Two complementary eigen-based geometric properties of a vortex

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In our previous paper [V. Kolář and J. Šístek, "Orbitally compact and loose vortex regions," Phys. Fluids **35**, 121708(2023)] the eigenvalue-based measure of orbital compactness of particle motion inside a vortex for compressible flows has been proposed. Here, a complementary eigenvector-based geometric property of cylindricity (three-dimensional /3D/ aspect) is introduced. This property is closely related to the local flow axisymmetry in the swirl plane (two-dimensional /2D/ aspect) which explains the positive response of the vortex-identification Δ-criterion, and closely related criteria swirling strength and Rortex (Liutex), for almost no vorticity and a large (without any limitation) rate-of-strain magnitude. A relatively high correlation between orbital compactness, cylindricity, and widely used vortex-identification criteria has been found for several flow examples.

The well-known vortex-identification Δ-criterion,[1–4] identifies a vortex by the existence of complex eigenvalues of the velocity-gradient tensor (VGT), that is, by a positive value of the discriminant Δ. Assuming complex eigenvalues of the VGT, the corresponding local instantaneous streamline pattern is spiraling or closed on the so-called swirl plane spanned by the complex eigenvectors in a local reference frame moving with the point.

In connection with the Δ-criterion, there is a fundamental statement in Chong et al.[1] (note that the term "rate-of-deformation tensor" is employed for the VGT): *"a vortex core is a region of space where the vorticity is sufficiently strong to cause the rate-of-strain tensor to be dominated by the rotation tensor, i.e., the rate-of-deformation tensor has complex eigenvalues."* This statement has been recalled in full many times in the literature dealing with vortices.[5–16] However, this statement—of the otherwise excellent paper by Chong et al.[1]—is incorrect as the first part of the statement does not represent a necessary condition for the second one.

According to the Δ-criterion, a vortex may be positively identified by negligible vorticity and extremely large (without any limitation) rate-of-strain magnitudes, and not necessarily by the relative vorticity dominance as stated above. What is behind the existence of a vortex according to the Δ-criterion for negligible vorticity is local flow axisymmetry in the swirl plane. It can be easily verified by considering the VGT of the deviatoric form representing negligible rigid body rotation aligned with an unlimited stretching (or contraction) in the swirl plane, and perhaps even combined with an arbitrary simple shear, that is, by considering the VGT of the following form:

$$\mathrm{VGT} = \begin{pmatrix} \mp S & -\omega & \varphi \\ \omega & \mp S & \psi \\ 0 & 0 & \pm 2S \end{pmatrix} \qquad (1)$$

or its transpose.

Vorticity, denoted as ω, is non-zero but negligible, strain rate S is arbitrarily large, both ω and S are positive parameters such that $S >> \omega$, a superimposed simple shear is given by arbitrary values of φ and ψ. The discriminant of the characteristic equation reads $\Delta = (II_\mathbf{D}/3)^3 + (III_\mathbf{D}/2)^2$. Here $II_\mathbf{D}$ and $III_\mathbf{D}$ denote the second and third invariants of the deviatoric (traceless) form $\mathbf{D}$ of an arbitrary VGT input. Considering the deviatoric form allows us to identify vortices in both compressible and incompressible flows.[17] On closer examination, by evaluating Δ for non-zero vorticity, $\omega > 0$, one finds that the VGT (1) has always complex eigenvalues $(S \pm \mathrm{i}\omega)$ for stretching or $(-S \pm \mathrm{i}\omega)$ for contraction in the swirl plane as $\Delta > 0$,

$$\Delta = \frac{1}{27}\omega^6 + \frac{2}{3}\omega^4 S^2 + 3\omega^2 S^4. \qquad (2)$$

The discriminant Δ is always positive, despite the negligible vorticity and despite the unlimited stretching or contraction ($S >> \omega$). Moreover, Δ is not a function of a superimposed simple shear appearing in (1) as φ and ψ do not enter $II_\mathbf{D}$ and $III_\mathbf{D}$, and hence do not enter the expression for Δ.

The same conclusion for almost no vorticity and an extremely large (without any limitation) rate-of-strain magnitude holds for the closely related vortex-identification criteria having a basis in the already mentioned Δ-criterion: swirling strength[18,19] and Rortex (Liutex).[20,21] Behind the positive response to vortex existence of all the three mentioned criteria stands the dominance of axisymmetry in the swirl plane as already shown through Eqs. (1) and (2). Hence, by no means, the vortex existence according to the Δ-criterion is based on the condition of high vorticity-to-strain-rate ratio. This was partially treated regarding Rortex (Liutex) and swirling strength in Refs.[22,23]

Assuming a standard double decomposition of the VGT in terms of the sum of vorticity tensor $\mathbf{\Omega}$ and strain-rate tensor $\mathbf{S}$, the vortex boundary is found for the ratio of the (Frobenius) norms $\|\mathbf{\Omega}\|/\|\mathbf{S}\|$ equal (everywhere at the boundary) to one for the Q-criterion[24] and at least to (approx. minimum value of the ratio $\|\mathbf{\Omega}\|/\|\mathbf{S}\|$ at the boundary of) 0.58 for the more tolerant λ_2-criterion,[25] see Fig. 3 in Ref.[19] Interestingly, the magnitude ratio $\|\mathbf{\Omega}\|/\|\mathbf{S}\|$ may turn out to be negligibly small, $\|\mathbf{\Omega}\| << \|\mathbf{S}\|$, for the recently proposed Rortex (Liutex) by Liu et al.[20] while the so-called Ω-criterion, also recently proposed by Liu et al.[26], is based on the idea that "vorticity overtakes deformation in vortex", that is, by the condition $\|\mathbf{\Omega}\| > \|\mathbf{S}\|$, the same condition as required for

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the widely used Q-criterion.[24] Weak and dissipating vortices may be, unlike the criteria Q and λ_2 requiring a significant value of the ratio $\|\boldsymbol{\Omega}\|/\|\mathbf{S}\|$, better covered by the most tolerant VGT-based criteria such as the complex-domain-based criteria (Δ-criterion, swirling strength, and Rortex/Liutex).

The local flow axisymmetry in the swirl plane is a 2D aspect while the *cylindrical axisymmetry* (shortly, *cylindricity*) of the flow is a 3D aspect. The perfect cylindricity conveniently expressed in the frame of strain-rate principal axes is described by the VGT of the form (1) without an additional superimposed simple shear ($\varphi = \psi = 0$), that is, by the tensor form

$$\text{VGT} = \begin{pmatrix} \mp S & -\omega & 0 \\ \omega & \mp S & 0 \\ 0 & 0 & \pm 2S \end{pmatrix}. \tag{3}$$

Such a local axisymmetry is fulfilled provided that (i) the strain-rate tensor (i.e. the symmetric part of the VGT) is axisymmetric as the strain rate of the submatrix $\begin{pmatrix} \mp S & -\omega \\ \omega & \mp S \end{pmatrix}$ represents a 2D uniform radial contraction or stretching (coupled with a perpendicular uniaxial stretching or contraction in the third direction) and (ii) the vorticity vector coincides with the axis of strain-rate axisymmetry (recall that the associated vorticity tensor is given by the antisymmetric part of the VGT).

Let us show that the necessary and sufficient condition for an arbitrary VGT-input with non-zero vorticity to be axisymmetric is the orthogonality of eigen-basis vectors. The conclusion obtained in 3D can be reduced to the 2D axisymmetric case. The orthogonality is understood with respect to the usual dot product for complex vectors. The orthogonality condition is equivalent to the condition that the VGT represents a so-called normal matrix as shown below. That is, redenoting the abbreviation VGT simply by $\mathbf{G}$, this condition reads

$$\mathbf{G}\mathbf{G}^T = \mathbf{G}^T\mathbf{G} \quad \text{or} \quad (\mathbf{G}\mathbf{G}^T - \mathbf{G}^T\mathbf{G}) = \mathbf{0}. \tag{4}$$

Let us examine the 3x3 matrix $(\mathbf{G}\mathbf{G}^T - \mathbf{G}^T\mathbf{G})$ in the reference frame of strain-rate principal axes for not necessarily divergence-free input $\mathbf{G}$ with non-zero vorticity (otherwise arbitrary)

$$\mathbf{G} = \begin{pmatrix} S_1 & -\omega_3 & \omega_2 \\ \omega_3 & S_2 & -\omega_1 \\ -\omega_2 & \omega_1 & S_3 \end{pmatrix}, \tag{5}$$

$$\mathbf{G}\mathbf{G}^T - \mathbf{G}^T\mathbf{G} = \begin{pmatrix} 0 & 2\omega_3(S_1 - S_2) & 2\omega_2(S_3 - S_1) \\ 2\omega_3(S_1 - S_2) & 0 & 2\omega_1(S_2 - S_3) \\ 2\omega_2(S_3 - S_1) & 2\omega_1(S_2 - S_3) & 0 \end{pmatrix}. \tag{6}$$

According to the structure of the symmetric matrix (6), the condition (4) is satisfied only by the perfect cylindricity of $\mathbf{G}$ expressed by (3). In this case, $S_1 = S_2 = \mp S$, $S_3 = \pm 2S$, $\omega_1 = \omega_2 = 0$, $\omega_3 = \omega$. Note that the principal strain-rate differences in (6) describe just a deviatoric motion assumed already for simplicity in (1), and in its special case (3).

Both structures (1) and (3) have a pair of complex conjugate eigenvalues. The existence of complex conjugate eigenvalues of the VGT in general ensures the largest volumetric region of a vortex core among the widely used local region-type criteria Q, λ_2, Δ, and λ_{ci} for all practical purposes.[19] Recall that the streamline pattern (in a local reference frame moving with the examined point) is spiraling or closed on a swirl plane spanned by the complex eigenvectors. Within the critical-point theory these points are elliptic ones (focus or centre). The eigen decomposition of the VGT in the domain characterized by complex conjugate eigenvalues (shortly, the *complex domain* indexed as CD) reads[12]

$$(\mathbf{G})_{\mathsf{CD}} = \begin{pmatrix} | & | & | \\ \mathbf{v}_{cr} + \mathrm{i}\mathbf{v}_{ci} & \mathbf{v}_{cr} - \mathrm{i}\mathbf{v}_{ci} & \mathbf{v}_r \\ | & | & | \end{pmatrix} \begin{pmatrix} \lambda_{cr} + \mathrm{i}\lambda_{ci} & 0 & 0 \\ 0 & \lambda_{cr} - \mathrm{i}\lambda_{ci} & 0 \\ 0 & 0 & \lambda_r \end{pmatrix} \begin{pmatrix} | & | & | \\ \mathbf{v}_{cr} + \mathrm{i}\mathbf{v}_{ci} & \mathbf{v}_{cr} - \mathrm{i}\mathbf{v}_{ci} & \mathbf{v}_r \\ | & | & | \end{pmatrix}^{-1} \tag{7}$$

where $(\lambda_{cr} \pm \mathrm{i}\lambda_{ci}, \lambda_r)$ denotes the eigenvalues and $(\mathbf{v}_{cr} \pm \mathrm{i}\mathbf{v}_{ci}, \mathbf{v}_r)$ stands for the associated eigenvectors. The condition that the discriminant of the characteristic equation for $\mathbf{G}$ is positive, $\Delta > 0$, implies that $\mathbf{G}$ has complex eigenvalues ($\lambda_{ci} > 0$). The time period for completing one revolution of the streamline is given by $2\pi/\lambda_{ci}$, and therefore, the quantity λ_{ci} is taken to be defined non-negative.[19] As mentioned earlier, for including compressible flows, the discriminant condition, $\Delta > 0$, holds for deviatoric quantities.[17] The swirling-strength criterion[18,19] is given by the imaginary part λ_{ci}. This criterion is one of the most widely used and studied even from the viewpoint of the dynamics of vortices, and its evolution equation has been recently derived.[27]

Here we deal with the departure from the full 3D cylindrical axisymmetry, that is, from the perfect cylindricity, of vortical flows. Previous studies[28,29] dealt with the 2D symmetry aspects in the swirl plane only, using the ratio of real-valued dual-eigenvectors representing the orthogonal elongated directions in the swirl plane. Thus, the dual-eigenvectors were employed in Refs.[28,29] as a substitute for a pair of complex conjugate eigenvectors. Further, the departure from the 3D cylindrical axisymmetry (perfect cylindricity) has been already investigated in Kolář and Šístek[30] though using much less satisfactory approach (partially based on dual eigenvectors) than the one described below.

The departure from the perfect cylindricity expressed by the inequality $(\mathbf{G}\mathbf{G}^T - \mathbf{G}^T\mathbf{G}) \neq \mathbf{0}$ can be examined through the non-orthogonality of eigen-basis vectors within the complex domain through the corresponding determinant of unit-eigenvector matrix. The determinant of unit-eigenvector matrix has its geometric interpretation which can be derived from the geometric interpretation of a determinant itself given by the associated volume of a parallelepiped, which is here geometrically defined by the three unit eigenvectors. The value of the determinant can be negative or even complex, hence the decisive quantity representing the volume is the absolute value of the determinant, and it ranges between 0 and 1. The former holds for (at least two) coincident unit eigenvectors, the latter for orthogonal unit eigenvectors. Consequently, in the present context, this volume serves as the cylindricity measure in the complex domain, and it is labelled as τ below,

$$\tau = |\mathrm{Det}(\text{unit-eigenvector matrix})|. \quad (8)$$

Extreme situations theoretically possible are as follows: $|\mathrm{Det}(\text{unit-eigenvector matrix})| = 0$, the corresponding eigenbasis volume is equal to 0, and the cylindricity $\tau = 0$ (i.e., its minimum value); $|\mathrm{Det}(\text{unit-eigenvector matrix})| = 1$, the corresponding eigenbasis volume is equal to 1, and the cylindricity $\tau = 1$ (i.e., its maximum value). This approach is directly applicable to the departure from the local flow axisymmetry in the swirl plane (spanned by the associated complex eigenvectors) where the two unit complex eigenvectors are decisive. The determinant of the corresponding 2D matrix represents the area of a parallelogram (again ranging between 0 and 1) instead of the whole eigenvector-based volume of a parallelepiped including the real eigenvector. Note that only perfect cylindricity, $\tau = 1$, is based on the eigenvector orthogonality. The difference $(1-\tau)$ expresses how far the local geometry is from the perfect cylindricity.

There is no limitation to apply the cylindricity as a local measure everywhere the Eulerian local region-type vortex-identification schemes based on the VGT can be effectively employed (within the "complex-domain envelope").

Alongside with the local *eigenvector-based* geometric property of a vortex, the already introduced cylindricity, we have to recall the recently proposed local *eigenvalue-based* geometric property of a vortex, namely the orbital compactness. The orbital compactness requires that the separation of swirling material points inside a vortex is bounded and remains small. The idea of orbital compactness introduced in Ref.[19] for incompressible flows has been recently extended and the measure of orbital compactness has been proposed in Ref.[31] for compressible flows. Orbitally compact and loose vortex regions have been distinguished. Their boundary is set very permissively and expressed through the introduced eigenvalue-based measure of orbital compactness. The investigation[31] shows that some vortex-identification criteria are too permissive, more or less ignoring the inherent vortex property of orbital compactness. What is important in the present context is that the property of orbital compactness has been introduced as eigenvalue-based geometric property, and the relation towards eigenvector-based cylindricity is basically complementary, as schematically depicted in Fig. 1 for an arbitrary VGT input within the complex domain.

The measure of orbital compactness is explicitly sensitive to the local compressibility impact through the explicit dependence on the first VGT invariant reflecting positive or negative uniform dilatation.[31] There is no limit to describe the extreme behavior of real flows (e.g., shock waves) provided that reliable local VGT data are available, for example, from detailed numerical simulations.

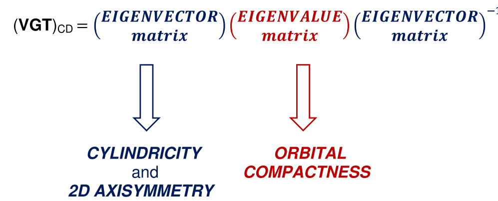

FIG. 1: Two complementary eigen-based geometric properties of a vortex derived from different parts of the eigen decomposition of the VGT within complex domain (marked by the subscript CD in the figure).

The parameter σ of orbital compactness for incompressible flows reads $\sigma = 1 - 3\,(\lambda_{cr}/\lambda_{ci})^2$ indicating compactness $\sigma = 1$ for $\lambda_{cr} = 0$. Only positive values of σ indicate that the examined point is a part of an orbitally compact vortex region. Any negative outcome represents an orbitally loose vortex region. More negative values of σ mean orbitally looser vortex regions.

The two complementary geometric properties are independent. For example, an extreme case of the perfect cylindricity ($\tau = 1$) may be combined with no orbital compactness and vice versa. All four extremal situations given by extreme values of both parameters for incompressible flow are stated below. Let us recall for simplicity the VGT structure (3) representing an incompressible flow under the already employed condition $S >> \omega$. As a result we obtain the case of perfect cylindricity, $\tau = 1$, but the measure of orbital compactness σ is extremely negative ($\sigma \to -\infty$) indicating an extremely orbitally loose vortex region. On the contrary, for the opposite condition $\omega >> S$, the VGT structure (3) results again in perfect cylindricity, $\tau = 1$, with an almost ideal orbital compactness ($\sigma \to 1$). Note that the limiting case $\sigma = 1$ represents a rigid-body rotation for incompressible flow. Moreover, the situation (1) for $\omega >> S$, now coupled with a much stronger simple shear, that is, satisfying $|\varphi| >> \omega >> S$ and/or $|\psi| >> \omega >> S$, leads to an almost ideal orbital compactness ($\sigma \to 1$) coupled with an almost zero cylindricity ($\tau \to 0$). And finally, the case (1) for $|\varphi| >> S >> \omega$ and/or $|\psi| >> S >> \omega$ gives an extremely negative σ ($\sigma \to -\infty$) indicating an extremely orbitally loose vortex region with an almost zero cylindricity ($\tau \to 0$).

The above configurations of extreme values of τ and σ are schematically depicted in Fig. 2. It should be emphasized that this schematic of streamlines stands behind the tensor forms (1) for the lower streamlines in Fig. 2 and (3) for the upper streamlines in Fig. 2. The picture would be different for different tensor structures and resulting extreme situations. For example, the usual shear appearing in the swirl plane, which may be considered as "inherent vortex shear", makes the local

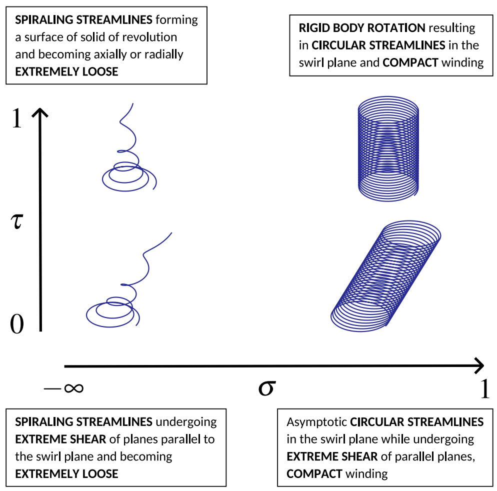

FIG. 2: Local streamline features at extreme situations for the given examples of VGT.

spiraling of streamlines in the swirl plane (or their projection onto the swirl plane) more elliptic and lowers the angle between the corresponding complex eigenvectors as well as the cylindricity outcome.

Recall that both complementary eigen-based vortex properties, orbital compactness and cylindricity, are defined exclusively for the complex domain ($\lambda_{ci} > 0$). Therefore, all the four extreme cases of the VGT discussed in the previous paragraph are still identified as part of a vortex according to the complex-domain Δ-criterion,[1–4] as well as according to the other Δ-based criteria: swirling strength[18,19] and Rortex (Liutex).[20,21] Also note that unlike the well-known vortex-identification criteria not based on the complex domain Q and λ_2, the swirling strength and Rortex (Liutex) suffer from the "disappearing vortex problem",[19,23,32] the clearly counterintuitive phenomenon appearing by increasing the vorticity magnitude for an otherwise fixed tensor configuration.

In this connection (and also in view of practical applications), it is necessary to keep in mind one issue for both geometric properties proposed: they are not defined outside the complex domain, and hence cannot provide a tool to resolve the "disappearing vortex problem" or any other problems extending beyond the boundary of the complex domain.

Both eigen-based vortex properties, though independent, are mostly well correlated with widely used vortex-identification criteria Q and λ_2 as shown in Fig. 3 using numerical data of four different flow situations. These flow situations can be summarized as follows (for more details, see Refs.[33–35]): a flow past a sphere (Re=300), a flow around an inclined flat plate (Re=300), a hairpin vortex of boundary-layer transition (Re=730, based on the displacement thickness), and the reconnection process of two Burgers vortices (circulation-based Re=10 000). The sphere and the plate problems were solved using the finite element method on unstructured meshes with 11 and 21×10^6 nodes, respectively, while the boundary-layer transition and the Burgers problems were solved using the finite difference method on structured grids with 2.3 and 2×10^6 grid points. To avoid numerical noise and identification of apparently false vortex regions, the evaluation is carried out within the rationally taken vortex envelope regions given by very low positive thresholds of λ_{ci} set for each flow situation. As already mentioned, the complex domain ensures the largest volumetric region of a vortex core among the widely used local region-type criteria.[19] These vortex envelopes are explicitly depicted for the four different flow situations in the first column of Fig. 3.

The following conclusion can be drawn: alongside with the application of the complex-domain-based criteria, it seems rational to require a certain positive level of the geometric measures proposed (e.g., as employed in Fig. 3) to avoid the over-identification of vortices in flow regions dominated by straining and shearing motions.

In this Letter, we have investigated two basic geometric properties of a vortex which are closely related to the eigen decomposition of the VGT in the domain characterized by complex conjugate eigenvalues. From the viewpoint of the eigen decomposition of the VGT, the two properties, the eigenvector-based cylindricity proposed here and the eigenvalue-based orbital compactness[31] proposed recently, can be understood to be mutually complementary.

Let us remark on the compressibility aspect. Unlike the eigenvalues and the eigenvalue-based orbital compactness, the eigenbasis vectors are independent of a non-zero isentropic compression or expansion given by a uniform dilatation and, consequently, the same holds for the eigenvector-based cylindricity measure.

The reliability of the eigen-based vortex characteristics is closely connected to the accuracy of the VGT, and therefore, these characteristics are more suitable for the local analysis of numerically simulated data (such as DNS data) than experimental ones. There are non-local vortex-identification methods that proved their usefulness for the interpretation of experimental data (such as PIV data), for example, Graftieaux et al.,[36] which are not as sensitive to the accuracy of spatial derivatives.

The applicability extent of the proposed measures in (high-Reynolds number) turbulent and transitional flows is, in general, just the same as the one of the Eulerian local region-type vortex-identification schemes based on the VGT, see, e.g., the review paper of Epps[12] and the relevant references therein.

The proposed geometric properties and measures should primarily restrict the relaxed assumption of the complex-domain-based criteria which theoretically find a vortex for almost no vorticity and an extremely large (without any limitation) rate-of-strain magnitude.

Regarding future possibilities, the combination of the obtained metrics (e.g., their product) may be plausible for incompressible flows and orbitally compact vortex regions[31] ($\sigma \geq 0$) due to their corresponding ranges between 0 and 1. In compressible flows, the orbital compactness may exceed 1

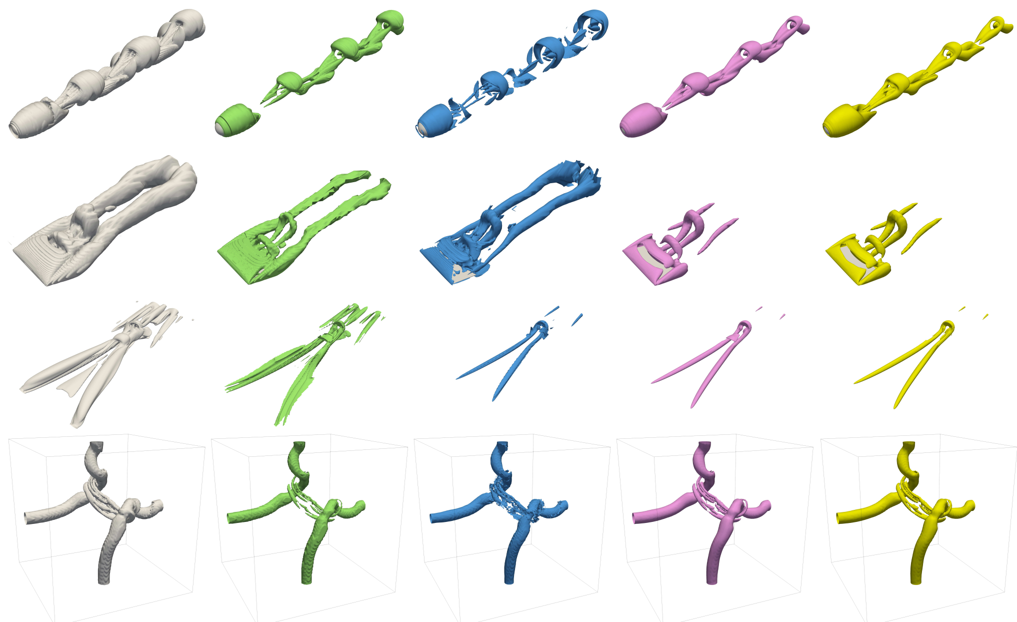

FIG. 3: From left to right: Vortex envelopes based on very low positive threshold of λ_{ci}, compactness (0.5), cylindricity (0.5), Q-criterion, and λ_2-criterion. The datasets are from top to bottom: a flow past a sphere, a flow around an inclined flat plate, a hairpin vortex of boundary-layer transition, and the reconnection process of two Burgers vortices.

without any limitation for higher compression rates while the cylindricity measure still keeps its "incompressible" range between 0 and 1. Consequently, their combination appears less suitable.

The VGT is a key quantity to describe and analyze the local flow kinematics. The velocity gradient analysis is one of the powerful tools in the research of vortices, see, among many others, the original vortex-identification paper by Chakraborty et al.[19], or, for example, the very recent paper of Arun and Colonius[37] dealing with the head-on vortex ring collision.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available from the authors upon reasonable request.

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