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Disappearing vortex problem in vortex identification: Non-existence for selected criteria

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A discontinuous outcome of vortex-identification methods called the disappearing vortex problem (DVP) has been already found for the swirling strength criterion and the Rortex (later renamed as Liutex) method. Here, the opposite property reflecting the situation that the DVP cannot be found for any input data, that is, the non-existence of the DVP, is examined and proved valid for selected criteria based on the velocity-gradient tensor including Q, lambda-2, and the triple decomposition method. For the Q-criterion and the triple decomposition method it is done directly, for lambda-2 it is shown using a proof by contradiction.

Vortex-identification methods can be classified according to the basic fluid-mechanical approach as Lagrangian or Eulerian, see the review by Epps.[1] Lagrangian methods are typically based on the analysis of fluid particle trajectories, such as the objective M_Z-criterion of Haller[2] or the finite-time Lyapunov exponent (FTLE). On the contrary, Eulerian methods analyze flow-field snapshots. They can be further sub-classified as local (pointwise) or non-local. The Eulerian local methods aim either at the vortex volumetric region (region-type schemes) or at the vortex-core skeleton (line-type schemes). Region-type methods include the widely used criteria Q by Hunt et al.,[3] λ_2 by Jeong and Hussain,[4] and the swirling strength by Zhou et al.,[5] all of which employ the velocity-gradient tensor. The present manuscript focuses on these Eulerian local region-type schemes. The last paragraph of this Letter briefly returns to the Lagrangian methods.

As already described by Kolář,[6] the preference for some vortex-identification schemes in contrast to others is given by their general properties and the ability to satisfy a number of natural requirements. Moreover, as noted by Xiong et al.,[7] the validation of the vortex definition using flow examples is not rigorous because of a prejudice of expecting what is a priori a vortex. Flow examples only suffice to invalidate but not support an idea. This is the main reason to study the properties of vortex-identification schemes along with their applications.

The so-called disappearing vortex problem (DVP) reflects the undesirable discontinuous outcome of some vortex-identification criteria based on the velocity-gradient tensor $\nabla\mathbf{u}$. The DVP can be expressed as follows: for a fixed tensor geometry of $\nabla\mathbf{u}$, fixed strain rate, and the only variable being vorticity-vector magnitude, according to Fig. 1, with increasing relative vorticity magnitude the vortex disappears and reappears again above a certain relative vorticity magnitude. Note that this discrepancy between the velocity-gradient input and the vortex-identification outcome is an inherent property of an identification method, and it should not be confused with a flow phenomenon of vortex occurrence. Figure 1 shows vorticity vector in the system of strain-rate principal axes (p_1, p_2, p_3) including principal strain rates (σ_1, σ_3, σ_3) and vorticity-vector configuration angles θ and ϕ. Without loss of generality, the flow depicted in Fig. 1 is radially converging.

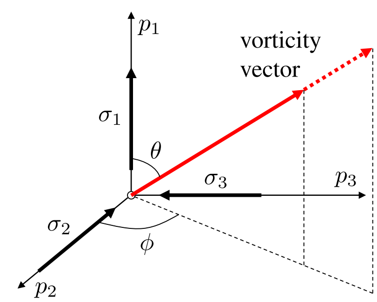

FIG. 1: Velocity-gradient configuration in the system of strain-rate principal axes. Reproduced from Phys. Fluids 32, 091702 (2020), with the permission of AIP Publishing.

This problem has been described in detail by Chakraborty et al.,[8] pp. 199–200, who found this "clearly counterintuitive" behavior for the swirling-strength criterion. They enhanced the swirling-strength criterion by requiring that the swirling material points inside a vortex have bounded separation remaining small and introduced an additional restriction parameter dealing with the orbital compactness of the fluid motion inside a vortex. The application of this new restriction parameter effectively eliminated the DVP described at the beginning. More specifically, the swirling strength is given by the imaginary part λ_{ci} of complex conjugate eigenvalues of the velocity-gradient tensor $\nabla\mathbf{u}$, $\lambda_{cr} \pm \mathrm{i}\lambda_{ci}$. The local parameter $\lambda_{cr}/\lambda_{ci}$ approximates the (inverse) measure of the non-local orbital compactness of the fluid motion inside a vortex and should be sufficiently limited.[8]

Later on, in the manner of the work of Chakraborty et al.,[8] the existence of the DVP has been shown by Kolář and Šístek[9] for the recently proposed Rortex (Liutex) method.[10,11]

The Rortex (Liutex) method is based on the idea of a vortex vector[10,11] and the DVP probably remains the most questionable feature of this new approach to vortex identification. Both already mentioned criteria, Rortex (Liutex) and the swirling strength, considered without an additional restriction parameter dealing with the orbital compactness, provide exactly the same vortex region, and the DVP occurs

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close to the approximate equality of strain-rate and vorticity magnitudes.[8,9] This parametric region is interesting from the vortex-identification viewpoint since this is near the vortex boundary where usually discrepancies among individual schemes occur.

For the Rortex (Liutex) method and the swirling strength criterion, there is actually a range of vorticity-vector configuration angles (identical for both criteria) for which the DVP exists.[9] One example of the DVP drawn from Fig. 2 by Kolář and Šístek[9]: for fixed configuration angles θ = 73° and ϕ = 45°, and fixed strain-rate ratio σ_1:σ_2:σ_3 = 1:(-0.4):(-0.6), the vortex-identification outcome for increasing vorticity-to-strain-rate magnitude ratio of 0.3, 0.6, 0.9, 1.2 has been found alternating as non-vortex, vortex, non-vortex, vortex, respectively. This phenomenon has been found[9] (for the vorticity vector in the first octant) for the range of θ, approx. $57^\circ \leq \theta < 90^\circ$, while there is no restriction for ϕ.

While the existence of the DVP can be inferred even from the existence of the only tensor configuration exhibiting this anomalous behavior, the evidence of the opposite, that is, the evidence of the non-existence of the DVP is a more subtle point as it should hold for all possible tensor configurations. It is shown below how to cope with this task for selected local $\nabla\mathbf{u}$-based vortex-identification criteria, namely for the widely used Q-criterion[3] and λ_2-criterion,[4] and for the triple decomposition method (TDM),[6] shown with a number of 3D applications, e.g., by Kolář and Šístek.[12] For details of these local criteria, see the comprehensive review by Epps.[1] Let us recall the coincidence of the Rortex (Liutex) method with the TDM in 2D flows, both methods being explicitly shear-eliminating vortex-identification methods in character. To show the non-existence of the DVP for the Q-criterion is straightforward, however, the evidence of the non-existence of the DVP for the other two vortex-identification schemes is more demanding.

The conventional double decomposition of $\nabla\mathbf{u}$ as the sum of strain-rate and vorticity tensors, $\nabla\mathbf{u} = \mathbf{S} + \mathbf{\Omega}$, is employed. Throughout the paper below, we are interested in the response of different identification methods to increasing vorticity-vector magnitude for the otherwise unchanged vorticity-vector configuration angles and unchanged strain rate, according to Fig. 1. Following the concept of the DVP, this vorticity change of $\nabla\mathbf{u}$ is expressed by the linear transformation of the form

$$(\nabla\mathbf{u})_{\mathrm{NEW}} = \mathbf{S} + \mathbf{\Omega}_{\mathrm{NEW}} = \mathbf{S} + (1+\alpha)\mathbf{\Omega} = \nabla\mathbf{u} + \alpha\mathbf{\Omega} \quad (1)$$

provided that the starting $\nabla\mathbf{u}$-input ($\alpha = 0$) identifies the examined point as a vortex and α is a positive real parameter. It should be noted that the input vorticity-vector configuration angles and the strain-rate tensor are otherwise arbitrary. The transformation is just what is schematically depicted in Fig. 1 by the vorticity-vector elongation (dashed line in red) and the unchanged strain rate (in black). Here, the relation between vorticity vector ω and the antisymmetric vorticity tensor $\mathbf{\Omega}$ in Eq. (1) can be recalled as $\omega/2 = (\Omega_{32}, \Omega_{13}, \Omega_{21})$.

Q-criterion[3]:

This widely used criterion identifies vortices of a 3D incompressible flow as the connected fluid regions of positive second invariant of $\nabla\mathbf{u}$. In other words, vortices are the regions in which the vorticity magnitude prevails over the strain-rate magnitude as the positive second invariant Q reads $Q = \left(\|\mathbf{\Omega}\|^2 - \|\mathbf{S}\|^2\right)/2 > 0$ where the magnitudes are defined by the Frobenius norm. The additional pressure condition[3] requiring that the pressure tends to a minimum inside these regions has been found arguable by Jeong and Hussain,[4] and therefore, has been usually omitted.

In the present DVP test, the impact of the transformation (1) is investigated. For the positive Q-criterion—the starting situation must be identified as a vortex—it is impossible for increasing relative vorticity magnitude through increasing the value of α in (1) to find any discontinuity in vortex identification in terms of vortex disappearance and reappearance as expressed by the DVP. The outcome is quite the opposite as the increasing vorticity magnitude continuously makes a vortex stronger for the otherwise unchanged vorticity-vector configuration angles and unchanged strain rate, hence this is in agreement with our intuition.

λ_2-criterion[4]:

For incompressible flow, a vortex is defined as a connected fluid region with two negative eigenvalues of the symmetric matrix $\mathbf{M} = \mathbf{S}^2 + \mathbf{\Omega}^2$, that is, if the eigenvalues are ordered, $\lambda_1 \geq \lambda_2 \geq \lambda_3$, a vortex is defined by the condition $\lambda_2 < 0$. Figure 3(b,c,d) by Chakraborty et al.[8] shows a situation of the λ_2-criterion for three specific strain-rate cases: for the case of axisymmetric (radially converging) strain rate as well as for its opposite case, planar strain rate, and for one intermediate case. For all the three strain-rate configurations depicted in Fig. 3(b,c,d) by Chakraborty et al.,[8] the tensor geometry is otherwise arbitrary. The results shown in Fig. 3(b,c,d)[8] indicate that the λ_2-criterion is a DVP-free criterion for the examined range of parameters. In addition, the eigenvalue expressions in Eq. (7) of the work of Cucitore, Quadrio and Baron[13] indicate for a rationally selected vortex axis that the λ_2-criterion operates by locally comparing strain rate to vorticity. Therefore, similarly as for the Q-criterion, the increasing vorticity magnitude continuously makes a vortex stronger, at least for the vortex axis selected in the manner described by the authors.[13]

Although the above indications are interesting and suggestive, they do not represent an exhaustive evidence that the λ_2-criterion does not suffer from the DVP for parameters not explicitly stated above. The full evidence is presented below using a proof by contradiction. Considering the DVP concept means that we start from the positive response to the λ_2-criterion (point inside a vortex) along with the assumption of an a priori existence of negative response to the λ_2-criterion (point outside a vortex) for a sufficiently high vorticity magnitude linearly increased according to transformation (1).

For a symmetric matrix $\mathbf{\Omega}^2$ of an antisymmetric tensor $\mathbf{\Omega}$, the eigenvalues ordered as $\kappa_1 \geq \kappa_2 \geq \kappa_3$ satisfy $\kappa_1 = 0 > \kappa_2 = \kappa_3$. The associated orthonormal eigenbasis is denoted $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$. For a point inside a vortex, for $\mathbf{M} = \mathbf{S}^2 + \mathbf{\Omega}^2$ with the eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and the corresponding orthonormal eigenbasis $(\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3)$, it holds $\lambda_2 < 0$. By Eq. (1), the transformation of $\mathbf{M} = \mathbf{S}^2 + \mathbf{\Omega}^2$ gives $\mathbf{M}_{\mathrm{NEW}} = \mathbf{S}^2 + \mathbf{\Omega}^2_{\mathrm{NEW}}$,

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where $\mathbf{\Omega}_{\mathrm{NEW}} = (1+\alpha)\mathbf{\Omega}$. By introducing $\beta = (2\alpha+\alpha^2) > 0$, it follows $\mathbf{M}_{\mathrm{NEW}} = \mathbf{M}+\beta\mathbf{\Omega}^2$. For the proof by contradiction, let us assume the existence of such $\beta > 0$ that for the eigenvalues $\mu_1 \geq \mu_2 \geq \mu_3$ of $\mathbf{M}_{\mathrm{NEW}}$, with the associated orthonormal eigenbasis $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$, it holds $\mu_2 \geq 0$. That is, by the last requirement, we assume that a certain vorticity-vector elongation results in the non-existence of a vortex identified by the given criterion.

The present proof by contradiction effectively combines the input information regarding *two negative* eigenvalues of $\mathbf{M}$ and the associated eigenvectors $(\mathbf{l}_2, \mathbf{l}_3)$ with the assumed outcome regarding *two non-negative* eigenvalues of $\mathbf{M}_{\mathrm{NEW}}$ and the associated eigenvectors $(\mathbf{m}_1, \mathbf{m}_2)$.

The planar subspace given by $(\mathbf{l}_2, \mathbf{l}_3)$ and the planar subspace given by $(\mathbf{m}_1, \mathbf{m}_2)$ share a line as an intersection (note that both planes go through the same origin of the vector space) which may be defined by a unit vector $\mathbf{s}$ at the origin. Now, let us examine how the transformations $\mathbf{M}$, $\mathbf{M}_{\mathrm{NEW}}$, and $\mathbf{\Omega}^2$ operate on the vector $\mathbf{s}$ in terms of scalar-product estimates as these estimates employ the eigenvalues of the given transformation matrices through the Rayleigh quotients.

The intersection unit vector $\mathbf{s}$ can be expressed in all the three introduced eigenbases as

$$\mathbf{s} = a_1\mathbf{k}_1 + a_2\mathbf{k}_2 + a_3\mathbf{k}_3 = b_2\mathbf{l}_2 + b_3\mathbf{l}_3 = c_1\mathbf{m}_1 + c_2\mathbf{m}_2. \quad (2)$$

Let us compare the estimates of the three following scalar products, namely $(\mathbf{\Omega}^2\mathbf{s})\cdot\mathbf{s}$, $(\mathbf{M}\mathbf{s})\cdot\mathbf{s}$, and $(\mathbf{M}_{\mathrm{NEW}}\mathbf{s})\cdot\mathbf{s}$ expressed as (recall that all the vectors are unit vectors by definition)

$$\begin{aligned}(\mathbf{\Omega}^2\mathbf{s})\cdot\mathbf{s} &= \quad (3)\\ &= (\kappa_1 a_1\mathbf{k}_1 + \kappa_2 a_2\mathbf{k}_2 + \kappa_3 a_3\mathbf{k}_3)\cdot(a_1\mathbf{k}_1 + a_2\mathbf{k}_2 + a_3\mathbf{k}_3)\\ &= \kappa_1 a_1^2 + \kappa_2 a_2^2 + \kappa_3 a_3^2\\ &\leq \kappa_1(a_1^2 + a_2^2 + a_3^2) = \kappa_1 = 0,\end{aligned}$$

$$\begin{aligned}(\mathbf{M}\mathbf{s})\cdot\mathbf{s} &= \quad (4)\\ &= (\lambda_2 b_2\mathbf{l}_2 + \lambda_3 b_3\mathbf{l}_3)\cdot(b_2\mathbf{l}_2 + b_3\mathbf{l}_3) = \lambda_2 b_2^2 + \lambda_3 b_3^2\\ &\leq \lambda_2(b_2^2 + b_3^2) = \lambda_2 < 0,\end{aligned}$$

$$\begin{aligned}(\mathbf{M}_{\mathrm{NEW}}\mathbf{s})\cdot\mathbf{s} &= \quad (5)\\ &= (\mu_1 c_1\mathbf{m}_1 + \mu_2 c_2\mathbf{m}_2)\cdot(c_1\mathbf{m}_1 + c_2\mathbf{m}_2) = \mu_1 c_1^2 + \mu_2 c_2^2\\ &\geq \mu_2(c_1^2 + c_2^2) = \mu_2 \geq 0.\end{aligned}$$

The last expression states non-negativity of $(\mathbf{M}_{\mathrm{NEW}}\mathbf{s})\cdot\mathbf{s}$, $(\mathbf{M}_{\mathrm{NEW}}\mathbf{s})\cdot\mathbf{s} \geq 0$, while Eqs. (3) and (4) suggest the opposite, negativity of $(\mathbf{M}_{\mathrm{NEW}}\mathbf{s})\cdot\mathbf{s}$, as the substitution from Eqs. (3) and (4) gives

$$\begin{aligned}(\mathbf{M}_{\mathrm{NEW}}\mathbf{s})\cdot\mathbf{s} &= ((\mathbf{M}+\beta\mathbf{\Omega}^2)\mathbf{s})\cdot\mathbf{s}\\ &= (\mathbf{M}\mathbf{s})\cdot\mathbf{s} + \beta(\mathbf{\Omega}^2\mathbf{s})\cdot\mathbf{s} \quad (6)\\ &< (0+\beta\cdot 0) = 0.\end{aligned}$$

The proof by contradiction is completed. Consequently, the non-existence of the DVP for the λ_2-criterion has been proved.

There is an underlying geometric interpretation of the above proof. Let us consider a plane, for which $\mathbf{s}$ is a unit normal, as a plane which divides the 3D space into positive (containing vector $\mathbf{s}$) and negative half-spaces. Then, according to Eqs. (3) and (4), the vectors $\mathbf{\Omega}^2\mathbf{s}$, $\mathbf{M}\mathbf{s}$ lie in the same negative half-space (strictly said, the vector $\mathbf{\Omega}^2\mathbf{s}$ is in the closed negative half-space including the dividing plane, while the vector $\mathbf{M}\mathbf{s}$ is in the open negative half-space). Therefore, their linear combination $(\mathbf{M}+\beta\mathbf{\Omega}^2)\mathbf{s}$ for positive β should be found in the open negative half-space, as shown schematically in Fig. 2. This is in contrast to the closed positive half-space required for $\mathbf{M}_{\mathrm{NEW}}\mathbf{s}$ through Eq. (5) by an a priori assumption $\mu_2 \geq 0$.

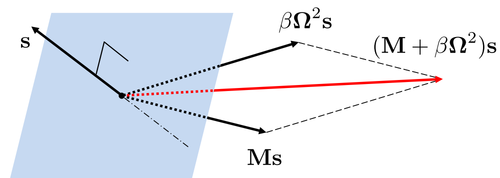

FIG. 2: Scheme of the images of the intersection unit vector $\mathbf{s}$.

TDM (triple decomposition method)[6]:

The third and last vortex-identification scheme to test for the existence of the DVP is the TDM briefly mentioned below. Recall that the conventional double decomposition of $\nabla\mathbf{u}$, $\nabla\mathbf{u} = \mathbf{S}+\mathbf{\Omega}$, is best viewed in the system of strain-rate principal axes showing symmetric and anti-symmetric parts of $\nabla\mathbf{u}$ separately. In this frame, the scalar quantity $|S_{12}\Omega_{12}| + |S_{23}\Omega_{23}| + |S_{31}\Omega_{31}|$ turns out to be zero. On the contrary, the TDM is viewed and performed in a rotated "basic reference frame" (BRF) where this scalar quantity is at its maximum. The TDM primarily aims at the extraction of the shear tensor $(\nabla\mathbf{u})_{\mathrm{SH}}$, so the TDM reads $\nabla\mathbf{u} = \mathbf{S}_{\mathrm{RES}} + \mathbf{\Omega}_{\mathrm{RES}} + (\nabla\mathbf{u})_{\mathrm{SH}}$, where each additive component, after its determination in the BRF, can be obtained in an arbitrary reference frame rotated with respect to the BRF under an orthogonal transformation. The extracted shear tensor $(\nabla\mathbf{u})_{\mathrm{SH}}$ is determined in the BRF in an indirect manner by defining the "complementary" residual tensor $(\nabla\mathbf{u})_{\mathrm{RES}} = \mathbf{S}_{\mathrm{RES}} + \mathbf{\Omega}_{\mathrm{RES}}$. Loosely speaking, an "overhang" causing the unbalance in magnitudes within off-diagonal pairs of $\nabla\mathbf{u}$ is virtually removed to generate $(\nabla\mathbf{u})_{\mathrm{SH}}$, and hence $(\nabla\mathbf{u})_{\mathrm{RES}}$ is defined in the form

$$\begin{aligned}(\nabla\mathbf{u})_{\mathrm{RES}} &= \mathbf{S}_{\mathrm{RES}} + \mathbf{\Omega}_{\mathrm{RES}}\\ &= \begin{pmatrix} u_x & (\operatorname{sgn} u_y)\mathrm{MIN}(|u_y|,|v_x|) & \bullet\\ (\operatorname{sgn} v_x)\mathrm{MIN}(|u_y|,|v_x|) & v_y & \bullet\\ \bullet & \bullet & w_z \end{pmatrix}.\end{aligned} \quad (7)$$

Here, the following notation is used: u, v, w are velocity components, subscripts x, y, z stand for partial derivatives. The remaining non-specified pairs of off-diagonal elements of the residual tensor in the right-hand side of Eq. (7) are constructed analogously as the specified one, each pair—if considered separately—being either symmetric or antisymmetric.

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A non-zero magnitude of $\mathbf{\Omega}_{\mathrm{RES}}$, $\|\mathbf{\Omega}_{\mathrm{RES}}\| > 0$, identifies the examined point as part of a vortex, both for incompressible and compressible fluids, and the quantity $\mathbf{\Omega}_{\mathrm{RES}}$ can be understood as local vortex intensity. The TDM can be viewed in a wider context. It has proved its usefulness in turbulence not only for characterizing vortices,[12,14,15] but also beyond vortex identification for detecting shear layers near the turbulent/non-turbulent interfaces,[16–18] for studying compressible and incompressible isotropic turbulence,[19,20] for energy stability analysis,[21] or even for answering the question “Do liquid drops roll or slide on inclined surfaces?”.[22]

To prove the non-existence of the DVP, let us again consider the transformation (1). A key aspect of the following analysis is that the transformation of the original $\nabla\mathbf{u}$-input data does not change the resulting BRF. This aspect can be seen from the structure of the BRF-decisive scalar quantity $|S_{12}\Omega_{12}| + |S_{23}\Omega_{23}| + |S_{31}\Omega_{31}|$ which is in the present case linearly increased in the same way for all rotated frames, namely by $(1+\alpha)$, and hence maximized in the same frame representing the BRF. More generally, the BRF is invariant to any transformation of the original input data $\nabla\mathbf{u}$ of the form $(\nabla\mathbf{u})_{\mathrm{NEW}} = p\mathbf{S} + r\mathbf{\Omega}$, where p and r are arbitrary real numbers. This basic property of the TDM and its BRF ensures that the sought change of balance between vorticity and strain rate can be evaluated directly in the same reference frame by considering each off-diagonal pair of $(\nabla\mathbf{u})_{\mathrm{NEW}}$ separately. Therefore, the resulting change of the criterial quantity used for vortex identification by the TDM, the residual vorticity tensor $\mathbf{\Omega}_{\mathrm{RES}}$, can be easily expressed element by element as shown below.

The TDM can be rewritten into four parts, two residual and two shear ones, by considering that the shear tensor $(\nabla\mathbf{u})_{\mathrm{SH}}$ has both symmetric and antisymmetric components (of the same magnitude) as $\nabla\mathbf{u} = (\nabla\mathbf{u})_{\mathrm{RES}} + (\nabla\mathbf{u})_{\mathrm{SH}} = (\mathbf{S}_{\mathrm{RES}} + \mathbf{\Omega}_{\mathrm{RES}}) + (\mathbf{S}_{\mathrm{SH}} + \mathbf{\Omega}_{\mathrm{SH}})$. A simple example shows how the transformation associated with the present DVP test operates. Let us assume the following illustrative $\nabla\mathbf{u}$-input data already determined in the BRF. Here, the three qualitatively different pairs of off-diagonal elements, if considered separately, represent strain-rate dominance (pair 12–21), vorticity dominance (pair 31–13), and their equilibrium in magnitudes (pair 23–32):

$$\begin{aligned}
\nabla\mathbf{u} &= \begin{pmatrix} A & 35 & -22 \\ 19 & B & 0 \\ 28 & -14 & C \end{pmatrix} \\
&= (\nabla\mathbf{u})_{\mathrm{RES}} + (\nabla\mathbf{u})_{\mathrm{SH}} = \begin{pmatrix} A & 19 & -22 \\ 19 & B & 0 \\ 22 & 0 & C \end{pmatrix} + \begin{pmatrix} 0 & 16 & 0 \\ 0 & 0 & 0 \\ 6 & -14 & 0 \end{pmatrix} \\
&= (\mathbf{S}_{\mathrm{RES}} + \mathbf{\Omega}_{\mathrm{RES}}) + (\mathbf{S}_{\mathrm{SH}} + \mathbf{\Omega}_{\mathrm{SH}}) \\
&= \begin{pmatrix} A & 19 & 0 \\ 19 & B & 0 \\ 0 & 0 & C \end{pmatrix} + \begin{pmatrix} 0 & 0 & -22 \\ 0 & 0 & 0 \\ 22 & 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 8 & 3 \\ 8 & 0 & -7 \\ 3 & -7 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 8 & -3 \\ -8 & 0 & 7 \\ 3 & -7 & 0 \end{pmatrix}.
\end{aligned} \tag{8}$$

Consequently, the new data $(\nabla\mathbf{u})_{\mathrm{NEW}}$ are obtained after the transformation (1) in a decomposed form as

$$\begin{aligned}
(\nabla\mathbf{u})_{\mathrm{NEW}} &= (\mathbf{S}_{\mathrm{RES}} + (1+\alpha)\mathbf{\Omega}_{\mathrm{RES}}) + (\mathbf{S}_{\mathrm{SH}} + (1+\alpha)\mathbf{\Omega}_{\mathrm{SH}}) \\
&= \begin{pmatrix} A & 19 & 0 \\ 19 & B & 0 \\ 0 & 0 & C \end{pmatrix} + (1+\alpha)\begin{pmatrix} 0 & 0 & -22 \\ 0 & 0 & 0 \\ 22 & 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 8 & 3 \\ 8 & 0 & -7 \\ 3 & -7 & 0 \end{pmatrix} + (1+\alpha)\begin{pmatrix} 0 & 8 & -3 \\ -8 & 0 & 7 \\ 3 & -7 & 0 \end{pmatrix}.
\end{aligned} \tag{9}$$

Note that the final matrices of (8) relate to the TDM components as indicated, though the matrices of (9) represent the intermediate result expressed through the original TDM components, but not necessarily representing the new corresponding TDM components. This requires an additional reevaluation after rearrangement as further discussed in detail.

By comparing part by part the original $\nabla\mathbf{u}$-input data from (8) and the rearranged ones from (9), the transformation works on the TDM components as follows. The diagonal elements A, B, and C forming a part of $\mathbf{S}_{\mathrm{RES}}$ remain invariant to the transformation. The original strain-rate dominated off-diagonal pair 12–21 may generate, though for sufficiently large α only, an additional non-zero residual vorticity due to a sufficiently large increase of the (original) shear vorticity compared against the unchanged available strain rate. The residual vorticity inferred from the original vorticity-dominated off-diagonal pair 31–13 is directly increased by a factor of $(1+\alpha)$ and further enhanced by an additional residual vorticity issued from a comparison of the magnified (original) shear vorticity with the unchanged shear strain-rate component. The original pair 23–32 representing a pure shear (original shear vorticity and original shear strain rate are in equilibrium in magnitudes) generates an additional residual vorticity due to the shear-vorticity increase resulting in an inevitable vorticity dominance. Finally, the last relevant situation—not considered explicitly within the above illustrative example—is the off-diagonal pair of $\nabla\mathbf{u}$ (considered again in the BRF) with exactly opposite values. Obviously, this off-diagonal pair represents nothing but the residual-vorticity component which is, after the above transformation, directly increased by a factor of $(1+\alpha)$. It should be emphasized that for the present DVP test, the starting input data generally must obtain at least one off-diagonal pair with vorticity dominance to provide a non-zero residual vorticity identifying the examined point as a part of a vortex (in our example, this is guaranteed by the pair 31–13).

Summing up, similarly as for the Q-criterion, the increasing vorticity magnitude for the otherwise unchanged vorticity-vector configuration angles and unchanged strain rate makes always a vortex, identified by the residual vorticity, stronger.

The following conclusion can be drawn from the above results: Unlike the Rortex (Liutex) method and the swirling strength criterion, the widely used Q-criterion and λ_2-criterion, as well as the residual vorticity of the TDM do not suffer from the undesirable discontinuous vortex-identification outcome reflected by the DVP.

A proper comparison of vortex-identification methods in-

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cludes both the evaluation of their performance on 3D data and the analysis of their inherent properties, which include the present interesting property of existence/non-existence of the DVP.

The final remark deals with the properties of Lagrangian criteria. The time-dependent Lagrangian methods considering the fluid particle trajectories are more difficult to analyze in the present DVP context than the Eulerian ones, and this task is beyond the scope of this short Letter. If the DVP-study is at all feasible for Lagrangian methods, it certainly needs a qualitatively different approach than just analyzing the velocity-gradient tensor as already shown for the Eulerian criteria. The difficulty lies in handling the Lagrangian quantities, as the time-dependent trajectories, the strain acceleration tensor, FTLE, etc. For example, the M_Z-criterion[2] deals with the strain acceleration tensor M over a zero-strain cone Z that travels with the trajectory, and hyperbolic and elliptic domains are distinguished. If M_Z remains positive definite, the trajectory is hyperbolic. Vortices, represented by elliptic domains, are defined as sets of fluid trajectories with indefinite M_Z. The M_Z-criterion is an objective criterion providing the same results in different rotating frames as it fulfils material objectivity or frame indifference (i.e., both translational and rotational independence). This objectivity property, missing by the Eulerian criteria discussed earlier, is an argument in favor of this criterion in situations where there is an unclear choice of a reference frame (for example, such as vortical flows in rotating tanks).

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY STATEMENT

Not applicable, the study did not report any data.

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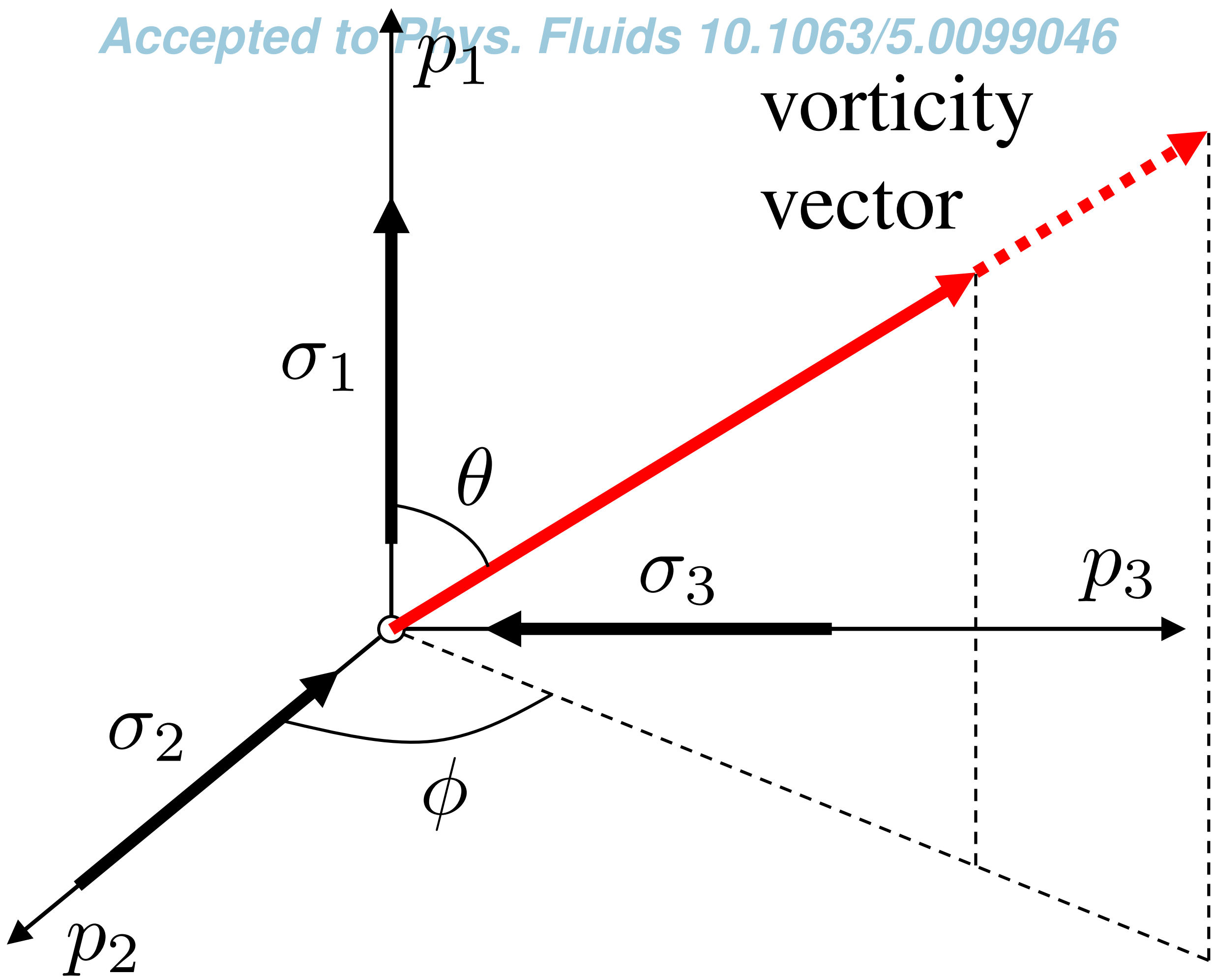
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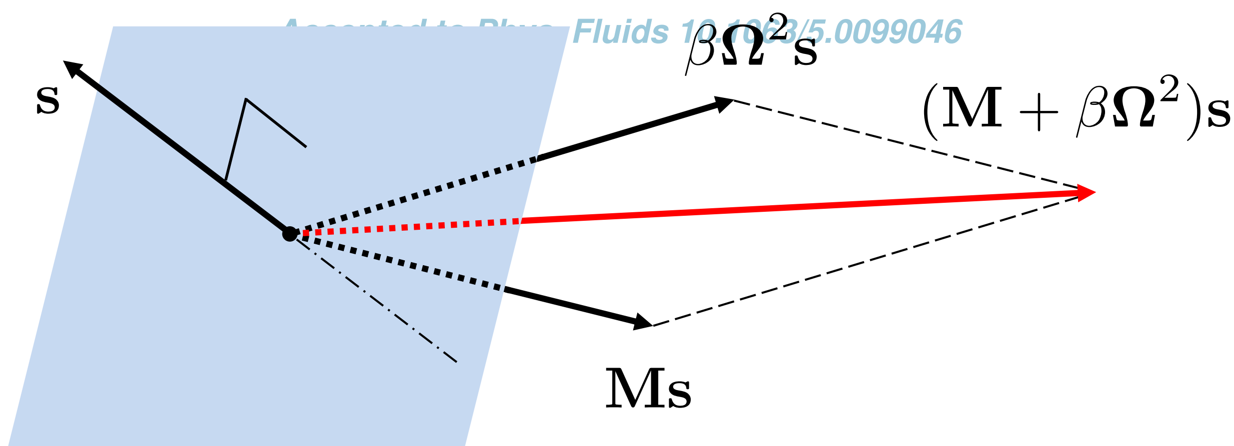
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