Discrete Maximum Principle for Higher-order Finite Elements in 1D

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Model problem

- \(-u'' = f\) in \(\Omega = (a, b)\); \(u(a) = u(b) = 0\)
Model problem

\[ -u'' = f \quad \text{in} \quad \Omega = (a, b); \quad u(a) = u(b) = 0 \]

\[ p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_{M-1} \quad p_M \]
\[ a = x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_{M-2} \quad x_{M-1} \quad b = x_M \]

\[ V_{hp} = \{ \nu_{hp} \in H^1_0(\Omega) : \nu_{hp}|_{K_i} \in P^p_i(K_i) \} \]

\[ a(u, \nu) = \int_a^b u' \nu' \, dx \quad (u, \nu) = \int_a^b uv \, dx \]

Find \( u_{hp} \in V_{hp} : a(u_{hp}, \nu_{hp}) = (f, \nu_{hp}) \quad \text{for all} \ \nu_{hp} \in V_{hp} \]
Discrete Maximum Principle (DMP)

Definition (DMP)

\[ f \geq 0 \text{ a.e. in } \Omega \quad \Rightarrow \quad u_{hp} \geq 0 \text{ in } \Omega. \]
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*Not valid for* \( p = 3, 5 \) *and* \( p \geq 7 \).

**Example** \((\Omega = (-1, 1), \ K_1, \ p_1 = 3, \ f(x) = 200e^{-10(x+1)} )\)
Discrete Green’s function

Definition
For $z \in \bar{\Omega}$ find $G_{hp,z} \in V_{hp} : a(w_{hp}, G_{hp,z}) = \delta_z(w_{hp}) \quad \forall w_{hp} \in V_{hp}$
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\[ u_{hp}(z) = \int_a^b f(x)G_{hp}(x,z) \, dx \quad G_{hp}(x,z) = G_{hp,z}(x) \]
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\( G_{hp}(x, z) = \sum_{j=1}^N \sum_{k=1}^N A_{jk}^{-1} \phi_k(x) \phi_j(z), \)

where \( \{\phi_1, \phi_2, \ldots, \phi_N\} \) is a basis of \( V_{hp} \) and \( A_{ij} = a(\phi_j, \phi_i) \).
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\( a(\cdot, \cdot) \) symmetric \( \Rightarrow \) \( G_{hp}(x, y) = G_{hp}(y, x) \)

\( a(\phi_j, \phi_i) = \delta_{ij} \Rightarrow \) \( G_{hp}(x, z) = \sum_{i=1}^{N} \phi_i(x) \phi_i(z) \)

\( A \) symmetric positive definite \( \Rightarrow \) \( G_{hp}(x, x) > 0 \, \forall x \in \Omega \)
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Definition
For \( z \in \overline{\Omega} \) find \( G_{hp,z} \in V_{hp} : a(w_{hp}, G_{hp,z}) = \delta_z(w_{hp}) \quad \forall w_{hp} \in V_{hp} \)

\[
\begin{align*}
\nabla u_{hp}(z) &= \int_a^b f(x) G_{hp}(x, z) \, dx \\
\nG_{hp}(x, z) &= \sum_{j=1}^N \sum_{k=1}^N A_{jk}^{-1} \phi_k(x) \phi_j(z), \\
\text{where } \{\phi_1, \phi_2, \ldots, \phi_N\} \text{ is a basis of } V_{hp} \text{ and } A_{ij} = a(\phi_j, \phi_i). \\
a(\cdot, \cdot) \text{ symmetric } \Rightarrow G_{hp}(x, y) = G_{hp}(y, x) \\
a(\phi_j, \phi_i) = \delta_{ij} \Rightarrow G_{hp}(x, z) = \sum_{i=1}^N \phi_i(x) \phi_i(z) \\
A \text{ symmetric positive definite } \Rightarrow G_{hp}(x, x) > 0 \quad \forall x \in \Omega \\
DMP \iff G_{hp}(x, z) \geq 0 \text{ in } \Omega^2
\end{align*}
\]
hp-FEM basis in 1D

\[ a = x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_{M-1} \quad x_M = b \]

\[ h_1 \quad h_2 \quad \ldots \quad h_M \]

\[ p_1 = 1 \quad p_2 = 2 \quad p_3 = 3 \]

\[ a = x_0 \quad x_1 \quad x_2 \quad x_3 = b \]

\[ \chi_{K_2} \quad \chi_{K_3} \]

\[ \Phi_M, \Phi_{M+1}, \ldots, \Phi_N \]
$hp$-FEM basis in 1D

\[ l_0(\xi) = \frac{(1 - \xi)}{2}, \quad \xi \in \hat{K} = [-1, 1] \]

\[ l_1(\xi) = \frac{(1 + \xi)}{2} \]

\[ l_j(\xi) = \frac{\sqrt{2j - 1}}{2} \int_{-1}^{\xi} P_{j-1}(x) \, dx \]

\[ \int_{-1}^{1} l'_i(\xi) l'_j(\xi) \, d\xi = \delta_{ij}, \quad i, j = 2, 3, \ldots \]

\[ l_j(\xi) = l_0(\xi) l_1(\xi) \kappa_j(\xi) \]

\[ \kappa_j(\xi) = \sqrt{\frac{2j - 1}{2}} \frac{4}{j(1 - j)} P'_{j-1}(\xi) \]

\[ \int_{-1}^{1} \frac{(1 - \xi^2)}{4} \kappa_i(\xi) \kappa_j(\xi) \, d\xi = \begin{cases} 0, & i \neq j \\ \frac{4}{j(j-1)}, & i = j \end{cases} \]
Stiffness Matrix

\[ A_{ij} = a(\phi'_j, \phi'_i) \]

\[ A = \begin{pmatrix} A^L & 0 \\ 0 & D \end{pmatrix} \]

\[ A^L = \begin{pmatrix}
\frac{1}{h_1} + \frac{1}{h_2} & -\frac{1}{h_2} & 0 & 0 & \cdots \\
-\frac{1}{h_2} & \frac{1}{h_2} + \frac{1}{h_3} & -\frac{1}{h_3} & 0 & \cdots \\
0 & -\frac{1}{h_3} & \frac{1}{h_3} + \frac{1}{h_4} & -\frac{1}{h_4} & \cdots \\
0 & 0 & -\frac{1}{h_4} & \frac{1}{h_4} + \frac{1}{h_5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \]

\[ D = \text{diag} \left( \frac{2}{h_1}, \ldots, \frac{2}{h_1}, \frac{2}{h_2}, \ldots, \frac{2}{h_2}, \ldots, \frac{2}{h_M}, \ldots, \frac{2}{h_M} \right) \]

\[ (p_1-1) \text{ times} \]

\[ (p_2-1) \text{ times} \]

\[ (p_M-1) \text{ times} \]
Stiffness Matrix

\[ A_{ij} = a(\phi_j', \phi_i') \quad A^{-1} = \begin{pmatrix} (A^L)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \]

\[
(A^L)^{-1} = \frac{1}{b - a} \\
\begin{pmatrix}
(x_1 - a)(b - x_1) & (x_1 - a)(b - x_2) & (x_1 - a)(b - x_3) & \ldots \\
(x_1 - a)(b - x_2) & (x_2 - a)(b - x_2) & (x_2 - a)(b - x_3) & \ldots \\
(x_1 - a)(b - x_3) & (x_2 - a)(b - x_3) & (x_3 - a)(b - x_3) & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

\[ D^{-1} = \text{diag} \left( \frac{h_1}{2}, \ldots, \frac{h_1}{2}, \frac{h_2}{2}, \ldots, \frac{h_2}{2}, \ldots, \frac{h_M}{2}, \ldots, \frac{h_M}{2} \right) \]

\( (p_1 - 1) \text{ times} \)
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\[ D^{-1} = \text{diag}\left( \frac{h_1}{2}, \ldots, \frac{h_1}{2}, \frac{h_2}{2}, \ldots, \frac{h_2}{2}, \ldots, \frac{h_M}{2}, \ldots, \frac{h_M}{2} \right) \]

\[ (p_1-1) \text{ times} \]

\[ (p_2-1) \text{ times} \]

\[ (p_M-1) \text{ times} \]

\[ G_{hp}(x, z) = \sum_{j=1}^{N} \sum_{k=1}^{N} A_{jk}^{-1} \phi_k(x) \phi_j(z) \]
DGF explicitely

\[ G_{hp}(x, z) = G^L_{hp}(x, z) + G^B_{hp}(x, z) \]

\[ G^L_{hp}(x, z) = \frac{1}{b - a} \left( \sum_{i=1}^{M-1} (x_i - a)(b - x_i)\phi_i(x)\phi_i(z) \right. \]
\[ \left. + \sum_{i=1}^{M-2} \sum_{j=i+1}^{M-1} (x_i - a)(b - x_j)[\phi_i(x)\phi_j(z) + \phi_j(x)\phi_i(z)] \right) \]
\[ \geq 0 \quad \forall [x, z] \in \Omega^2 \]

\[ G^B_{hp}(x, z) = \sum_{k=M}^{N} D_{kk}^{-1} \phi_k(x)\phi_k(z) \geq 0 \text{ in } \Omega^2 \]
DGF example
DGF example
Restriction to $K_i^2$

\[ K_i = [x_{i-1}, x_i] \]

\[ G_{hp}(x, z)|_{K_i^2} = \]
\[ \frac{(x_{i-1} - a)(b - x_{i-1})}{b - a} \phi_{i-1}(x)\phi_{i-1}(z) + \frac{(x_i - a)(b - x_i)}{b - a} \phi_i(x)\phi_i(z) \]
\[ + \frac{(x_{i-1} - a)(b - x_i)}{b - a} [\phi_i(x)\phi_{i-1}(z) + \phi_{i-1}(x)\phi_i(z)] \]
\[ + \frac{x_i - x_{i-1}}{2} G_{Bhp}(x, z)|_{K_i^2} \]

\[ G_{hp}^B(x, z)|_{K_i^2} = \sum_{k=0}^{p-2} \phi_{k+B}(x)\phi_{k+B+k}(z) \]
Technical modifications

- Transformation \( (x, z) \in K_i^2 \mapsto (\xi, \eta) \in [-1, 1]^2 = \hat{K}^2 \)
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- Transformation \((x, z) \in K_i^2 \leftrightarrow (\xi, \eta) \in [-1, 1]^2 = \hat{K}^2\)

- \(H = x_i - x_{i-1}\)

- \(H_{\text{rel}} = \frac{H}{b - a}\)
Technical modifications

- Transformation $\mathbf{K}_i^2 \ni (x, z) \mapsto (\xi, \eta) \in [-1, 1]^2 = \hat{K}^2$

- $H = x_i - x_{i-1}$

- $H_{rel} = \frac{H}{b-a}$

- $\exists t \in [0, 1] :$

$$x_{i-1} = (1 - t)a + t(b - H)$$

$$x_i = (1 - t)(a + H) + tb \quad t \in [0, 1]$$
Relative Critical Element Length

\[ \frac{\hat{G}_{hp}(\xi, \eta)}{H} = t(1 - t) \frac{(1 - H_{rel})^2}{H_{rel}} \]

\[ + tl_0(\xi)l_0(\eta) \left[ 1 - H_{rel} + \frac{1}{2} l_1(\xi)l_1(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta) \right] \]

\[ + (1 - t)l_1(\xi)l_1(\eta) \left[ 1 - H_{rel} + \frac{1}{2} l_0(\xi)l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta) \right] \]

Definition

\[ H_{rel}^*(1) = 1 \]

\[ H_{rel}^*(p) = 1 + \frac{1}{2} \min_{(\xi, \eta) \in \hat{K}^2} l_0(\xi)l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta) \]

\[ = 1 + \frac{1}{2} \min_{(\xi, \eta) \in \hat{K}^2} l_1(\xi)l_1(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta) \text{ for } p \geq 2 \]
Subcritical Elements $\Rightarrow G_{hp} \geq 0$

Theorem

If $a \leq x_{i-1} < x_i \leq b$ and \( \frac{x_i - x_{i-1}}{b - a} = H_{\text{rel}} \leq H_{\text{rel}}^*(p) \), then $\hat{G}_{hp}(\xi, \eta) \geq 0$ for all $[\xi, \eta] \in \hat{K}^2 = [-1, 1]^2$. 
Main Result

Theorem

If the partition \( a = x_0 < x_1 < \ldots < x_M = b \) of the domain \( \Omega = (a,b) \) satisfies the condition

\[
\frac{x_i - x_{i-1}}{b - a} \leq H_{\text{rel}}^*(p_i) \quad \text{for all } i = 1, 2, \ldots, M,
\]

where \( p_i \geq 1 \) is the polynomial degree assigned to the element \( K_i = [x_{i-1}, x_i] \), then the problem satisfies the discrete maximum principle (i.e., \( u_{hp} \geq 0 \) in \( \Omega \) for arbitrary \( f \in L^2(\Omega) \) which is nonnegative a.e. in \( \Omega \)).
Value of $H_{\text{rel}}^*(p)$

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<th>$p$</th>
<th>$H_{\text{rel}}^*(p)$</th>
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<tr>
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\[
H_{\text{rel}}^*(p) = 1 + \frac{1}{2} \min_{(\xi,\eta)\in \hat{K}^2} l_0(\xi)l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi)\kappa_k(\eta)
\]
Value of $H^*_{rel}(p)$

\[ H^*_{rel}(p) = 1 + \frac{1}{2} \min_{(\xi, \eta) \in \hat{K}^2} l_0(\xi) l_0(\eta) \sum_{k=2}^{p} \kappa_k(\xi) \kappa_k(\eta) \]
Conclusions

▶ 1D Poisson equation + Dirichlet BC ⇒ DMP if $H_{rel} \leq 9/10$

(http:\\www.math.utep.edu/preprints)
Conclusions

- 1D Poisson equation + Dirichlet BC $\Rightarrow$ DMP if $H_{rel} \leq 9/10$ (http:\\www.math.utep.edu/preprints)

- 1D Poisson equation + mixed BC
  $\Rightarrow$ DMP on arbitrary $hp$-mesh.
Conclusions

- 1D Poisson equation + Dirichlet BC ⇒ DMP if \( H_{rel} \leq 9/10 \) ([http://www.math.utep.edu/preprints](http://www.math.utep.edu/preprints))

- 1D Poisson equation + mixed BC
  ⇒ DMP on arbitrary \( hp \)-mesh.

- Generalization

\[-(au')' = f, \quad a \text{ is piecewise constant}\]
Thank you for your attention

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