ and all y, we have $x(y+z) = xy + xz$.

Theorem 2.12 (Q). *$\mathcal{A}_5$ is inductive.*

Proof. Clearly, 0 is in $\mathcal{A}_5$.

Suppose $z \in \mathcal{A}_5$ and $x \in \mathcal{A}_4$. We have:

$$\begin{array}{rcl} x(y+\mathsf{S}z) & = & x\mathsf{S}(y+z) \\ & = & x(y+z)+x \\ & = & (xy+xz)+x \\ & = & xy+(xz+x) \\ & = & xy+x\mathsf{S}z. \end{array}$$ ❑

We define $\mathcal{A}_6$ as the class of all z such that, for all x and y in , we have $(xy)z = x(yz)$.

Theorem 2.13 (Q)**.** $\mathcal{A}_6$ *is inductive.*

Proof. Clearly, $0 \in \mathcal{A}_6$.

Suppose $z \in \mathcal{A}_6$, $y \in \mathcal{A}_5$, $x \in \mathcal{A}_4$. We find:

$$(xy)\mathsf{S}z = (xy)z + xy = x(yz) + xy = x(yz+y) = x(y\mathsf{S}z).$$ ❑

Let $\mathsf{mul}(\mathcal{B})$ be the class of y in $\mathcal{B}$ such that, for all x in $\mathcal{B}$, we have xy is in $\mathcal{B}$. Let $\mathcal{A}^*$ be the intersection of $\mathcal{A}_2, \ldots, \mathcal{A}_6$.

We easily show that the z such $1 \cdot z = z$ form an inductive class. If we add the intersection with this class to the definition of $\mathcal{A}^*$, we can drop the demand that $y \in \mathcal{B}$ in the definition of mul.

Theorem 2.14. *Suppose we have $\mathcal{B} \subseteq \mathcal{A}^*$ and $\mathcal{B}$ is inductive and $\mathcal{B}$ is closed under plus. Then, $\mathsf{mul}(\mathcal{B})$ is closed under multiplication. Moreover, if $\mathcal{B}$ is a cut, then so is $\mathsf{mul}(\mathcal{B})$.*

Proof. Clearly, $0 \in \mathcal{B}$.

Suppose y is in $\mathsf{mul}(\mathcal{B})$. Consider any x in $\mathcal{B}$. Then, $x\mathsf{S}y = xy + x \in \mathcal{B}$.

Suppose y and z are in $\mathsf{mul}(\mathcal{B})$. Consider any x in $\mathcal{B}$. Then, $x(y+z) = xy+xz \in \mathcal{B}$.

Suppose y and z are in $\mathsf{mul}(\mathcal{B})$. Then, $x(yz) = (xy)z \in \mathcal{B}$.

Let $\mathcal{B}$ be a cut. Suppose $y \leqslant z$ and $z \in \mathsf{mul}(\mathcal{B})$. Say $u + y = z$. Then, for any $x \in \mathcal{B}$, we have $xu + xy = x(u + y) \in \mathcal{B}$. So, $xy \in \mathcal{B}$. ❑

Now the most elegant way of proceeding is to relativise to $\mathcal{A}^\circ := \mathsf{mul}(\mathsf{add}(\mathsf{dc}(\mathcal{A}^*)))$. Inside $\mathcal{A}^\circ$, we have Q plus the associativity of plus and times, plus the commutativity of plus, plus left distributivity. (We remind the reader that $\mathcal{A}^\circ$ is closed under predecessor, since $x \leqslant \mathsf{S}x$.) Let us call this theory Q°. Moreover, we have:

Theorem 2.15. $\mathsf{Q} \vdash (\mathsf{cut}(\mathcal{B}))^{\mathcal{A}^\circ} \leftrightarrow \mathsf{cut}(\mathcal{B}\mathcal{A}^\circ)$.

Proof. The right-to-left direction is easy. Suppose $(\mathsf{cut}(\mathcal{B}))^{\mathcal{A}^\circ}$, $x \leqslant y$ and $y \in \mathcal{B}\mathcal{A}^\circ$. Since $y \in \mathcal{A}^\circ$ and $\mathcal{A}^\circ$ is downward closed, we find that $x \in \mathcal{A}^\circ$, By the absoluteness of $\leqslant$, it follows that $(x \leqslant y)^{\mathcal{A}^\circ}$ and, hence $x \in \mathcal{B}\mathcal{A}^\circ$. ❑

So we can proceed with our bootstap inside $\mathcal{A}^\circ$ using Q° and we are guaranteed that internal cuts will be also external ones at the level of the orginal theory.

3. A Cut in a Cut need not be a Cut

A *pro-cut* or cut^+ is a cut that is closed under $+$ and $\times$ and satisfies H&P's Q^+, i.e., we have associativity of plus and times and left-distributivity inside the cut.

Can there be two formulas $\mathcal{I}$ and $\mathcal{J}$ such that $\mathsf{Q} \vdash \mathsf{cut}^+(\mathcal{I})$ and $\mathsf{Q} \vdash (\mathsf{cut}^+(\mathcal{J}))^{\mathcal{I}}$, but $\mathsf{Q} \nvdash \mathsf{cut}(\mathcal{J}\mathcal{I})$?

In this section, we give a simple model, say $\mathfrak{M}_0$, to show that the answer is yes. The domain of $\mathfrak{M}_0$ is the natural numbers plus three new elements $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$. We define the operations $+$ and $\times$. The 0 of our model is the 0 of the natural numbers and S is defined as $+1$. In the tables below m and n range of natural numbers. The operations on the natural numbers are the usual ones.

$+$	n	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$
m	$m+n$	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$
$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{c}$
$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{a}$	$\mathfrak{c}$

$\times$	0	$n+1$	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$
0	0	0	0	0	0
$m+1$	0	$(m+1)\times(n+1)$	$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{a}$
$\mathfrak{a}$	0	$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{a}$
$\mathfrak{b}$	0	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$
$\mathfrak{c}$	0	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$

It is easily verified that we have a defined a model of Q. Moreover, we have $n < \mathfrak{a}$, $n < \mathfrak{b}$, $n < \mathfrak{c}$, $\mathfrak{a} \leqslant \mathfrak{b}$, $\mathfrak{a} < \mathfrak{c}$, $\mathfrak{b} \leqslant \mathfrak{a}$. We note that $\leqslant$ is transitive, hence a preordering.

Addition is not associative. We have:

$$\mathfrak{c} + (\mathfrak{c} + \mathfrak{b}) = \mathfrak{c} + \mathfrak{a} = \mathfrak{c} \neq \mathfrak{a} = \mathfrak{c} + \mathfrak{b} = (\mathfrak{c} + \mathfrak{c}) + \mathfrak{b}.$$

Multiplication is associative. I did not check whether we have left distributivity. Leon Probst checked that we have both right and left distributivity.

Let α be the conjunction of the axioms of Q^+. Let $\beta(x, y, z)$ be the universal closure of the conjunction of the addition and multiplication tables where:

- x has the role of $\mathfrak{a}$;
- y has the role of $\mathfrak{b}$;
- z has the role of $\mathfrak{c}$;
- we have α relativised to the elements distinct from x, y, z.

Let $\beta^* := \exists x\, \exists y\, \exists z\, \beta(x, y, z)$. Suppose β^*. Let

$$\mathcal{K} := \{u \mid \exists x\, \exists y\, \exists z\, (\beta(x, y, z) \wedge u \neq z)\}.$$

Then, $\mathcal{K}$ is a cut. In our model, the cut $\mathcal{K}$ is ω plus $\mathfrak{a}$, $\mathfrak{b}$. We can easily show that both plus and times are associative in $\mathcal{K}$ and that we have left distributivity. So, $\mathcal{K}$ is a pro-cut.

We know that, in Q, we can effectively construct a pro-cut $\mathcal{P}$. We define $\mathcal{I} := \mathcal{K}\langle\beta^*\rangle\mathcal{P}$, i.e., if we have β^*, then $\mathcal{I}$ behaves like $\mathcal{K}$, otherwise, $\mathcal{I}$ behaves like $\mathcal{P}$. It follows that Q proves $\mathcal{I}$ is a pro-cut.

It is a bit strange that we seem to need the correct construction to show that another attempt at construction is incorrect. Can this be avoided?

We define $\mathcal{J}$ inside $\mathcal{I}$ as follows. We define $\gamma(x, y)$ similarly to $\beta(x, y, z)$ with x in the role of $\mathfrak{a}$ and y in the role of $\mathfrak{b}$. Let $\gamma^* := \exists x\, \exists y\, \gamma(x, y)$. Assuming γ^*, we see that $\mathcal{Q} := \{v \mid \exists x\, \exists y\, (\gamma(x, y) \wedge v \neq y)\}$ is a pro-cut. Moreover, $\mathcal{Q}$ will satisfy Q^+.

We define $\mathcal{J} := \mathcal{Q}\langle\gamma^*\rangle\mathsf{ID}$. Here ID is the identical cut.

We find that Q^+ proves that $\mathcal{J}$ is a pro-cut and, hence, that Q proves that $\mathcal{J}$ is a pro-cut inside $\mathcal{I}$.

Finally, we see that $\mathcal{J}\mathcal{I}$ is not Q-provably a cut, since, in $\mathcal{M}_0$, we have $\mathfrak{b} \leqslant \mathfrak{a}$.

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