

Logic in Computer Science II

4th lesson

the graphs of proofs

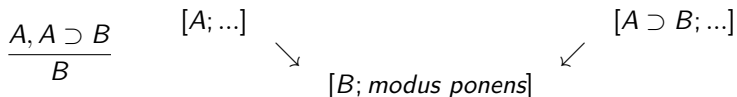
- ▶ directed acyclic graph (DAG)
- ▶ nodes = labeled by
 1. formulas or sequents and
 2. rules applied
- ▶ arrows = indicate which assumptions used
- ▶ sources = axioms
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Example



trees and DAGs

Two forms of proofs

1. general, DAG-like
2. tree-like, useful for analyzing proofs

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A similar distinction for Boolean circuits:

1. general Boolean **circuits**, DAG-like
2. tree-like, propositional **formulas**

Basic fact

If

$$\models \alpha(\vec{q}) \vee \beta(\vec{r})$$

where $\vec{q} \cap \vec{r} = \emptyset$, then

$$\models \alpha(\vec{q}) \quad \text{or} \quad \models \beta(\vec{r}).$$

Proof.

Suppose we have assignments $\vec{q} \mapsto \vec{a}$, $\vec{r} \mapsto \vec{b}$ such that $\alpha(\vec{a}) = 0$ and $\beta(\vec{b}) = 0$, then $\alpha(\vec{a}) \vee \beta(\vec{b}) = 0$. □

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Hence if

$$\models \alpha(\vec{p}, \vec{q}) \vee \beta(\vec{p}, \vec{r})$$

and $\vec{p} \mapsto \vec{c}$, then

$$\models \alpha(\vec{c}, \vec{q}) \quad \text{or} \quad \models \beta(\vec{c}, \vec{r}).$$

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From a **given proof** P

$$P \vdash \alpha(\vec{p}, \vec{q}) \vee \beta(\vec{p}, \vec{r}),$$

and an assignment \vec{c} to common variables we may decide which of the formulas $\alpha(\vec{c}, \vec{q}), \beta(\vec{c}, \vec{r})$ is a tautology.

Definition

If the above can be done in polynomial time, then we say that the proof system has **feasible interpolation**.

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Definition

If the above can be done in polynomial time, then we say that the proof system has **feasible interpolation**.

For some weak proof systems this is indeed possible.

Feasible interpolation in terms of disjoint NP-sets:

Given a proof P of

$$A \cap B = \emptyset$$

and given $c \in A \cup B$, then it is possible to decide whether

$$c \in A \quad \text{or} \quad c \in B.$$

Formally, we need a **sequence** of **polynomial size** proofs P_n of tautologies $\alpha_n(\vec{p}_n, \vec{q}_n) \vee \beta_n(\vec{p}_n, \vec{r}_n)$ representing the complements of sets A, B .

feasible interpolation for cut-free proofs

Theorem

Given a *tree-like* cut-free proof

$$P \vdash \neg\alpha(\vec{p}, \vec{q}) \rightarrow \beta(\vec{p}, \vec{r})$$

we can construct in *polynomial time* a *formula* $I(\vec{p})$ s.t.

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Hence given $\vec{p} \mapsto \vec{a}$, we can decide in *polynomial time* which of the two is true

$$\models \alpha(\vec{a}, \vec{q}) \quad \text{or} \quad \models \beta(\vec{a}, \vec{r}).$$

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feasible interpolation for Resolution

Theorem

Given a *Resolution* proof P of contradiction from a set of clauses $\{A_i(\vec{p}, \vec{q})\}_i \cup \{B_j(\vec{p}, \vec{r})\}_j$, in symbols:

$$P : \{A_i(\vec{p}, \vec{q})\}_i \cup \{B_j(\vec{p}, \vec{r})\}_j \rightarrow \perp,$$

we can construct in *polynomial time* a *circuit* C s.t. for all assignments \vec{a}

$$C(\vec{a}) = 0 \rightarrow \{A_i(\vec{p}, \vec{q})\}_i \text{ is unsatisfiable}$$

$$C(\vec{a}) = 1 \rightarrow \{B_j(\vec{p}, \vec{r})\}_j \text{ is unsatisfiable}$$

splitting Resolution proofs

Theorem

Given a *Resolution* proof P of contradiction

$$P : \{A_i(\vec{p}, \vec{q})\}_i \cup \{B_j(\vec{p}, \vec{r})\}_j \rightarrow \perp,$$

and an assignment for $\vec{p} \mapsto \vec{a}$, we can construct in *polynomial time* two proofs

- ▶ P^q a proof from $\{A_i(\vec{a}, \vec{q})\}_i$,
- ▶ P^r a proof from $\{B_j(\vec{a}, \vec{r})\}_j$,

such that *one of them is a proof of contradiction*.

proof

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- ▶ the initial clauses are OK
- ▶ a mixed clause appears when we resolve a q-clause with an r-clause
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Let $\vec{p} \mapsto \vec{a}$. We gradually transform the clause from the proof

$C \mapsto C'$ as follows:

- ▶ if we resolve w.r.t. some q_i or r_i in the given proof, we do the same;

- ▶ if we resolve w.r.t. some p_i then, if $a : p_i \mapsto 0$, then

$$\frac{\Gamma \vee p, \quad \Delta \vee \neg p}{\Gamma \vee \Delta} \mapsto \frac{\Gamma' \vee p, \quad \Delta' \vee \neg p}{\Gamma'}$$

otherwise

$$\mapsto \frac{\Gamma' \vee p, \quad \Delta' \vee \neg p}{\Delta'}$$

- ▶ this is not a logically valid derivation;
- ▶ if $C \mapsto C'$, then $C' \subseteq C$;
- ▶ hence $\perp \mapsto \perp$.

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Next substitute \vec{a} and $C' \mapsto C''$:

- ▶ if C' has a true literal, then $C'' := \top$
- ▶ otherwise $C'' := C'$ -less literals from \vec{p} .

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Claim The resulting set of clauses is a valid Resolutions proof of \perp .

- ▶ if $a : p_i \mapsto 0$, then

$$\frac{\Gamma' \vee p, \quad \Delta' \vee \neg p}{\Gamma'} \mapsto \frac{\Gamma'', \quad \top}{\Gamma''}$$

- ▶ if we resolve with q or r and neither $C'_1 \mapsto \top$ nor $C'_2 \mapsto \top$, then

$$\frac{C'_1, \quad C'_2}{C'} \mapsto \frac{C''_1, \quad C''_2}{C''}$$

- ▶ if we resolve with q or r and $C'_1 \mapsto \top$ then

$$\frac{C'_1, \quad C'_2}{C'} \mapsto \frac{\top, \quad C'_2}{\top}$$

- ▶ etc.



applications of feasible interpolation

1. program verification
2. lower bounds on the complexity of proofs

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Theorem

Suppose that $\mathbf{NP} \cap \mathbf{coNP} \not\subseteq \mathbf{P}/\text{poly}$. Then there are sequences of tautologies that do not have polynomial size proofs in any propositional proof system that has feasible interpolation.

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It suffices to assume that there exist two disjoint \mathbf{NP} sets that cannot be separated by a set in \mathbf{P}/poly .

\mathbf{P}/poly = the nonuniform version of \mathbf{P} = sets definable by polynomial size Boolean circuits.

Proof.

Let A, B be disjoint **NP** sets that cannot be separated by a set in **P/poly**. Let

$$A := \{\bar{u} \mid \exists \bar{v} \neg \alpha_n(\bar{u}, \bar{v}), n \in \mathbb{N}\},$$

$$B := \{\bar{u} \mid \exists \bar{w} \neg \beta_n(\bar{u}, \bar{w}), n \in \mathbb{N}\}$$

Then the sequence of formulas

$$\alpha_n(\bar{u}, \bar{v}) \vee \beta_n(\bar{u}, \bar{w})$$

expresses that $A \cap B = \emptyset$. Hence they are **tautologies**.

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Let \mathcal{P} be a proof system with feasible interpolation and suppose \mathcal{P} has polynomial size proofs P_n of these tautologies. By feasible interpolation, for every \bar{a} , we can decide **in polynomial time** whether

$$\alpha_n(\bar{a}, \bar{v}) \quad \text{or} \quad \beta_n(\bar{a}, \bar{w})$$

is a tautology, i.e., whether $\bar{a} \notin A$ or $\bar{a} \notin B$.

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From **polynomial time algorithm** we can construct **polynomial size**

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Monotone Interpolation: if \bar{u} occurs

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- ▶ only negatively in $\beta(\vec{p}, \vec{r})$,

then there exists a **monotone** polynomial size circuit $C(\vec{p})$ s.t.

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We do have exponential lower bounds on monotone circuits separating disjoint **NP** sets, hence we can prove lower bounds in this way.

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*If the factoring problem is not solvable in polynomial time, then Frege systems, sequent calculi with cut and natural deduction system do **not** have feasible interpolation.*

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Factoring is the problem to find nontrivial factors of a given composed integer.

proof theory of 1st order logic

(See Buss's chapter in Handbook)

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Syntax

proof theory of 1st order logic

(See Buss's chapter in Handbook)

Syntax

Primitive concepts

- ▶ relation and function symbols R, S, \dots, f, g, \dots
- ▶ the equality sign $=$
- ▶ variables x, y, \dots (for elements) and constants c, d, \dots
- ▶ propositional connectives \neg, \wedge, \dots
- ▶ quantifiers \forall, \exists
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Terms and formulas

- ▶ terms t, s, \dots , e.g., $f(c, g(d))$
- ▶ **atomic formulas** $R(t_1, \dots, t_n)$, $t_1 = t_2$, where t_i are terms
- ▶ general formulas may have free variables
- ▶ **sentences** = formulas with no free variables
- ▶ **prenex formulas/sentences** = all quantifiers are in the prefix

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I suppose that you know what a well-formed formula is, what the scope of a

Semantics

Fact [attributed to A. Tarski] There is a well defined relation of satisfaction of a formula $\phi(x_1, \dots, x_n)$ by elements a_1, \dots, a_n in a model M , which is denoted by

$$M \models \phi[a_1, \dots, a_n].$$

Proof.

Define inductively on the complexity of terms and formulas. □

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Definition

A sentence ϕ is logically valid, if for every model M (of appropriate signature) $M \models \phi$.

Hilbert-style calculus

Frege system for propositional axioms and rules

+ quantifier axioms and rules:

Hilbert-style calculus

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Axioms (I am now using \rightarrow for implication.)

$$\phi(t) \rightarrow \exists x.\phi(x) \quad (\forall x.\phi(x)) \rightarrow \phi(t)$$

t is a term **not containing any bound variables.**

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Rules

$$\frac{\phi(x) \rightarrow \psi}{(\exists x.\phi(x)) \rightarrow \psi} \quad \frac{\psi \rightarrow \phi(x)}{\psi \rightarrow \forall x.\phi(x)}$$

where x is not free in ψ .

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Proofs are sequences of **formulas.**

Formalizations with MP only and sentences are known.

axioms of equality

See Buss's chapter.

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Exercise

1. *Derive the axiom of the nonempty domain*

$$\exists x(x = x)$$

2. *Can one prove that the domain is nonempty without using equality? How can one state such an axiom?*

the sequent calculus

Useful convention: a, b, \dots free variables, x, y, \dots bounded variables.

Notation: \Rightarrow for the arrow in sequents.

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Quantifier rules

$$\text{(weak)} \quad \frac{\Gamma \Rightarrow \Delta, \phi(t)}{\Gamma \Rightarrow \Delta, \exists x.\phi(x)} \quad \frac{\phi(t), \Gamma \Rightarrow \Delta}{\forall x.\phi(x), \Gamma \Rightarrow \Delta}$$

where t is a term not containing any bound variables.

$$\text{(strong)} \quad \frac{\Gamma \Rightarrow \Delta, \phi(a)}{\Gamma \Rightarrow \Delta, \forall x.\phi(x)} \quad \frac{\phi(a), \Gamma \Rightarrow \Delta}{\exists x.\phi(x), \Gamma \Rightarrow \Delta}$$

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where a does not occur in ψ .

Axioms of equality: same, but stated as sequents (See Buss's chapter)

examples of wrong applications

$$\frac{\Rightarrow \forall x (f(x) = f(x))}{\Rightarrow \exists y \forall x (f(x) = y)}$$

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$$\frac{a = b \Rightarrow a = b}{a = b \Rightarrow \forall x (x = b)}$$

Natural Deduction

quantifier rules

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quantifier rules

$$\forall\text{-intro} \quad \frac{A(b)}{(\forall x)A(x)}$$

$$\forall\text{-elim} \quad \frac{(\forall x)A(x)}{A(t)}$$

$$\exists\text{-intro} \quad \frac{A(t)}{(\exists x)A(x)}$$

$$\exists\text{-elim} \quad \frac{(\exists x)A(x) \quad [A(b)] \quad B}{B}$$

In \forall -intro, b must not occur in any non-discharged hypothesis.

In \exists -elim, b must not occur in any non-discharged hypothesis except for the hypothesis of this rule.

Lesson 5

cut-elimination in the sequent calculus

Preprocessing:

- ▶ put the proof into a tree-like form
- ▶ ensure **the free variable normal form** — use distinct free variables whenever possible

Lesson 5

cut-elimination in the sequent calculus

Preprocessing:

- ▶ put the proof into a tree-like form
- ▶ ensure **the free variable normal form** — use distinct free variables whenever possible

Caveat:

- ▶ When transforming the proof watch for possible conflicts of free variables in the strong q. rules!
- ▶ Also do not forget about contractions!

example

$P_1(a, b)$

$P_2(s, t)$

$$\begin{array}{c}
 \dots \\
 \hline
 A(a), A(b), \Gamma \rightarrow \Delta \\
 \hline
 \exists x A(x), A(b), \Gamma \rightarrow \Delta \\
 \hline
 \exists x A(x), \exists x A(x), \Gamma \rightarrow \Delta \\
 \hline
 \text{contraction} \frac{\exists x A(x), \exists x A(x), \Gamma \rightarrow \Delta}{\exists x A(x), \Gamma \rightarrow \Delta} \\
 \hline
 \text{cut} \frac{\exists x A(x), \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}
 \end{array}
 \qquad
 \begin{array}{c}
 \dots \\
 \hline
 \Gamma \rightarrow A(s), A(t), \Delta \\
 \hline
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 \hline
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 \hline
 \Gamma \rightarrow \Delta
 \end{array}$$

example

$$\begin{array}{c}
 P_1(a, b) \qquad \qquad \qquad P_2(s, t) \\
 \dots \\
 \frac{A(a), A(b), \Gamma \rightarrow \Delta}{\exists x A(x), A(b), \Gamma \rightarrow \Delta} \\
 \frac{\exists x A(x), \exists x A(x), \Gamma \rightarrow \Delta}{\exists x A(x), \Gamma \rightarrow \Delta} \\
 \text{contraction} \\
 \text{cut} \\
 \frac{\dots}{\Gamma \rightarrow \Delta}
 \end{array}
 \qquad
 \begin{array}{c}
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 \frac{\dots}{\Gamma \rightarrow \Delta}
 \end{array}$$

To eliminate the cut with $\exists x A(x)$ we replace subproof $P(a, b)$ with two subproofs obtained from $P_1(a, b)$ by substituting $a \mapsto s, b \mapsto s$ in the first subproof, and $a \mapsto t, b \mapsto t$ in the second. Then use two cuts one with $A(s)$ and the other with $A(t)$.

$$\begin{array}{c}
 \dots \\
 \frac{A(s), A(s), \Gamma \rightarrow \Delta}{A(s), \Gamma \rightarrow \Delta} \\
 \text{cut} \\
 \frac{\dots}{\Gamma \rightarrow \Delta}
 \end{array}
 \qquad
 \begin{array}{c}
 \dots \\
 \frac{A(t), A(t), \Gamma \rightarrow \Delta}{A(t), \Gamma \rightarrow \Delta} \\
 \text{cut} \\
 \frac{\dots}{\Gamma \rightarrow \Delta}
 \end{array}$$

Notice:

1. The number of new cuts is the number of different terms occurring in A , which is 2 in the example.
2. This is because the contraction in the right hand part is not used.

Definition

A is a **generalized subformula** of B if it is a substitution instance of a subformula of B .

Proposition

Every formula in a cut-free proof is a generalized subformula of a formula in the last sequent.

mid-sequent theorem

Theorem

Suppose ϕ is a provable sentence in a *prenex form*. Then there exists a (cut-free) proof of $\rightarrow \phi$ in which there is a sequent $\rightarrow \Delta$ (*the mid-sequent*) such that

- ▶ there are no quantifier rules above $\rightarrow \Delta$ (thus the mid-sequent does not contain quantifiers)
- ▶ there are only quantifier rules and structural rules below $\rightarrow \Delta$.

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1. Take a cut-free proof in the free-variable normal form.
2. Whenever a propositional rule is below a quantifier rule, switch the rules.



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Simple idea, tedious verification.

digression — some history

Gerhard Gentzen (1909-1945)

- ▶ calculus of natural deduction, sequent calculus
- ▶ cut-elimination theorem
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Jacques Herbrand (1908-1931)

- ▶ algebraic number fields
- ▶ logic – Herbrand's theorem
- ▶ computability theory – the Gödel-Herbrand recursive functions

Herbrand's Theorem

Theorem (basic version)

Let A be an existential sentence

$$\exists x_1 \dots \exists x_n \phi(x_1, \dots, x_n)$$

(ϕ an open, i.e., quantifier-free formula). Then TFAE

1. A is logically valid (\equiv provable)
2. there exist terms t_{ij} , $i = 1, \dots, n$, $j = 1, \dots, m$ in the language of A such that

$$\bigvee_{j=1}^m \phi(t_{1j}, \dots, t_{nj})$$

is a *propositional tautology*.

Proof.

Let $\rightarrow \Gamma$ be the mid-sequent in a proof of $\rightarrow A$, then $\rightarrow \Gamma$ is

$$\rightarrow \phi(t_{11}, \dots, t_{n1}), \dots, \phi(t_{1m}, \dots, t_{nm})$$



exercise

Prove the following generalization:

Theorem

Let A be a $\forall\exists$ prenex sentence

$$\forall y_1 \dots \forall y_k \exists x_1 \dots \exists x_n \phi(y_1, \dots, y_k, x_1, \dots, x_n)$$

Then TFAE

1. A is logically valid
2. there exist terms t_{ij} , $i = 1, \dots, n$, $j = 1, \dots, m$ in the language of A such that

$$\bigvee_{j=1}^m \phi(a_1, \dots, a_k, t_{1j}, \dots, t_{nj})$$

is a *propositional tautology*.

Hint: use constants instead of free variables.

example

Let P be predicate, 0 a constant, and S a unary function. We will write $S^n x$ for S n -times iterated.

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The following is a logically true sentence for every concrete n :

$$(P(0) \wedge \forall x(P(x) \rightarrow P(Sx))) \rightarrow P(S^n 0)$$

We can prove it in $O(\log n)$ steps by deriving gradually

$$\forall x(P(x) \rightarrow P(S^2 x)), \forall x(P(x) \rightarrow P(S^4 x)), \forall x(P(x) \rightarrow P(S^8 x)), \dots$$

from $\forall x(P(x) \rightarrow P(Sx))$.

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from $\forall x(P(x) \rightarrow P(Sx))$.

Write it as an existential formula:

$$\exists x(\neg P(0) \vee (P(x) \wedge \neg P(Sx)) \vee P(S^n 0))$$

example, contd

The mid-sequent is $\rightarrow \Delta$ where Δ contains all

$$\neg P(0) \vee (P(S^i 0) \wedge \neg P(S^{i+1} 0)) \vee P(S^n 0), \quad i = 0, \dots, n-1.$$

Applying \exists -right rule to terms $t := S^i 0$ we get

$$\exists x (\neg P(0) \vee (P(x) \wedge \neg P(Sx)) \vee P(S^n 0))$$

from each of the formulas from Δ . Then we contract to a single formula.

exercise

Why do we need all terms $0, S0, SS0, SSS0, \dots, S^n 0$?

Theorem

TFAE:

1. $\exists x\forall y.\phi(x, y)$ is logically valid,
2. there exist terms t_1, \dots, t_n such that

$$\phi(t_1, b_1) \vee \phi(t_2(b_1), b_2) \vee \dots \vee \phi(t_n(b_1, \dots, b_{n-1}), b_n)$$

is a propositional tautology, where $t_i(b_1, \dots, b_{i-1})$ *may only contain* some b_1, \dots, b_{i-1} .

Interpretation: *Teacher-Student Game*

- ▶ Teacher asks student to find t such that $\forall y. \phi(t, y)$ holds true.
- ▶ Student tries t_1 , Teacher gives a counterexample b_1 ;
 $\neg\phi(t_1, b_1)$
- ▶ knowing b_1 , Student tries t_2 , Teacher gives a counterexample b_2 , $\neg\phi(t_2, b_2)$;
- ▶ etc.
- ▶ eventually, for some $i \leq n$, there is no counterexample, hence t_i is a solution.

Proof.

1. \rightarrow 2. Let

$$\rightarrow \phi(t_1, b_1), \phi(t_2, b_2), \dots, \phi(t_n, b_n)$$

be the mid-sequent of a proof of $\exists x \forall y. \phi(x, y)$.

- ▶ Let $\phi(t_n, b_n)$ be the formula to which the first \forall -rule is applied. Then none of t_1, \dots, t_n contains b_n . (We could apply \forall -rule if b_n were in t_n , but then we would not be able to apply \exists -rule to t_n .)
- ▶ Let $\phi(t_{n-1}, b_{n-1})$ be the formula to which the next \forall -rule is applied. Then none of t_1, \dots, t_{n-1} contains b_{n-1} .
- ▶ ...
- ▶ Let $\phi(t_1, b_1)$ be the formula to which the last \forall -rule is applied. Then t_1 does not contain any b_1, \dots, b_n .

2. \rightarrow 1. Write the disjunction as the sequent

$$\rightarrow \phi(t_1, b_1), \phi(t_2(b_1), b_2), \dots, \phi(t_n(b_1, \dots, b_{n-1}), b_n)$$

- ▶ Introduce \forall for b_n , then \exists for t_n ,
- ▶ introduce \forall for b_{n-1} , then \exists for t_{n-1} ,
- ▶ etc.
- ▶ contract.



Exercise

Extend the previous theorem from $\exists\forall$ to $\forall\exists$

the general Herbrand theorem

The previous theorem can be extended to complex prefixes, but for automated theorem proving we need a different approach.

Therefore we will use new function symbols, [Herbrand function](#)¹ symbols, to reduce a general prenex formula to an existential one.

¹Some authors call these functions “Skolem” and distinguish two kinds of Skolem function. We will call one class “Herbrand functions” and the other “Skolem functions”.

the general Herbrand theorem

Example

Consider $A := \exists x \forall y \exists z \forall u. \phi(x, y, z, u)$. We translate A to

$$He(A) := \exists x \exists z. \phi(x, f(x), z, g(x, z))$$

where f, g are *new function symbols*.

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where f, g are *new function symbols*. Think of f and g as *counterexamples* in case A is not true.

(If A is false, let f be such that for all x , $\exists z \forall u. \phi(x, f(x), z, u)$ is false, etc.)

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(If A is false, let f be such that for all x , $\exists z \forall u. \phi(x, f(x), z, u)$ is false, etc.)

If A is true, *no counterexample* is possible, hence $He(A)$ is also true.

In general, for a prenex formula A , $He(A)$ is obtained by

1. omitting all \forall and
2. substituting the term $f(x_1, \dots, x_k)$ for every y universally quantified, where f is a new function symbol and x_1, \dots, x_k are the existentially quantified variables before the universal quantifier $\forall y$.

N.B. if A starts with \forall , we use “nullary” function symbols, i.e., *constants*.

Theorem (Herbrand's Theorem)

Let A be a prenex sentence, let

$$He(A) := \exists x_1 \dots \exists x_n \psi(x_1, \dots, x_n),$$

where the Herbrand functions are implicit in ψ . Then A is logically valid iff there exist terms t_{ij} , $i = 1, \dots, n$, $j = 1, \dots, m$, in the language of $He(A)$ such that

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Proof.

We only need to show that $\vdash A$ iff $\vdash He(A)$.

1. One can easily show that in fact $\vdash A \rightarrow He(A)$ ([exercise!](#)).
2. If $\vdash He(A)$ then $\vdash A$ — see below.



Skolem functions

Skolem functions and $Sk(A)$ are dual to Herbrand functions and $He(A)$.

Example

$$Sk(\forall x \exists y \forall z \exists u. \phi(x, y, z, u)) := \forall x \forall z. \phi(x, f(x), z, g(x, z)).$$

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Lemma

Let $M \models A$. Then one can extend M with functions interpreting the Skolem functions of $Sk(A)$ so that in the extended model $M' \models Sk(A)$.

Proof.

Consider the sentence above.

- ▶ For every $a \in M$, define $f^M(a) = b$ by choosing some d such that $M \models \forall z \exists u \phi(a, b, z, u)$.
- ▶ For every $a, c \in M$, define $g^M(a, c) = d$ by choosing some d such that $M \models \phi(a, f(a), c, d)$.

Thus we get $M' \models \phi(a, f(a), c, g(a, c))$ for every $a, c \in M$. □

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We now prove that $\vdash He(A)$ implies $\vdash A$ by proving the contrapositive implication.

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Assume $\not\vdash A$. Let $M \models \neg A$. Then $M' \models Sk(\neg A)$. But $\vdash Sk(\neg A) \equiv \neg He(A)$. Hence $M' \models \neg He(A)$, which means $\not\vdash He(A)$. □